Fixed Rate Mortgages: Valuation and Closed Form Approximations

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Abstract—**This article considers an amortized fixed rate mortgage where the borrower has the choice of prepayment. Asymptotic behaviors of the contract value are fully analyzed. Based on a known fact about the optimal prepayment boundary, two analytical approximations to the value of the mortgage contract are derived. Numerical experiments are carried out to validate the accuracy of the approximation formulas. In general, the relative error is less than 4%.**

Keywords: fixed rate mortgage, mortgage valuation, asymptotic analysis, approximation

1 Introduction and main result

We consider an amortized mortgage contract with a given duration T (years) and a fixed mortgage interest rate c (year^{-1}), where the borrower is allowed to make prepayment, i.e., to settle the loan balance $M(\tau)$ at any time $\tau \le T$, where

$$M(\tau) = \frac{m}{c}\Big\{1 - e^{c(\tau - T)}\Big\},$$

which is the uniquely determined by the differential equation $dM(\tau) = cM(\tau)dt - md\tau$ with $M(T) = 0$ (see [4] or [11]). Here m is the rate of payment in dollars that the borrower pays back to the bank (lender) per unit time.

For banks or mortgage companies who are holding a large pool of such contracts with different nominal loan balances, different maturity dates, or different payment schedules, it is crucial for them to know the fair value of such contracts. If these mortgages constitute substantial portion of a bank's total assets, the market value of these contracts may largely determine its credit rating and refinancing cost. Having this said, it is not trivial for the bank to determine a fair value of such a contract because it is the borrower, rather than the bank, who has the choice to respond rationally according to the market reality. Only after the market value of each individual contract is known, the bank can construct hedging strategies or make appropriate financial planing for risk management purposes.

Assume the borrower always has sufficient working capital to settle the loan, then at any moment τ while the contact is in effect, he can either close the contract by paying off $M(\tau)$ or invest in the market with the amount of $M(\tau)$ less the current obligatory payment of m per unit time, earning an instantaneous return rate of, say, r_τ. Here we assume r_τ follows Vasicek model [16], which is described by the stochastic differential equation

$$dr_\tau = k(\theta - r_\tau)d\tau + \sigma\, dW_\tau$$

where k, θ, and σ are assumed to be positive known constants and W_τ is the standard Wiener process. Here the units for k, θ, σ, and W_t are year^{-1}, year^{-1}, year$^{-3/2}$ and year$^{1/2}$ respectively.

For convenience, we use the time to maturity date of the contract, $t := T - \tau$, instead of real time τ, and introduce a function $V(r,t)$ being the expected value of the contract at time t and current market return rate $r_t = x$. Without loss of generality, we assume $m = 1$. Mathematically we have a free boundary problem where the free boundary $x = h(t)$ denotes the optimal market interest rate level at which the borrower should close the contract. For each $(t,x) \in [0,\infty) \cup (-\infty,+\infty)$ being fixed, we are to find a (h,V) such that

$$\begin{cases} \mathbf{L}(V) = 1 & \text{for } \ x > h(t), t > 0, \\ V(x,t) = M(t) & \text{for } \ x \le h(t), t > 0, \\ V_x(x,t) = 0 & \text{for } \ x \le h(t), t > 0, \\ V(x,0) = 0 & \forall\ x \ge h(0) = c. \end{cases} \tag{1}$$

where the operator $\mathbf{L}$ is defined as

$$\mathbf{L}(V) = \frac{\partial V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - k(\theta - x)\frac{\partial V}{\partial x} + xV. \tag{2}$$

Similar problems have been discussed from option pricing viewpoint in [6, 8, 13, 12, 7, 3]. The free boundary problem formulation of (1) is obtained from standard mathematical finance theory [17, 14, 15] and variational inequalities [10], and is detailed in [11, 4] . In particular, it is proved in [11] that (1) is well-posed, possessing a unique solution which is smooth up to to the free boundary $x = h(t)$. In [4], asymptotic behaviors of $h(t)$ have

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been analyzed and a global approximation formula for $h(t)$, valid for all $t > 0$, is derived, namely,

$$h(t) = c - (c - R^*)\sqrt{1 - e^{-2[\frac{\kappa\sigma}{c-R^*}]^2 t}}, \tag{3}$$

where κ is the unique root to the integral equation

$$\sqrt{\pi} = \int_0^{\kappa} \frac{e^{-z^2}(\kappa^2 - z^2)^4(18\kappa^2 + 2z^2)}{(\kappa^2 + z^2)^5} dz, \tag{4}$$

with numerical value $\kappa = 0.3343641440309...$, and R^* is determined parametrically in terms of the Hermite functions H by

$$\begin{cases} R^* = \theta - \dfrac{\sigma^2}{k^2} + \dfrac{\sigma}{\sqrt{k}} x^*, \\ \\ c = \theta - \dfrac{\sigma^2}{k^2} + \dfrac{\sigma}{\sqrt{k}} \dfrac{\int_{x^*}^{\infty} yH(\mu; y)e^{-y^2+ay}dy}{\int_{x^*}^{\infty} H(\mu; y)e^{-y^2+ay}dy}, \end{cases} \tag{5}$$

$$a := \frac{\sigma}{k^{3/2}}, \qquad \mu := \frac{\sigma^2 - 2k^2\theta}{2k^3}.$$

