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A New Asymptotic Solution for Third Order More Critically Damped Nonlinear Systems

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***Abstract*— A third order nonlinear differential equation modeling a more critically damped system is considered. A new perturbation technique based on the Krylov-Bogoliubov-Mitropolskii (KBM) method is developed for obtaining the transient response in the presence of different damping forces as well as for different sets initial conditions. For large eigenvalues, the technique presented in this article gives better results than the technique presented by Shamsul.**

***Index Terms*— Perturbation, Asymptotic Solution, More Critically Damped Systems.**

I. INTRODUCTION

The Krylov-Bogoliubov-Mitropolskii (KBM) [2], [4] method is one of the most convenient and extensively used methods to study nonlinear differential systems with small nonlinearites. Originally, the method was developed by Krylov and Bogoliubov [2] for obtaining the periodic solutions of second order nonlinear differential systems. Later, the method was amplified and justified mathematically by Bogoliubov and Mitropolskii [4]. Popov [8] extended the KBM method to damped oscillatory nonlinear processes in which strong linear damping forces were active. Murty *et al*. [5] and Shamsul [14] extended the KBM method for solving over-damped nonlinear systems. Sattar [11] examined an asymptotic solution of second order critically damped nonlinear systems.

First, Osiniskii [7] investigated the solution of third order nonlinear systems by Bogoliubov's method imposing some restrictions on the parameters and thus the solution was over-simplified and ultimately gave incorrect results. Mulholland [6] removed the restrictions imposed by Osiniskii and found the desired solution. Bojadziev [3] and Sattar [12] respectively investigated solutions of the similar type of three dimensional damped and over-damped systems. Shamsul [16] examined a solution of third order over-damped nonlinear systems, when certain relations exist among the eigenvalues of the corresponding linear systems.

Shamsul and Sattar [13] have extended Bogoliubov's asymptotic method for obtaining the solution of third order critically damped nonlinear systems. Shamsul [15] has also investigated solutions of third order critically damped nonlinear systems whose unequal eigenvalues are in integral multiple and in the same article [15] Shamsul also extended the KBM method to solve more critically damped systems.

In the present article, we have investigated a new asymptotic solution for third order more critically damped nonlinear systems. For numerically large eigenvalue, the results obtained by the solution, presented in this article are better than the results obtained by Shamsul [15] and show good coincidence with numerical results.

II. THE METHOD

Let us consider a weakly nonlinear system governed by the third order ordinary differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = -\varepsilon\, f(x,\dot{x},\ddot{x}) \qquad (1)$$

where over-dots denote the derivatives of x with respect to t; k_1, k_2, k_3 are constants, ε is the small parameter and f is the given nonlinear function. When $\varepsilon = 0$, the equation (1) becomes linear and since the system is more critically damped, suppose the eigenvalues of the corresponding linear equation are $-\lambda, -\lambda, -\lambda$. Therefore, the solution of the linear equation is

$$x(t,0) = (a_0 + b_0\, t + c_0\, t^2)\, e^{-\lambda t} \qquad (2)$$

where a_0, b_0, c_0 are constants of integration.

When $\varepsilon \neq 0$, following [17], an asymptotic solution of the system (1) is presented in the form

$$x(t,\varepsilon) = (a + b\,t + c\,t^2)\, e^{-\lambda t} + \varepsilon\, u_1(a,b,c,t) + \cdots \qquad (3)$$

where each a, b, c are functions of t and satisfy the following first order differential equation

$$\begin{aligned} \dot{a}(t) &= \varepsilon\ A_1(a,b,c,t) + .\cdots \\ \dot{b}(t) &= \varepsilon\ B_1(a,b,c,t) + \cdots \\ \dot{c}(t) &= \varepsilon\ C_1(a,b,c,t) + \cdots \end{aligned} \qquad (4)$$

Confining only a first few terms $1, 2, 3, \cdots, n$ in the series expansion of (3) and (4), we evaluate the functions u_i and A_i, B_i, C_i, $i = 1, 2, 3, \cdots, n$ such that a, b, c appearing in (3) and (4) satisfy the given system (1) with an accuracy of order ε^{n+1}. In order to determine these unknown functions, following the KBM method, Murty [5] assumed that the correction terms, u_i, $i = 1, 2, \cdots, n$ must exclude the fundamental terms, since these are included in the series expansion (3) at order ε^0. Theoretically, the solution can obtain up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first (see also Murty [5] for details).

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Differentiating the equation (3), three times with respect to t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ in the original equation (1), utilizing the relations presented in (4) and finally equating the coefficients of ε, we obtain

$$e^{-\lambda t}\left\{\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+t\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+t^2\frac{\partial^2 C_1}{\partial t^2}\right\}+\left(\frac{\partial}{\partial t}+\lambda\right)^3 u_1=-f^{(0)}(a,b,c,t) \quad (5)$$

where $f^{(0)}(a,b,c,t)=f(x_0,\dot{x}_0,\ddot{x}_0)$ and $x_0=(a+bt+ct^2)e^{-\lambda t}$.