While it is useful, from the borrower's point of view, to know the optimal early exercise boundary, the bank or the contract holder, on the hand, is more interested to know the market value of the contract, at given time t and interest level x. In this article, we start with the integral representation of $V(x,t)$, derive asymptotic expansions of $V(x,t)$ for both x large and $x \to h(t)$. In particular, we provide an exact asymptotic expansion of $V(x,t)$ for small x in terms of confluent hypergeometric functions. Interpolating these results we obtain two global approximation formulas for V, valid for all $y := (x - h(t)) \in \mathbf{R}$. The first formula (Formula 1) is

$$V(y,t) \;\; = \;\; a\chi_{(-\infty,0]} + a\mathrm{Erfcx}(\frac{a}{\sqrt{\pi}}y)\chi_{(0,\infty)}, \tag{6}$$

where $\chi(x)$ is the indicator function defined as

$$\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \bar{A} \end{cases},$$

Erfcx(x) is the scaled complementary error function defined as

$$\mathrm{Erfcx}(x) = e^{x^2}\mathrm{Erfc}(x),$$

and

$$a \;\; := \;\; \frac{1}{c}(1 - e^{-ct}).$$

And the second formula (Formula 2) is

$$\begin{aligned} V(y,t) = a\chi_{(-\infty,0]} + \Big[\Sigma_{i=1}^{i=2}(P_i \\ + \lambda_i y)\mathrm{Erfcx}(Q_i y^2)\Big]\chi_{(0,\infty)} \end{aligned} \tag{7}$$

where $a, \chi,$ and Erfcx are the same as in Formula 1, and $P_i, Q_i > 0, \lambda_i, i = 1, 2$ are given by a solution of the algebraic system

$$\begin{cases} \dfrac{\lambda_1}{Q_1} + \dfrac{\lambda_2}{Q_2} = \sqrt{\pi} \\ P_1 + P_2 = a \\ \lambda_1 + \lambda_2 = 0 \\ -\dfrac{2}{\sqrt{\pi}}(P_1Q_1 + P_2Q_2) = -\dfrac{1}{\sigma^2}(1 - e^{-ct})(1 - \dfrac{h(t)}{c}) \\ -\frac{2}{\sqrt{\pi}}(\lambda_1 Q_1 + \lambda_2 Q_2) = \frac{1}{3\sigma^2}[a - k(\theta - h(t))b)] \end{cases}.$$

The accuracy of our approximation formulas are validated by numerical experiments with a variety of parameters. The asymptotic analysis and the two analytical approximation formulas, together with the existing result of (1.3), provide useful tools for the industry practitioners in addition to theoretical interests.

2 full asymptotic analysis of V

In this section we first derive, by analyzing the PDE in (1.1) directly, the Taylor expansion of $V(x,t)$ for small x up to the third order. By formulating the integral representation of $V(x,t)$ with the free boundary $h(t)$ embedded, we shall obtain an asymptotic representation of $V(x,t)$ for large value of x in terms of confluent hypergeometric function of the first kind, the leading order term of which turns out to be $\frac{1}{x}$.

2.1 Asymptotic expansion of $V(x,t)$ for small x

We first notice, by regularity, that V_x exists and is continuous for all $x \in \mathbf{R}$. In particular, $\lim_{x\to h(t)_+} V_x = V_x(h(t),t) \equiv 0$. Now let $V \to M(t)$ in the PDE in (1), we have

$$\lim_{x\to h(t)_+} V_{xx} = \lim_{x\to h(t)_+} V_{xx} = -\frac{2}{\sigma^2}(1 - e^{-ct})(1 - \frac{x}{c}).$$

Also one notice that in the set where $V < M(t)$,

$$\frac{\partial V_x}{\partial (t)} - \frac{\sigma^2}{2}\frac{\partial^2 V_x}{\partial x^2} - k(\theta - x)\frac{\partial V_x}{\partial x} + (x+k)V_x = -V.$$

Let $x \to h(t)^+$, we have that

$$\lim_{x\to h(t)_+} V_{xxx} = \frac{2}{\sigma^2}\{k(\theta - x)[\frac{2}{\sigma^2}(1-e^{-ct})(1-\frac{h(t)}{c})] + \frac{1}{c}(1-e^{-ct})\}$$

Then we have the Taylor expansion for $V(x,t)$, up to third order term, near $x = h(t)$

$$V = \frac{1}{c}(1-e^{-ct}) - \frac{1}{\sigma^2}(1-e^{-ct})(1-\frac{h(t)}{c})(x-h(t))^2 + \frac{1}{3\sigma^2}\{\frac{2k}{\sigma^2}(\theta - x)(1-\frac{h(t)}{c}) + \frac{1}{c}\}(x-h(t))^3$$

2.2 Integration representation of $V(x,t)$

Let

$$\begin{cases} u(y,s) = \dfrac{2\sqrt{\pi}ck^{\frac{3}{2}}}{m\sigma}\Big\{M(t) - V(x,t)\Big\} \\ \qquad \times e^{-\frac{k}{\sigma^2}[x+\frac{\sigma^2}{2k^2}-\theta]^2 - [k+\frac{\sigma^2}{2k^2}-\theta]t} \\ h(t) = \eta(s) \end{cases}, \tag{8}$$

where

$$y = \frac{\sqrt{k}e^{kt}}{\sigma}\Big[x + \frac{\sigma^2}{k^2} - \theta\Big], \quad s = e^{2kt}. \tag{9}$$

One can verify [4] that, under above change of variables, the problem (1) for (V,h) is equivalent to the following system for (W,η)

$$\begin{cases} W_s - W_{yy} = f(y,s) & \text{if } y > \eta(s), s > 1, \\ W(y,s) = 0 & \text{if } y \le \eta(s), s > 1, \\ W_y(y,s) = 0 & \text{if } y \le \eta(s), s > 1, \\ W(y,1) = 0 & \forall y \ge \eta(1) = \omega. \end{cases} \tag{10}$$

where

$$f(y,s) = \sqrt{\pi}(s^{\frac{c}{2k}} - 1)s^{-2-\frac{\sigma^2}{4k^3}-\frac{c-\theta}{2k}} (y - \frac{\sqrt{k}}{\sigma}(c - \theta + \frac{\sigma^2}{k^2})\sqrt{s})e^{-(\frac{y}{\sqrt{s}} - \frac{\sigma}{2k^{3/2}})^2},$$
$$\omega = \frac{\sigma}{k^{\frac{3}{2}}} + \frac{(c-\theta)k^{\frac{1}{2}}}{\sigma}.$$

Since the fundamental solution associated with the heat operator $\partial_s - \frac{1}{4}\partial^2_{yy}$ is known as $\frac{e^{-y^2/s}}{\sqrt{\pi s}} := \Gamma(y,s)$. Using Green's identity, the solution W to the differential equation in (10) can be expressed as

$$W(y,s) = \int_1^s \int_{\eta(\xi)}^{\infty} \Gamma(y-\rho, s-\xi) f(\rho,\xi)\, d\rho d\xi \tag{11}$$

Translate the integral representation for W in (2.4) into the integral representation for V, and simplify terms, we have

$$V(x,t) = M(t) - \int_0^t \int_{h(\tau)}^{\infty} \frac{1}{\sqrt{\pi\alpha}} e^{\beta_1 + \frac{1-s}{k}y - \frac{[\beta_2 + x - ys]^2}{\alpha}} (1-\frac{y}{c})(e^{-c\tau} - 1)dyd\tau, \tag{12}$$

where $s, \alpha, \beta_1, \beta_2$ are functions of $(x, y; t, \tau)$ defined by

$$s = e^{k(t-\tau)},$$
$$\alpha = \frac{\sigma^2}{\theta}(s^2 - 1),$$
$$\beta_1 = \frac{3\sigma^2}{4k^3}s^2 + \frac{1}{k}(\theta - \frac{\sigma^2}{k^2})s + (-\theta + \frac{\sigma^2}{2k^2} + k)(t-\tau) + \frac{1}{k}(-\theta + \frac{\sigma^2}{4k^2}),$$
$$\beta_2 = \frac{\sigma^2}{2k^2}s^2 + (\theta - \frac{\sigma^2}{k^2})s + (-\theta + \frac{\sigma^2}{2k^2}).$$

2.3 Leading Order Expansion of V for Large x

Here we are to derive the large x asymptotic representation of V in terms of confluent hypergeometric functions. Let $V(x,t) = M(t) - U(x,t)$. The inside integral on the right hand side of (12) is a convolution of a linear function of y and an exponential of a quadratic form of y, which can be explicitly calculated. And the result of such a calculation gives

$$U(x,t) = \frac{1}{2c\sqrt{\pi}}\int_0^t (1-e^{-c\tau})e^{\beta}\Big\{\frac{\sigma\sqrt{1-s^2}}{\sqrt{k}}e^{-\hat{h}^2} + \sqrt{\pi}[(-c+\theta-\frac{\sigma^2}{2k^2})+(x-\theta+\frac{\sigma^2}{k^2})s^{-1} - \frac{\sigma^2}{2k^2}s^{-2}]\mathrm{Erfc}(\hat{h})\Big\}d\tau, \quad (13)$$

where

$$\beta = (-\theta+\frac{\sigma^2}{2k^2})(t-\tau)+(\frac{\theta}{k}-\frac{3\sigma^2}{4k^3}-\frac{x}{k}) + (-\frac{\theta}{k}+\frac{\sigma^2}{k^3}+\frac{x}{k})s^{-1}-\frac{\sigma^2}{4k^3}s^{-2},$$

$$\hat{h} = \begin{cases} \frac{s}{\sqrt{\alpha}}\{(h(\tau)-\theta+\frac{\sigma^2}{2k^2})-(-\theta+\frac{\sigma^2}{k^2}+x)s^{-1} + \frac{\sigma^2}{2k^2}s^{-2}\}, & \text{if} \quad \tau\neq t; \\ 0, & \text{if} \quad \tau = t. \end{cases}$$

To find the asymptotic expansion of $U(x,t)$ for x large, one notices that $U(x,t)$ is essentially a sum of two integrals, and also

$$\lim_{x\to\infty}\hat{h} = -\infty,$$

thus (13) reduces to

$$\begin{aligned} U(x,t) &= \frac{1}{c}\int_0^t (1-e^{-c\tau})e^{\beta}[(-c+\theta-\frac{\sigma^2}{2k^2})+(x-\theta + \frac{\sigma^2}{k^2})s^{-1}-\frac{\sigma^2}{2k^2}s^{-2}]d\tau \\ &= I - II, \end{aligned}$$

where

$$I = \frac{1}{c}\int_0^t e^{\beta}[(-c+\theta-\frac{\sigma^2}{2k^2}) + (x-\theta+\frac{\sigma^2}{k^2})s^{-1}-\frac{\sigma^2}{2k^2}s^{-2}]d\tau, \quad (14)$$

$$II = \frac{1}{c}e^{-ct}\int_0^t e^{c(t-\tau)+\beta}[(-c+\theta-\frac{\sigma^2}{2k^2}) + (x-\theta+\frac{\sigma^2}{k^2})s^{-1}-\frac{\sigma^2}{2k^2}s^{-2}]d\tau. \quad (15)$$