In this article, we have expanded the functional $f^{(0)}$ in the Taylor's series of the form (see also [16], [18] for details)

$$f^{(0)}=\sum_{i=1}^{\infty}F_0(a,b,c)e^{-i\lambda t}+t\sum_{i=1}^{\infty}F_1(a,b,c)e^{-i\lambda t}+t^2\sum_{i=1}^{\infty}F_2(a,b,c)e^{-i\lambda t}+t^3\sum_{i=1}^{\infty}F_3(a,b,c)e^{-i\lambda t}+\cdots \quad (6)$$

Substituting the value of $f^{(0)}$ from (6) into (5), we obtain

$$e^{-\lambda t}\left\{\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+t\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+t^2\frac{\partial^2 C_1}{\partial t^2}\right\}+\left(\frac{\partial}{\partial t}+\lambda\right)^3 u_1=-\sum_{i=1}^{\infty}F_0(a,b,c)e^{-i\lambda t}-t\sum_{i=1}^{\infty}F_1(a,b,c)e^{-i\lambda t}-t^2\sum_{i=1}^{\infty}F_2(a,b,c)e^{-i\lambda t}-t^3\sum_{i=1}^{\infty}F_3(a,b,c)e^{-i\lambda t}-\cdots \quad (7)$$

According to the KBM method (see also [5], [9], [10], [14], [18] for details) u_1 does not contain the fundamental terms (the solution (2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. Therefore, equation (7) can be separated for unknown functions A_1, B_1, C_1 and u_1 (see also [1], [18] for details) in the following way:

$$e^{-\lambda t}\frac{\partial^2 C_1}{\partial t^2}=-\sum_{i=1}^{\infty}F_2(a,b,c)e^{-i\lambda t} \quad (8)$$

$$e^{-\lambda t}\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-\sum_{i=1}^{\infty}F_1(a,b,c)e^{-i\lambda t} \quad (9)$$

$$e^{-\lambda t}\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)=-\sum_{i=1}^{\infty}F_0(a,b,c)e^{-i\lambda t} \quad (10)$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)^3 u_1=-t^3\sum_{i=1}^{\infty}F_3(a,b,c)e^{-i\lambda t}-\cdots \quad (11)$$

Now, solving the equation (8), we obtain

$$C_1=-\sum_{i=1}^{\infty}F_2(a,b,c)e^{-(i-1)\lambda t}\times\left[(i-1)\lambda\right]^{-2} \quad (12)$$

Putting the value of C_1 from (12) into equation (9) and solving, we obtain

$$B_1=-\left\{\sum_{i=1}^{\infty}F_1(a,b,c)\left[(i-1)\lambda\right]^{-2}+6\sum_{i=1}^{\infty}F_2(a,b,c)\left[(i-1)\lambda\right]^{-3}\right\}e^{-(i-1)\lambda t} \quad (13)$$

Substituting the value of B_1 and C_1 from (13) and (12) respectively into equation (10) and solving, we obtain

$$A_1=-\left\{\sum_{i=1}^{\infty}F_0(a,b,c)\left[(i-1)\lambda\right]^{-2}+3\sum_{i=1}^{\infty}F_1(a,b,c)\times\left[(i-1)\lambda\right]^{-3}+12\sum_{i=1}^{\infty}F_2(a,b,c)\left[(i-1)\lambda\right]^{-4}\right\}e^{-(i-1)\lambda t} \quad (14)$$

Equation (11) is a third order non-homogeneous linear differential equation; so, we can solve the equation (11) for u_1 by the well-known operator method.

Substituting the values of A_1, B_1 and C_1 into the equation (4) and then integrating, we obtain the values of a, b and c.

Thus, the determination of the first order improved solution is completed.

III. EXAMPLE

As an example of the above method, we have considered a nonlinear mechanical system with internal friction and relaxation (see also [7], [13], [15])

$$m\ddot{x}+\sigma=0$$
$$\sigma+\gamma\dot{\sigma}=\alpha x+\beta\dot{x}+s x^3, \quad s<1 \quad (15)$$

Here, x is the deformation, σ is the stress, m is the mass of the system, α, β, γ and s are constants. The terms with coefficients α and s represent respectively the linear and nonlinear elasticity, the term with coefficient β corresponds to the linear viscous damping and the term with coefficient γ reflects the linear relaxation. In the case of small internal friction, one can neglect the effect of relaxation. However, there exist phenomena in which the influence of relaxation is significant, such as, the plastic materials, and the study of such cases based on the assumption of lack of relaxation may severely limit their closeness to the reality. By a little effort, the above system (15) can be reducing to the third order nonlinear differential system as:

$$\dddot{x}+\gamma^{-1}\ddot{x}+\beta\gamma^{-1}m^{-1}\dot{x}+\alpha\gamma^{-1}m^{-1}x=-s\gamma^{-1}m^{-1}x^3 \quad (16)$$