Observe that

$$\frac{\partial}{\partial\tau}[c(t-\tau)+\beta] = (\theta-c-\frac{\sigma^2}{2k^2})+(x-\theta+\frac{\sigma^2}{k^2})s^{-1}-\frac{\sigma^2}{2k^2}s^{-2},$$

so we can apply integral by parts to evaluate II directly, and get

$$II = \frac{1}{c}e^{-ct}e^{c(t-\tau)+\beta}\Big|_{\tau=0}^{\tau=t} = \frac{1}{c}e^{-ct}-\frac{1}{c}e^{\beta(0,t)}$$

To evaluate I, we first write $I = III + IV$, where

$$III = \frac{1}{c}\int_0^t e^{\beta}(-c)d\tau = -\int_0^t e^{\beta}d\tau, \quad (16)$$

$$IV = \frac{1}{c}\int_0^t e^{\beta}[(\theta-\frac{\sigma^2}{2k^2}) + (x-\theta+\frac{\sigma^2}{k^2})s^{-1}-\frac{\sigma^2}{2k^2}s^{-2}]d\tau. \quad (17)$$

For (17), we can simply evaluate in the same way we evaluate II, thus have

$$IV = \frac{1}{c}\int_0^t e^{\beta}\frac{\partial\beta}{\partial\tau}d\tau = \frac{1}{c}e^{\beta}\Big|_{\tau=0}^{\tau=t} = \frac{1}{c}-\frac{1}{c}e^{\beta(0,t)}. \quad (18)$$

Now the asymptotic behavior of V for large x is essentially determined by the parametric integral

$$V(x,t) = \int_0^t e^{(-\theta+\frac{\sigma^2}{2k^2})(t-\tau)+(\frac{\theta}{k}-\frac{3\sigma^2}{4k^3}-\frac{x}{k})+(-\frac{\theta}{k}+\frac{\sigma^2}{k^3}+\frac{x}{k})s^{-1}-\frac{\sigma^2}{4k^3}s^{-2}}d\tau.$$

For x large, it becomes

$$\begin{aligned} V(x,t) &= \frac{1}{k}e^{-\frac{x}{k}}\int_{\tau=0}^{\tau=t} e^{\frac{x}{k}s^{-1}}ds^{-1} \\ &= \frac{1}{x}e^{-\frac{x}{k}}\int_{\tau=0}^{\tau=t} e^{\frac{x}{k}s^{-1}}d(\frac{x}{k}s^{-1}) \\ &= \frac{1}{x}e^{-\frac{x}{k}}[e^{\frac{x}{k}}-e^{\frac{x}{k}e^{-kt}}] \\ &= \frac{1}{x}[1-e^{-\frac{x}{k}(1-e^{-kt})}] \\ &\sim \frac{1}{x}\quad . \end{aligned}$$

2.4 Representation of V for Large x

Here we use integral by parts to derive a more accurate representation of $V(x,t)$ for large x in terms of the confluent hypergeometric functions of the first kind. Because we are interested in the expansion of $V(x,t)$ as $x\to\infty$, we can neglect the factor $-\frac{\sigma^2}{4k^3}s^{-2}$ in (2.12), thus have

$$
\begin{aligned}
V(x,t) &= e^{\frac{\theta}{k}-\frac{3\sigma^2}{4k^3}-\frac{x}{k}}\int_0^t e^{(\frac{\sigma^2}{2k^2}-\theta)(t-\tau)+(\frac{\sigma^2}{k^3}-\frac{\theta}{k}-\frac{x}{k})s^{-1}}d\tau\\
&= e^{\frac{\theta}{k}-\frac{3\sigma^2}{4k^3}-\frac{x}{k}}\int_0^t e^{(-\theta+\frac{\sigma^2}{2k^2})z+(-\frac{\theta}{k}+\frac{\sigma^2}{k^3}-\frac{x}{k})e^{-kz}}dz,
\end{aligned}
$$

by letting $t-\tau=z$. To proceed, as in general, we have that for given $a,b,c>0$,

$$
\begin{aligned}
\int_0^t e^{-ay+be^{-cy}}dy &= -\frac{1}{a}\int_0^t e^{be^{-cy}}[e^{-ay}]'dy\\
&= (-1)\frac{1}{a}e^{-at+be^{-ct}}+\frac{1}{a}e^{b}\\
&\quad+(-1)\frac{bc}{a}\int_0^t e^{-(a+c)y+be^{-cy}}dy.
\end{aligned}
$$

The process of integration by parts can be repeated using the recursive identity

$$
\begin{aligned}
&\int_0^t e^{-(a+nc)y+be^{-cy}}dy = (-1)\frac{1}{a+nc}e^{-(a+nc)t+be^{-ct}}\\
&+\frac{1}{a+nc}e^{b}+(-1)\frac{bc}{a+nc}\int_0^t e^{-(a+(n+1)y)+be^{-cy}}dy,
\end{aligned}
$$

which leads to

$$
\int_0^t e^{-ay+be^{-cy}}dy = P+Q+R,
$$

where

$$
\begin{cases}
P = \frac{1}{a}+(-1)\frac{bc}{a(a+c)}+...+(-1)^n\frac{(bc)^n}{a(a+c)...(a+nc)}\\
Q = (-1)\frac{1}{a}e^{-at+be^{-ct}}+(-1)^2\frac{bc}{a(a+c)}e^{-(a+c)t+be^{-ct}}\\
\qquad +...+\frac{(-1)^{n+1}(bc)^n}{a(a+c)...(a+nc)}e^{-(a+nc)t+be^{-ct}}\\
R = \frac{(-1)^{n+1}(bc)^{n+1}}{a(a+c)...(a+nc)}\int_0^t e^{-[a+(n+1)c]y+be^{-cy}}dy
\end{cases}
$$

It is apparent the tail definite integral R vanishes as $n\to\infty$. Also one notices [5] that

$$
\sum_{n=1}^{n=\infty}\frac{(-1)^n b^n}{(a/c+1)(a/c+2)...(a/c+n)}e^{b} = M(1,\frac{a}{c}+1,-b),
$$

where $M(p,q,z)$ is the confluent hypergeometric function of the first kind of order p,q. Thus we have

$$
\begin{aligned}
\int_0^t e^{-ay+be^{-cy}}dy = \frac{1}{a}M(1,\frac{a}{c}+1,-b)\\
-\frac{e^{-at}}{a}M(1,\frac{a}{c}+1,-be^{-ct}).
\end{aligned}
\tag{19}
$$

In terms of our problem, we have

$$
\begin{aligned}
V(x,t) = &\frac{e^{-(\theta-\frac{\sigma^2}{2k^2})t}}{\theta-\frac{\sigma^2}{2k^2}}\\
&\times M(1,\frac{\theta}{k}-\frac{\sigma^2}{2k^3}+1,(\frac{\theta}{k}-\frac{\sigma^2}{k^3}+\frac{x}{k})e^{-kt})\\
&-e^{\frac{\theta}{k}-\frac{3\sigma^2}{4k^3}-\frac{x}{k}}\times\Big[\frac{1}{\theta-\frac{\sigma^2}{2k^2}}\\
&M(1,\frac{\theta}{k}-\frac{\sigma^2}{2k^3}+1,\frac{\theta}{k}-\frac{\sigma^2}{k^3}+\frac{x}{k})\Big]
\end{aligned}
. \tag{20}
$$

It is straightforward to verify, by the series representation of $M(\mu,\nu,x)$ [5], that the leading order term of (20) is $\frac{1}{x}$.

3 Global Analytical Approximations

In this section we shall interpolate the asymptotic expansions of $V(x)$ for large x and small x to derive highly accurate approximation formulas. Define $y := x-h(t)$. From the asymptotic analysis in the previous section, we have that

$$
\lim_{y\to 0}V(y,t) = a+by^2+cy^3, \tag{21}
$$

$$
\lim_{y\to\infty}V(y,t) = \frac{1}{y}, \tag{22}
$$

where $a=a(t)$, $b=b(t)$ and $c=c(t)$ are define by

$$
\begin{aligned}
a &:= \frac{1}{c}(1-e^{-ct}),\\
b &:= -\frac{1}{\sigma^2}(1-e^{-ct})(1-\frac{h(t)}{c}),\\
c &:= \frac{1}{3\sigma^2}[a-k(\theta-h(t))b)].
\end{aligned}
$$

3.1 Approximation One

We first postulate that

$$
V(y,t) = ae^{py^2}\mathrm{Erfc}(qy), \qquad \forall y\geq 0. \tag{23}
$$

Note that such a postulation automatically satisfies the condition $\lim_{y\to 0}V=a$. To meet the requirement at $y\to\infty$, recall (see [5], for instance) that

$$
\begin{aligned}
\mathrm{Erfc}(y)\sim\frac{1}{\sqrt{\pi}}\frac{e^{-y^2}}{y}(1-\frac{1}{2y^2}+\frac{1\cdot 3}{(2y^2)^2}-...),\\
y\to\infty.
\end{aligned}
\tag{24}
$$

Using the asymptotic expansion of complementary function to approximate (23), we have

$$\lim_{y\to\infty} V(y,t) \quad = \quad \frac{1}{\sqrt{\pi}}\frac{a}{qy}e^{(p-q^2)y}. \tag{25}$$

Compare (22) with (25),we have

$$q = \frac{a}{\sqrt{\pi}}, \quad p = \frac{a^2}{\pi}.$$

In terms of original variables, we have the following global analytical approximation (Formula 1):

$$V(x,t) = ae^{\frac{a^2}{\pi}y^2}\mathrm{Erfc}(\frac{a}{\sqrt{\pi}}y), \tag{26}$$

where $a := \frac{1}{c}(1-e^{-ct})$. Numerical plots show that Formula 1 is accurate enough from practitioner's point of view. But there is room for improvement, namely, the approximation (26) only takes care of the leading order limit at $x \to h(t)$, but not all of the asymptotic behaviors specified in (21).