It is clear that the equation (16) is a particular case of the nonlinear system (1). Therefore, comparing (1) and (16), we obtain $k_1=\gamma^{-1}$, $k_2=\beta\gamma^{-1}m^{-1}$, $k_3=\alpha\gamma^{-1}m^{-1}$, $\varepsilon=s\gamma^{-1}m^{-1}$ and $f=x^3$. And we obtain

$$f^{(0)}=\left\{a^3+t\,3a^2b+t^2\left(3ab^2+3a^2c\right)+b^3t^3+6t^3abc+3t^4bc^2+3t^4ca^2+3t^5bc^2+t^6c^3\right\}e^{-3\lambda t}$$

So, for equation (16), equations (8)-(11) respectively become

$$e^{-\lambda t}\frac{\partial^2 C_1}{\partial t^2}=-\left(3ab^2+3a^2c\right)e^{-3\lambda t} \quad (17)$$

$$e^{-\lambda t}\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-3a^2be^{-3\lambda t} \quad (18)$$

$$e^{-\lambda t}\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)=-a^3e^{-3\lambda t} \tag{19}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)^3 u_1=-\left(b^3t^3+6t^3\,abc+3t^4b\,c^2+3t^4c\,a^2+3t^5\,bc^2+t^6c^3\right)e^{-3\lambda t} \tag{20}$$

The solution of the equation (17) is

$$C_1=-(3/4)P^2\left(ab^2+a^2\,c\right)e^{-2\lambda t},\quad \text{where } P=\lambda^{-1} \tag{21}$$

Putting the value of C_1 from equation (21) into the equation (18) and solving, we obtain

$$B_1=-\left\{(9/4)P^3\left(ab^2+a^2\,c\right)+(3/4)P^2\,a^2b\right\}e^{-2\lambda t} \tag{22}$$

Substituting the value of B_1 and C_1 from the equations (22) and (21) respectively, into the equation (19) and solving, we obtain

$$A_1=\left\{-(9/4)P^3(ab^2+a^2c)-(9/8)P^3a^2b-(1/4)P^2a^3\right\}e^{-2\lambda t} \tag{23}$$

And the solution of the equation (20) is

$$\begin{aligned}u_1=e^{-3\lambda t}\Big\{&(b^3+6\,abc)\left(r_1\,t^3+r_2\,t^2+r_3\,t+r_4\right)\\&+(b\,c^2+c\,a^2)\left(r_5\,t^4+r_6\,t^3+r_7\,t^2+r_8\,t+r_9\right)\\&+bc^2\left(r_{10}\,t^5+r_{11}\,t^4+r_{12}\,t^3+r_{13}\,t^2+r_{14}\,t+r_{15}\right)\\&+c^3\left(r_{16}\,t^6+r_{17}\,t^5+r_{18}\,t^4+r_{19}\,t^3+r_{20}\,t^2+r_{21}\,t+r_{22}\right)\Big\}\end{aligned} \tag{24}$$

where $r_1=(1/8)P^3$, $r_2=r_1\times(9/2)P$, $r_3=r_1\times 9P^2$, $r_4=r_1\times(15/2)P^3$, $r_5=(3/8)P^3$, $r_6=r_5\times 6P$, $r_7=r_5\times 18P^2$, $r_8=r_5\times 30P^3$, $r_9=r_5\times(45/2)P^4$, $r_{10}=r_5\times(15/2)P$, $r_{11}=r_5\times 30P^2$, $r_{12}=r_5\times 75P^3$, $r_{13}=r_5\times(225/2)P^4$, $r_{14}=r_5\times(315/4)P^5$, $r_{15}=r_1\times 9P$, $r_{16}=r_1\times 45P^2$, $r_{17}=r_1\times 150P^3$, $r_{18}=r_1\times(135/4)P^4$, $r_{19}=r_1\times(945/2)P^5$, $r_{20}=r_1\times 315P^6$.

Substituting the values of A_1, B_1, C_1 from the equation (23), (22) and (21) into (4), we obtain

$$\begin{aligned}\dot a&=-\varepsilon\left\{(9/4)P^3(ab^2+a^2c)+(9/8)P^3a^2b+(1/4)P^2a^3\right\}e^{-2\lambda t}\\\dot b&=-\varepsilon\left\{(9/4)P^3\left(ab^2+a^2\,c\right)+(3/4)P^2\,a^2b\right\}e^{-2\lambda t}\\\dot c&=-\varepsilon\,(3/4)P^2\left(ab^2+a^2\,c\right)e^{-2\lambda t}\end{aligned} \tag{25}$$

The equations of (25) have no exact solution. But, since $\dot a, \dot b, \dot c$ are proportional to the small parameter ε, so they change slowly with time t. So, it is logical to replace a, b, c by their respective values obtained in the linear case (i.e. the values of a, b, c obtained, when $\varepsilon=0$) in the right hand side of the equations of (25). This type of replacement was first made by Murty [5] to solve similar type of nonlinear equations.

Therefore, solving the equations of (25), we obtain

$$\begin{aligned}a&=a_0+\varepsilon(1/16)P^3\left\{18P(a_0\,b_0^2+a_0^2\,c_0)+9P\,a_0^2\,b_0+2\,a_0^3\right\}\left(e^{-2\lambda t}-1\right)\\b&=b_0+\varepsilon(3/8)P^3\left\{3P\left(a_0b_0^2+a_0^2c_0\right)+a_0^2\,b_0\right\}\left(e^{-2\lambda t}-1\right)\\c&=c_0+\varepsilon(3/8)P^3\left(a_0b_0^2+a_0^2\,c_0\right)\left(e^{-2\lambda t}-1\right)\end{aligned} \tag{26}$$

Consequently, we obtain the first order improved solution of the equation (16) as:

$$x(t,\varepsilon)=(a+b\,t+c\,t^2)\,e^{-\lambda t}+\varepsilon\,u_1 \tag{27}$$

where a, b, c are given by the equation (26) and u_1 given by (24).

IV. DISCUSSION OF THE METHOD OF SHAMSUL [15]

Shamsul [15] found an asymptotic solution of the nonlinear system (1) which is identical to the form, as we have considered in the equation (3) and the variational equations are also identical to the form as we have considered in the equation (4).

We have extended the functional $f^{(0)}$ in the Taylor's series of the form which is given by the equation (6), but Shamsul [15] expanded the functional $f^{(0)}$ in the Taylor's series about $t=\frac{-b}{c}$. i.e. in powers of $(b+c\,t)$. Therefore, he obtained

$$f^{(0)}=F_0(a,t)+F_1(a,t)(b+c\,t)+F_2(a,t)(b+c\,t)^2+\cdots \tag{28}$$

where $F_0, F_1, \cdots, F_n$ do not contain the terms of the form $t, t^2, t^3, \cdots, t^n$.

Following the KBM [2], [4] method, Murty [5], Sattar [11], Shamsul and Sattar [13], Shamsul, in article [15] assumed that u_1 does not contain the terms with $(b+c\,t)^0$ and $(b+c\,t)^1$ of $f^{(0)}$, since these are already included in the series expansion (3) at order ε^0.

Therefore, putting the values of $f^{(0)}$ from (28) into (5) and equating the coefficients of t^0, t^1 and $t^r, r\ge 2$, he obtained

$$e^{-\lambda t}\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)=-F_0 \tag{29}$$

$$e^{-\lambda t}\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-b\,F_1 \tag{30}$$

$$e^{-\lambda t}\,\frac{\partial^2 C_1}{\partial t^2}=-cF_1 \tag{31}$$

and $$\left(\frac{\partial}{\partial t}+\lambda\right)^3 u_1=-F_2\,(b+c\,t)t^2-\cdots \tag{32}$$

Solving the equations (29)-(32), he obtained the unknown functions A_1, B_1, C_1 and u_1. Finally, substituting the values of A_1, B_1, C_1 in the equation (4) and integrating them, Shamsul [15] obtained the values of a, b and c. This completes the determination of the solution of the system (1).

Therefore, for the example (16), Shamsul [15] obtained, $F_0=a^3e^{-3\lambda t}$, $F_1=a^2e^{-3\lambda t}$, $F_2=a\,e^{-3\lambda t}$ and $F_2=e^{-3\lambda t}$.

Substituting the values of F_0 and F_1 into equations (29)-(31) and then solving, he obtained

$$
\begin{aligned}
A_1 &= -\frac{a^2}{8}\left(\frac{2a}{\lambda^2}+\frac{9b}{\lambda^3}+\frac{18c}{\lambda^4}\right)e^{-2\lambda t} \\
B_1 &= -\frac{3a^2}{8}\left(\frac{b}{\lambda^2}+\frac{3c}{\lambda^3}\right)e^{-2\lambda t} \\
C_1 &= -\frac{3a^2c}{4\lambda^2}e^{-2\lambda t}
\end{aligned} \tag{33}
$$

Again putting the values of F_2 and F_3 into equation (32) and then solving, he obtained

$$
\begin{aligned}
u_1 = \frac{e^{-3\lambda t}}{8\lambda^3}\Bigg\{ & \frac{9ab^2}{\lambda^2}+\frac{15}{2\lambda^3}\left(b^3+6abc\right) \\
& +\frac{135}{\lambda^4}\left(b^2c+ac^2\right)+\frac{945}{\lambda^5}bc^2+\frac{315}{\lambda^6}c^3 \\
& +\Bigg(\frac{9ab^2}{\lambda}+\frac{9}{\lambda^2}\left(b^3+6abc\right)+\frac{90}{\lambda^3}\left(b^2c+ac^2\right) \\
& +\frac{675}{4\lambda^4}bc^2+\frac{945}{2\lambda^5}c^3\Bigg)t+\Bigg(3ab^2+\frac{9}{2\lambda}\left(b^3+6abc\right) \\
& +\frac{54}{\lambda}\left(b^2c+ac^2\right)+\frac{225}{\lambda^2}bc^2+\frac{675}{8\lambda^3}c^3\Bigg)t^2\Bigg\}
\end{aligned} \tag{34}
$$