3.2 Approximation Two

Aiming to improve the accuracy of the approximation formula, we further postulate that

$$\begin{aligned} V(x,t) = (P_1+\lambda_1 x)\mathrm{Erfcx}(Q_1x^2) \\ + (P_2+\lambda_2 x)\mathrm{Erfcx}(Q_2x^2) \end{aligned} \tag{27}$$

where Erfcx is the scaled complementary error function defined as

$$\mathrm{Erfcx}(z) : \quad = \quad e^{z^2}\mathrm{Erfc}(z),$$

with asymptotic expansion

$$\begin{aligned} \mathrm{Erfcx}(z) &\sim \frac{1}{z}, &&\text{for large } z>0; \\ \mathrm{Erfcx}(z) &\sim (1+z^4)(1-\frac{2}{\sqrt{\pi}}x^2), &&\text{for small } z>0. \end{aligned}$$

Use the above asymptotic expansions to approximate (27), match terms with (21) and (22), we have

$$\begin{cases} \dfrac{\lambda_1}{Q_1}+\dfrac{\lambda_2}{Q_2} = \sqrt{\pi} \\ P_1+P_2 = a \\ \lambda_1+\lambda_2 = 0 \\ -\dfrac{2}{\sqrt{\pi}}(P_1Q_1+P_2Q_2) = b \\ -\frac{2}{\sqrt{\pi}}(\lambda_1Q_1+\lambda_2Q_2) = c \\ Q_1 > 0 \\ Q_2 > 0 \end{cases} \tag{28}$$

Note that $Q_1 > 0$ and Q_2 are required because otherwise the condition for large x expansion of V will not be satisfied. This algebraic system can be solved by expressing P_1, P_2 and Q_1 in terms of Q_2, yielding

$$\sqrt{\pi}P_2Q_2^2 + [P_1\lambda_1 - P_2\lambda_2 + \frac{\pi}{2}b]Q_2 - \frac{\sqrt{\pi}}{2}\lambda b = 0. \tag{29}$$

Then we get

$$Q_2 = \frac{-(P_1\lambda_1 - P_2\lambda_2 + \frac{\pi}{2}b) + \sqrt{\Delta}}{2\sqrt{\pi}P_2}, \tag{30}$$

where

$$\Delta := [P_1\lambda_1 - P_2\lambda_2 + \frac{\pi}{2}b]^2 + 2\pi P_2\lambda_2 b$$

One notices that (28) has more number of variables than number of equations, thus may assume multiple roots. To simplify, we let $P_1 = P_2 = a/2$, then (28) reduces to

$$\begin{cases} \dfrac{\lambda_1}{Q_1}+\dfrac{\lambda_2}{Q_2} = \sqrt{\pi} \\ \lambda_1+\lambda_2 = 0 \\ Q_1+Q_2 = -\dfrac{\sqrt{\pi}b}{a} \\ \lambda_1Q_1+\lambda_2Q_2 = -\frac{\sqrt{\pi}c}{2} \\ Q_1 > 0 \\ Q_2 > 0 \end{cases} \tag{31}$$

In this case,

$$Q_2 = \frac{-(a\lambda_1 + \frac{\pi}{2}b) + \sqrt{(a\lambda_1+\frac{\pi}{2}b)^2 - \pi a\lambda_1 b}}{\sqrt{\pi}a},$$

$$Q_1 = \frac{(a\lambda_1 - \frac{\pi}{2}b) - \sqrt{(a\lambda_1-\frac{\pi}{2}b)^2 + \pi a\lambda_1 b}}{\sqrt{\pi}a},$$

Indeed one can verify that $Q_1, Q_2 > 0$. Once Q_1 and Q_2 are determined, λ_1 and $\lambda 2$ can be numerically solved using, say, standard Matlab fslove package.

4 Numerical Examples

Here we provide some numerical examples to validate the accuracy of our approximations Formula 1 and Formula 2. In each of the following figures, we plot both the true numerical solution of V and the analytical approximation of V using our approximation formulas. Tested with a variety of parameters, the relative error of Formula 2 is less than 4%, where

$$\text{relative error} := \frac{\max_{x\in\mathbf{R}} |\text{true solution-analytical approximation}|}{|V(\infty)-V(-\infty)|}.$$

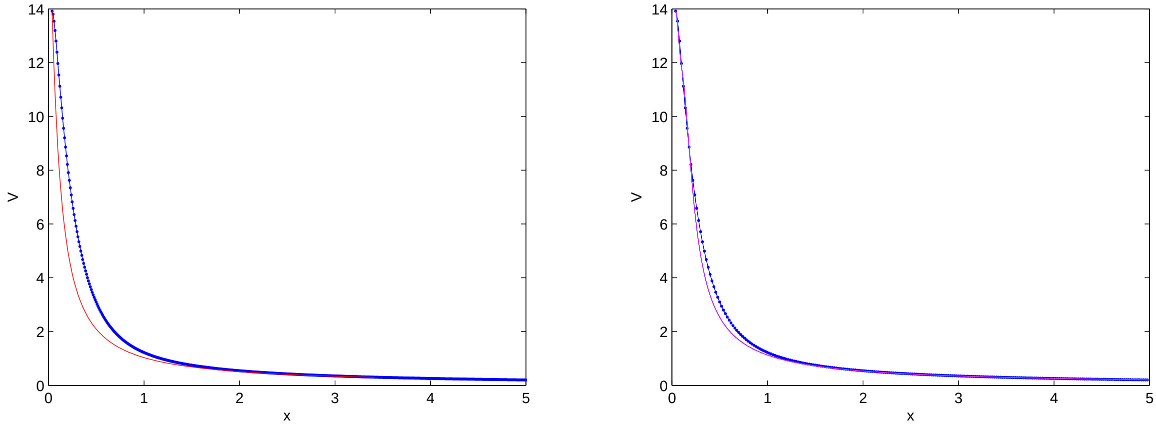

Figure 1: The dotted curve is the true numerical solution. The plain curve is the global approximation using Formula 1 (left) and Formula 2 (right). Here $c = 0.06, \theta = 0.05,\ \sigma = 0.015, k = 0.15, R^* = 0.0372, t = 30, h(t) = 0.0384$.