Therefore, substituting the values of A_1, B_1 and C_1 into (4), he obtained

$$
\begin{aligned}
\dot{a} &= -\varepsilon\,\frac{a^2}{8}\left(\frac{2a}{\lambda^2}+\frac{9b}{\lambda^3}+\frac{18c}{\lambda^4}\right)e^{-2\lambda t} \\
\dot{b} &= -\varepsilon\,\frac{3a^2}{8}\left(\frac{b}{\lambda^2}+\frac{3c}{\lambda^3}\right)e^{-2\lambda t} \\
\dot{c} &= -\varepsilon\,\frac{3a^2c}{4\lambda^2}e^{-2\lambda t}
\end{aligned} \tag{35}
$$

Solving equation (35), he obtained

$$
\begin{aligned}
a &= a_0+\varepsilon\,\frac{a_0^2}{16}\left(\frac{2a_0}{\lambda^3}+\frac{9b_0}{\lambda^4}+\frac{18c_0}{\lambda^5}\right)\left(e^{-2\lambda t}-1\right) \\
b &= b_0+\varepsilon\,\frac{3a_0^2}{8}\left(\frac{b_0}{\lambda^3}+\frac{3c_0}{\lambda^4}\right)\left(e^{-2\lambda t}-1\right) \\
c &= c_0+\varepsilon\,\frac{3a_0^2c_0}{8\lambda^3}\left(e^{-2\lambda t}-1\right)
\end{aligned} \tag{36}
$$

Thus, Shamsul [15] obtained the first order improved solution of the more critically damped nonlinear system (16) as:

$$x(t,\varepsilon)=(a+bt+ct^2)e^{-\lambda t}+\varepsilon u_1+O(\varepsilon^2) \tag{37}$$

where a, b, c are given by (36) and u_1 is given by (34).

V. General Discussion and Results

General Discussion: Shamsul [15] expanded the functional $f^{(0)}$ in powers of $(b+ct)$. i.e. he expanded $f^{(0)}$ at $t=\frac{-b}{c}$ (**see in discussion of Shamsul [15]**). On the other hand, in this article, we have expanded the functional $f^{(0)}$ in power of t . *i. e.* we have extended $f^{(0)}$, at $t=0$. Simply, one may claim that, our expansion equation (6) is a special case of the expansion equation (28) presented by Shamsul [15]. i.e. the expansion equation (6) can be obtained by putting $b=0$ in the expansion equation (28). But, this claim is not true, because, if one put $b=0$ in the expansion equation (28), then it takes the form $f^{(0)}=F_0(a,t)+F_1(a,t)\,c\,t+F_2(a,t)\,c^2\,t^2+\cdots$, which is not at all identical to the expansion equation (6) as well as the solution (3) reduces to $x(t,\varepsilon)=(a+ct^2)\,e^{-\lambda t}+\varepsilon\,u_1(a,c,t)+\cdots$, which is not the actual solution of the nonlinear system (1). Therefore, our expansion equation (6) is fully independent. As a result our variational equation (25) is different from the variational equation (35) of Shamsul [15] and our correction term u_1 given by (24) is different from Shamsul's [15] correction term u_1 given by (34) (**see in discussion of Shamsul [15]**). Although the variational equation as well as correction term are different, but for different set of initial conditions as well as for different eigenvalues our solution gives desired results.

Results: Based on the KBM method an asymptotic solution of third order more critically damped nonlinear systems has been found in this article. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we some times compare the approximate results to the numerical results (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer the work of Murty [5].

First of all, we have computed $x(t,\varepsilon)$ by (27) (designated by x) in which a, b, c are computed by (26) and u_1 is computed by (24) together with the initial conditions $a_0=0.3$, $b_0=0.3$, $c_0=0.0$ for $\lambda=1.0$, $\lambda=2.0$, $\lambda=3.0$ and $\lambda=4.0$, and the results are presented in the second column of the **Table-1**, **Table-2**, **Table-3** and **Table-4** respectively. Then $x(t,\varepsilon)$ has again been calculated by (37) (designated by $x_{SA[15]}$) in which a, b, c are computed by (36) and u_1 is computed by (34) together with the same set of initial condition as well as with same eigenvalues and the results are presented in the fifth column of the **Table-1**, **Table-2**, **Table-3** and **Table-4.** Column three and six show the corresponding numerical results calculated by the fourth order Runge-Kutta method. Corresponding percentage errors have also been calculated and are presented in the fourth and seventh column of the **Table-1**, **Table-2**, **Table-3** and **Table-4**. The first column shows the various values of time t .

Table-1: For $a_0 = 0.3$, $b_0 = 0.3$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 1.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.302531	0.302531	0.0000	0.305569	0.305569	0.0000
0.5	0.271054	0.271560	0.1863	0.274089	0.274132	0.0157
1.0	0.217856	0.218792	0.4278	0.220531	0.220556	0.0113
1.5	0.164637	0.165825	0.7164	0.166868	0.166873	0.0030
2.0	0.119568	0.120819	1.0354	0.121361	0.121352	-0.0074
2.5	0.084458	0.085632	1.3709	0.085854	0.085838	-0.0186
3.0	0.058448	0.059469	1.7168	0.059508	0.059490	-0.0303
3.5	0.039818	0.040659	2.0684	0.040606	0.040588	-0.0443
4.0	0.026791	0.027456	2.4221	0.027366	0.027351	-0.0548

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

Table-2: For $a_0 = 0.3$, $b_0 = 0.3$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 2.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.300040	0.300040	0.0000	0.300134	0.300134	0.0000
0.5	0.165417	0.165516	0.0598	0.165499	0.165506	0.0042
1.0	0.081112	0.081214	0.1256	0.081168	0.081134	0.0419
1.5	0.037291	0.037363	0.1927	0.037324	0.037284	-0.1073
2.0	0.016459	0.016502	0.2606	0.016477	0.016446	-0.1885
2.5	0.007063	0.007086	0.3246	0.007072	0.007053	-0.2694
3.0	0.002969	0.002981	0.4025	0.002973	0.002963	-0.3375
3.5	0.001228	0.001234	0.4862	0.001231	0.001225	-0.4898
4.0	0.000502	0.000505	0.5941	0.000503	0.000500	-0.6000

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

Table-3: For $a_0 = 0.3$, $b_0 = 0.3$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 3.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.300003	0.300003	0.00000	0.300016	0.300016	0.00000
0.5	0.100390	0.100309	0.08075	0.100399	0.100298	0.10069
1.0	0.029864	0.029785	0.26523	0.029869	0.029772	0.32580
1.5	0.008329	0.008288	0.49469	0.008331	0.008281	0.60379
2.0	0.002230	0.002214	0.72267	0.002231	0.002211	0.90456
2.5	0.000580	0.000575	0.86956	0.000581	0.000574	1.21951
3.5	0.000037	0.000037	0.00000	0.000037	0.000036	2.77777
4.0	0.000009	0.000009	0.00000	0.000009	0.000009	0.00000

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

Table-4: For $a_0 = 0.3$, $b_0 = 0.3$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 4.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.300001	0.300001	0.0000	0.300004	0.300004	0.0000
0.2	0.161751	0.161727	-0.0148	0.161754	0.161727	-0.0166
0.4	0.084791	0.084761	-0.0353	0.084793	0.084757	-0.0427
0.6	0.043541	0.043515	-0.0597	0.043543	0.043511	-0.0735
0.8	0.022010	0.021990	-0.0909	0.022011	0.021987	-0.1091
1.0	0.010988	0.010975	-0.1184	0.010989	0.010973	-0.1458
1.2	0.005431	0.005423	-0.1475	0.005431	0.005421	-0.1844
1.4	0.002662	0.002657	-0.1881	0.002662	0.002656	-0.2259
1.6	0.001296	0.001293	-0.2302	0.001296	0.001292	-0.3096
1.8	0.000627	0.000626	-0.1597	0.000627	0.000625	-0.3200
2.0	0.000302	0.000301	-0.3322	0.000302	0.000301	-0.3322

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

In **Table-1**, we see that, our percentage errors are greater than the percentage errors obtained by Shamsul [15], but from **Table-2**, **Table-3** and **Table-4,** we see that our percentage errors are smaller than the percentage errors obtained by Shamsul [15]. i.e. if the eigenvalue increases numerically then our errors decrease and Shamsul's [15] errors increase.

We again have computed $x(t,\varepsilon)$ by (27) (designated by x) in which a, b, c are computed by (26) and u_1 is computed by (24) together with the initial conditions $a_0 = 0.8$, $b_0 = 0.7$, $c_0 = 0.0$ for $\lambda = 1.0$, $\lambda = 3.0$, $\lambda = 4.0$ and $\lambda = 5.0$, and the results are respectively presented in the second column of the **Table-5**, **Table-6**, **Table-7** and **Table-8**. Then $x(t,\varepsilon)$ has been calculated by (37) (designated by $x_{SA[15]}$) in which a, b, c are computed by (36) and u_1 is computed by (34) together with the same set of initial condition as well as with same eigenvalues and the results are presented in the fifth column of the **Table-5**, **Table-6**, **Table-7** and **Table-8.** Column three and six show the corresponding numerical results calculated by the fourth order Runge-Kutta method. Corresponding percentage errors have also been calculated and are presented in the fourth and seventh column of the **Table-5**, **Table-6**, **Table-7** and **Table-8**. The first column shows the various values of time t .