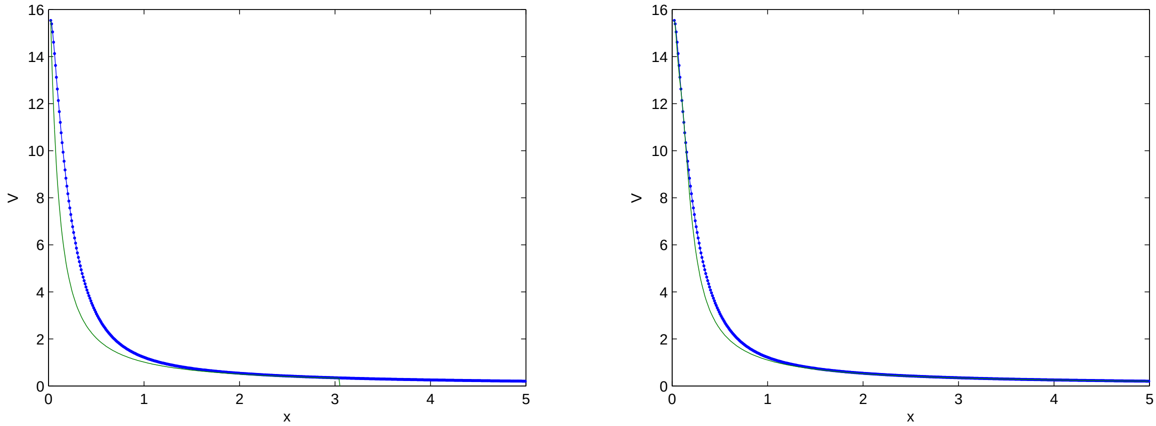

Figure 2: The dotted curve is the true numerical solution. The plain curve is the global approximation using Formula 1 (left) and Formula 2 (right). Here $c = 0.05, \theta = 0.05,\ \sigma = 0.015, k = 0.15, R^* = 0.0199, t = 30, h(t) = 0.0231$.

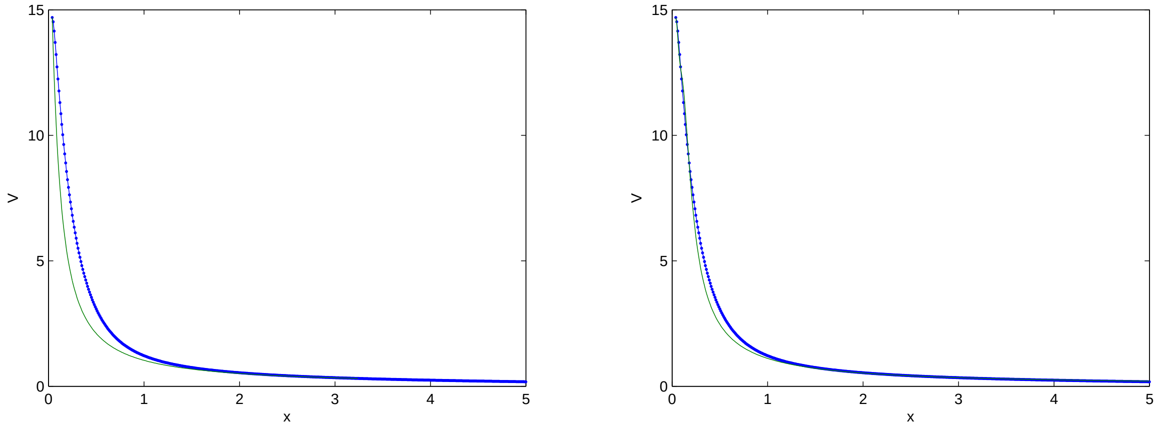

Figure 3: The dotted curve is the true numerical solution. The plain curve is the global approximation using Formula 1 (left) and Formula 2 (right). Here $c = 0.055, \theta = 0.05,\ \sigma = 0.010, k = 0.15, R^* = 0.0383, t = 30, h(t) = 0.0395$.