Table-5: For $a_0 = 0.8$, $b_0 = 0.7$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 1.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.832156	0.832156	0.0000	0.876256	0.876256	0.0000
0.2	0.759032	0.766521	0.9770	0.810497	0.811849	0.1665
0.4	0.691176	0.701535	1.4766	0.742935	0.744364	0.1919
0.6	0.625920	0.638124	1.9124	0.675279	0.676326	0.1548
0.8	0.563060	0.577094	2.4318	0.609081	0.609642	0.0920
1.0	0.503124	0.519090	3.0757	0.545578	0.545690	0.0205
1.2	0.446730	0.464585	3.8432	0.485668	0.485416	-0.0519
1.4	0.394357	0.413885	4.7182	0.429936	0.429412	-0.1220
1.6	0.346296	0.367153	5.6870	0.378701	0.377990	-0.1881
1.8	0.302650	0.324425	6.7118	0.332078	0.331249	-0.2502
2.0	0.263376	0.285642	7.7950	0.290019	0.289129	-0.3078

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

Table-6: For $a_0 = 0.8$, $b_0 = 0.7$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 3.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.800044	0.800044	0.00000	0.800226	0.800226	0.00000
0.5	0.256295	0.256253	0.01639	0.256426	0.256113	0.12221
1.0	0.074559	0.074553	0.00804	0.074623	0.074362	0.35098
1.5	0.020510	0.020512	0.00975	0.020535	0.020407	0.62723
2.0	0.005440	0.005442	0.03675	0.005449	0.005399	0.92609
2.5	0.001406	0.001407	0.07107	0.001409	0.001392	1.22126
3.0	0.000357	0.000357	0.00000	0.000358	0.000352	1.70454
3.5	0.000089	0.000089	0.00000	0.000089	0.000088	1.13636
4.0	0.000022	0.000022	0.00000	0.000022	0.000022	0.00000

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

Table-7: For $a_0 = 0.8$, $b_0 = 0.7$, $c_0 = 0.0$, $\varepsilon = 0.1$ and $\lambda = 4.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.800008	0.800008	0.0000	0.800051	0.800051	0.0000
0.2	0.422250	0.422202	-0.0113	0.422289	0.422203	-0.0204
0.4	0.217961	0.217930	-0.0142	0.217991	0.217887	-0.0477
0.6	0.110623	0.110611	-0.0109	0.110644	0.110556	-0.0796
0.8	0.055407	0.055404	-0.0054	0.055420	0.055356	-0.1156
1.0	0.027457	0.027458	0.0036	0.027465	0.027423	-0.1532
1.2	0.013488	0.013490	0.0148	0.013493	0.013467	-0.1931
1.4	0.006577	0.006579	0.0304	0.006580	0.006565	-0.2285
1.6	0.003188	0.003189	0.0314	0.003189	0.003180	-0.2830
1.8	0.001537	0.001538	0.0650	0.001537	0.001533	-0.2609
2.0	0.000737	0.000738	0.1355	0.000738	0.000735	-0.4082

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

Table-8: For $a_0 = 0.8,\ b_0 = 0.7,\ c_0 = 0.0,\ \varepsilon = 0.1$ and $\lambda = 5.0$.

1	2	3	4	5	6	7
t	x	x^*	Error%	$x_{SA[15]}$	$x^*_{SA[15]}$	Error%
0.0	0.800002	0.800002	0.0000	0.800016	0.800016	0.0000
0.2	0.345759	0.345598	-0.0466	0.345771	0.345593	-0.0515
0.4	0.146135	0.145994	-0.0966	0.146143	0.145976	-0.1144
0.6	0.060727	0.060637	-0.1484	0.060732	0.060619	-0.1864
0.8	0.024903	0.024853	-0.2012	0.024906	0.024841	-0.2617
1.0	0.010104	0.010078	-0.2580	0.010105	0.010071	-0.3376
1.2	0.004064	0.004051	-0.3209	0.004065	0.004047	-0.4448
1.4	0.001623	0.001617	-0.3711	0.001623	0.001615	-0.4954
1.6	0.000644	0.000641	-0.4680	0.000644	0.000640	-0.6250
1.8	0.000254	0.000253	-0.3953	0.000254	0.000252	-0.7937
2.0	0.000100	0.000099	-1.0101	0.000100	0.000099	-1.0101

x is computed by (27) and $x_{SA[15]}$ is computed by (37); x^* and $x^*_{SA[15]}$ denote the corresponding numerical results.

In **Table-5**, we observe that, our percentage errors are greater than the percentage errors obtained by Shamsul [15], but from **Table-6**, **Table-7**, and **Table-8**; we notice that our percentage errors are smaller than the percentage errors obtained by Shamsul [15]. i. e. if the eigenvalue increases numerically then our errors decrease and Shamsul's [15] errors increase.

Therefore, in the Tables above, it is seen that, if the eigenvalue is small, then Shamsul's [15] technique gives better results. On the other hand, if the numerical size of the eigenvalue is large, our technique gives better results.

If $k_1 k_2 = 9k_3$, the system (1) undergoes more critically damped and the relation between the eigenvalue and the constants is $\lambda^6 = (1/9)k_1 k_2\, k_3$. Thus for the example (16), we have $\lambda^6 = \alpha\,\beta\left[9\,m^2\gamma^3\right]^{-1}$. From this relation, we see that, the numerical value of the eigenvalue is large if the mass m and the coefficient of relaxation γ are small. Conversely if the mass m and the coefficient of relaxation γ are large then the numerical value of the eigenvalue is small. Inasmuch as, our techniques give better results for large eigenvalue, therefore, our technique is suitable for light (not heavy) mass systems.

VI. Conclusion

The KBM method has been extended in this article for obtaining the solution of third order more critically damped nonlinear systems. For large eigenvalue, the solution obtained in this article gives more accurate results than the results obtained by the solution of Shamsul [15]. Since, for light mass m and for small coefficient of relaxation γ, the eigenvalue become large, therefore, for small coefficient of relaxation and for light mass systems our technique is more efficient than the technique of Shamsul [15].

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REFERENCES

[1] M. Ali Akbar, M. Shamsul Alam and M. A. Sattar, "A Simple Technique for Obtaining Certain Over-damped Solutions of an *n*-th Order Nonlinear Differential Equation", *Soochow J. Math.*, vol. 31(2), 2005, pp. 291-299.

[2] N. N. Bogoliubov and Y. Mitropolskii, "*Asymptotic Methods in the Theory of Nonlinear Oscillations*", Gordan and Breach, New York, 1961.

[3] G. N. Bojadziev, "Damped Nonlinear Oscillations Modeled by a 3-dimensional Differential System", *Acta Mechanica*,, vol. 48, 1983, pp. 193-201.

[4] N. N. Krylov and N. N. Bogoliubov, "*Introduction to Nonlinear Mechanics*", Princeton University Press, New Jersey, 1947.

[5] I. S. N. Murty, B. L. Deekshatulu and G. Krishna, "On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems", *J. Frank. Inst*., vol. 288, 1969, pp. 49-65.

[6] R. J. Mulholland, "Nonlinear Oscillations of Third Order Differential Equation", *Int. J. Nonlinear Mechanics*, vol. 6**,** 1971**,** pp. 279-294.

[7] Z. Osiniskii, "Longitudinal, Torsional and Bending Vibrations of a Uniform Bar with Nonlinear Internal Friction and Relaxation", *Nonlinear Vibration Problems*, vol. 4, 1962, pp. 159-166.

[8] I. P. Popov, "A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian)", *Dokl. Akad. USSR* **,** vol. 3, 1956, pp. 308-310.

[9] Mo. Rokibul Islam, M. Ali Akbar and M. Samsuzzoha, "A New Technique for Third Order Critically Damped Nonlinear Systems", *J. Appl. Sci. Res*., vol. 4(6), pp. 695-706, 2008.

[10] Mo. Rokibul Islam, M. Sharif Uddin, M. Ali Akbar, M. Azmol Huda and S. M. S. Hossain, " A New Technique for Fourth Order Critically Damped Nonlinear Systems with Some Conditions", *Bull. Cal. Math. Soc.*, to be published.

[11] M. A. Sattar, "An asymptotic Method for Second Order Critically Damped Nonlinear Equations", *J. Frank. Inst.*, vol. 321, 1986, pp. 109-113.

[12] M. A. Sattar, "An Asymptotic Method for Three-dimensional Over-damped Nonlinear Systems", *J. Bangladesh Math. Soc.*, vol. 13**,** 1993, pp. 1-8.

[13] M. Shamsul Alam and M. A. Sattar, "An Asymptotic Method for Third Order Critically Damped Nonlinear Equations", *J. Math. Phy. Sci.*, vol. 30, 1996, pp. 291-298.

[14] M. Shamsul Alam, "Asymptotic Methods for Second Order Over-damped and Critically Damped Nonlinear Systems", *Soochow J. Math.* vol. 27, 2001, pp. 187-200.

[15] M. Shamsul Alam, "Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems", *Soochow J. Math.*, vol. 28, 2002, pp. 65-80.

[16] M. Shamsul Alam, "On Some Special Conditions of Third Order Over-damped Nonlinear System", *Indian J. pure appl. Math.*, vol. 33**,** 2002, pp. 727-742.

[17] M. Shamsul Alam, "A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving *n*-th Order Nonlinear Systems", *J. Frank. Inst.*, vol. 339, 2002, pp. 239-248.

[18] M. Shamsul Alam, "On Some Special Conditions of Over-damped Nonlinear Systems", *Soochow J. Math*., vol. 29**,** 2003, pp. 181-190..