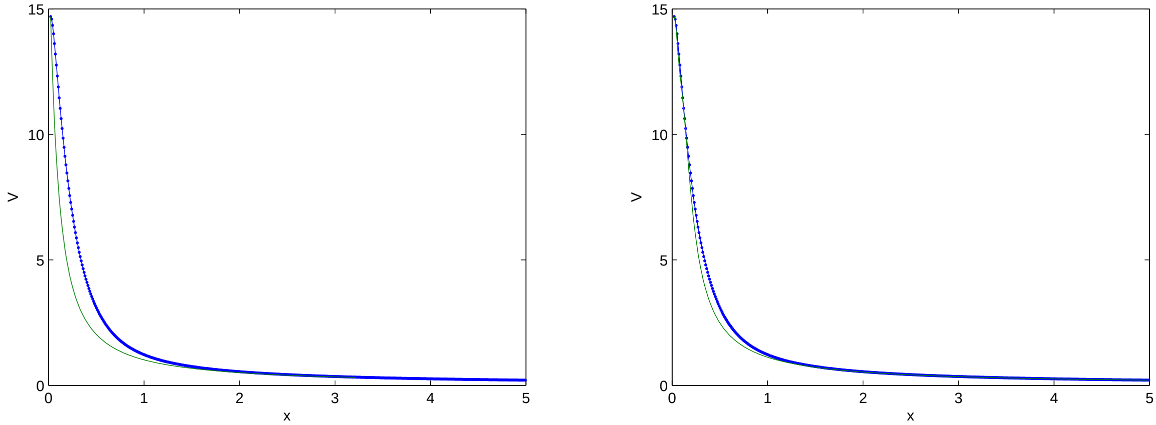

Figure 4: The dotted curve is the true numerical solution. The plain curve is the global approximation using Formula 1 (left) and Formula 2 (right). Here $c = 0.055, \theta = 0.05, \sigma = 0.020, k = 0.15, R^* = 0.0201, t = 30, h(t) = 0.0226$.

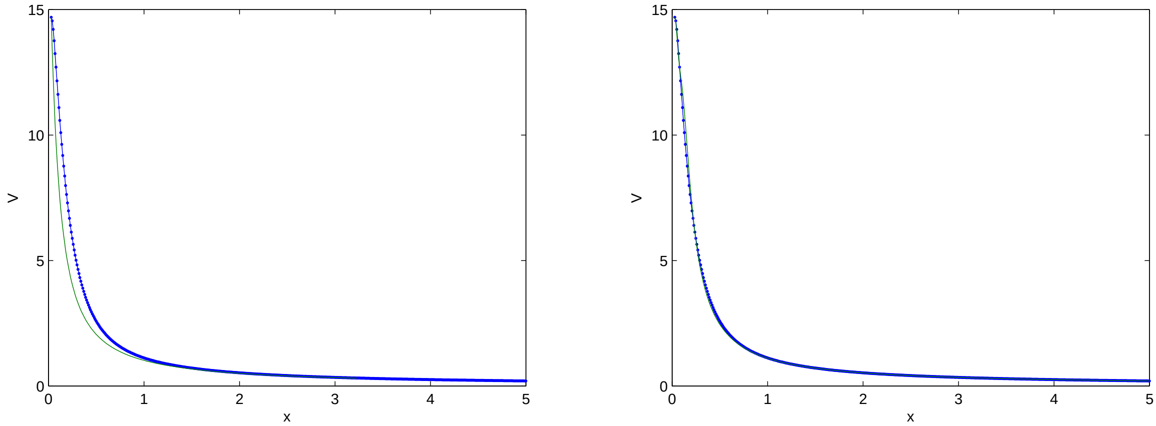

Figure 5: The dotted curve is the true numerical solution. The plain curve is the global approximation using Formula 1 (left) and Formula 2 (right). Here $c = 0.055, \theta = 0.05, \sigma = 0.020, k = 0.15, R^* = 0.0266, t = 30, h(t) = 0.0290$.

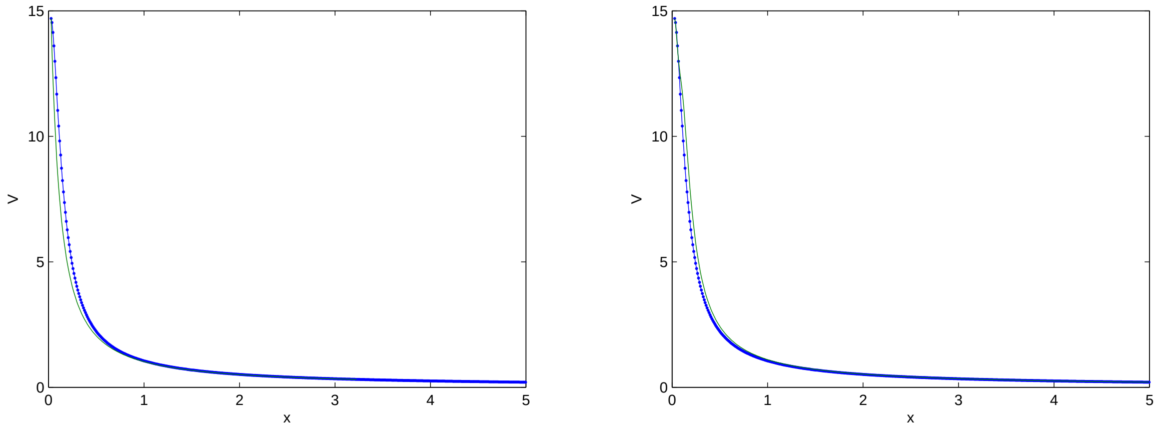

Figure 6: The dotted curve is the true numerical solution. The plain curve is the global approximation using Formula 1 (left) and Formula 2 (right). Here $c = 0.055, \theta = 0.05, \sigma = 0.015, k = 0.05, R^* = 0.0237, t = 30, h(t) = 0.0269$.

5 Conclusion

A standard type of amortized fixed rate mortgage is formulated in terms of parametric integral equations and analyzed asymptotically. Two global analytical approximations for the contract value have been derived using novelty interpolations of the asymptotic expansions. Numerical simulations are provided to validate the accuracy of the analytical formulas.

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