Kernel Polynomial of 2-Orthogonal Sequence

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Abstract—**In this paper, the construction of the kernel polynomial of 2-orthogonal polynomials is given. Properties of this polynomial are invertigated. We prove in particular that this polynomial conserves the 2-orthogonality, the strictly 2-quasi-orthogonality, the 2-weakly-orthogonality. On the other hand we prove that it also preserves the classical 2-orthogonality properties under some conditions.**

Keywords: vd- orthogonal polynomials, d- strictly-quasi-orthogonality, semi classical d-orthogonal polynomials, recurrence relations, Christoffel-Darboux identitie.

1 Introduction

Let $\{B_n\}_{n\geq 0}$be any orthogonal polynomial sequence (OPS), and λ a complex number such that $B_n(\lambda) \neq 0,\ n \geq 1$, its Kernel polynomial $\left\{B^*_{n,\lambda}\right\}_{n\geq 0}$ has been studied by Chihara [3] , [4] and Maroni [12] and has been completed by Kwon and all [8]. It has been shown in [8], that $(x-\lambda)\, B^*_{n,\lambda}(x)$ can be written in the form of a linear combination of $B_n(x)$ and $B_{n-1}(x)$, that is

$$(x-\lambda)\, B^*_{n,\lambda}(x) = B_{n+1}(x) - \alpha_n(\lambda)\, B_n(x)$$

where $\alpha_n(\lambda) = B_{n+1}(\lambda)/B_n(\lambda)$

From this fact, Kwon and all [8] proved that for any monic OPS $\{B_n\}_{n\geq 0}$ with respect to the form σ and for any complex number λ with $B_n(\lambda) \neq 0,\ n \geq 1$, its Kernel polynomial $\left\{B^*_{n,\lambda}\right\}_{n\geq 0}$is also an OPS with respect to the form $(x-\lambda)\,\sigma$.

In this work, we construct the Kernel polynomial of a 2−OPS, that we denote by $\left\{B^*_{n,y,z}\right\}_{n\geq 0}$, as we are able to write $(x-y)(x-z)\, B^*_{n,y,z}(x)$ in the form of linear combination of $B_{n+2}(x), B_{n+1}(x)$ and $B_n(x)$, that is

$$\begin{aligned}(x-y)(x-z)\, B^*_{n,y,z}(x) &= B_{n+2}(x) - \alpha_n B_{n+1}(x)\\ &\quad + \delta_n B_n(x)\end{aligned}$$

where α_n and δ_n are complex numbers. We also show that this kernel polynomial keeps the 2−orthogonality, the strictly 2−quasi-orthogonality and the 2−weakly-orthogonality properties. Finaly, if $\{B_n\}_{n\geq 0}$ is a classical 2−OPS, its kernel polynomial is also a classical 2−OPS under some conditions that we will be given later.

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2 Fundamental Results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$, equipped with its natural inductive limit topology; and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}$ on $f \in \mathcal{P}'$.

In particular, we denote by $(u)_n = \langle u, x^n\rangle, n \geq 0$, the moments of u,where $\langle .,.\rangle$ is the dual brakets between the vector space of polynomials with complex coefficients and its dual.

By a polynomial set (PS), we mean a sequence of monic polynomials $\{B_n\}_{n\geq 0}$ which $\deg B_n(x) = n$ for all n, where, $B_n(x) = x^n + ..., n \geq 0$. Let $\{B_n\}_{n\geq 0}$be a polynomial set; there exists a sequence of linear functionals $\{\mathcal{L}_n\}_{n\geq 0}$, such that:

$$\mathcal{L}_n(B_m) = \langle \mathcal{L}_n, B_m\rangle = \delta_{nm}, \quad n, m \geq 0 \tag{2.1}$$

The sequence $\{\mathcal{L}_n\}_{n\geq 0}$ is called the dual sequence of $\{B_n\}_{n\geq 0}$; it is unique [4] , [6] .

Lemma 1 [5] , [11] . *Let $f \in \mathcal{P}'$ and q be a positive integer. f satisfies*

$$f(P_{q-1}) \neq 0 \quad \textit{and} \quad f(P_n) = 0, \qquad n \geq q$$

if there exist $\lambda_\nu \in \mathbb{C}$, for $0 \leq \nu \leq q-1$, with $\lambda_{q-1} \neq 0$, such that

$$f = \sum_{\nu=0}^{q-1} \lambda_\nu \mathcal{L}_\nu$$

Proposition 2 [11] *If $\{\mathcal{L}_n\}_{n\geq 0}$ $\left(\text{resp. } \left\{\breve{\mathcal{L}}_n\right\}_{n\geq 0}\right)$ is the dual sequence of $\{B_n\}_{n\geq 0}$ $\left(\text{resp. } \{Q_n\}_{n\geq 0}\right)$ $\left(\text{where } Q_n(x) = \frac{1}{n+1} DB_{n+1}(x)\right)$ then we have*

$$D\breve{\mathcal{L}}_n = -(n+1)\,\mathcal{L}_{n+1}, \quad n \geq 0 \tag{2.2}$$

Let us consider d linear functionals $\Gamma_1, \Gamma_2, ..., \Gamma_d$ $(d \geq 1)$.

Definition 1 [5], [11] *Let* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_d)^T$ *be a* $d-$*linear form defined on the vector space of polynomials on* $\mathbb{C}$. *A sequence* $\{B_n\}_{n\geq 0}$ *is said to be a* $d-$*dimensional orthogonal polynomial sequence, or simply* $d-$*orthogonal sequence* ($d-OPS$) *with respect to* Γ, *if it satisfies:*

$$\langle \Gamma_\alpha, x^m B_n(x)\rangle = 0, \quad n \geq md+\alpha, \quad m \geq 0 \tag{2.3}$$

$$\langle \Gamma_\alpha x^m B_{md+\alpha-1}(x)\rangle \neq 0, \quad m \geq 0 \tag{2.4}$$

for each integer α *with* $1 \leq \alpha \leq d$.

Remark 1 *(1) When* $d = 1$, *we meet again the ordinary regular orthogonality. In this case* $\{B_n\}_{n\geq 0}$ *is an orthogonal polynomial sequence (OPS).*

(2) The inequality (2.4) *is the regularity condition. In this case, the* $d-$*dimensional functional* Γ *is called regular. It is not unique. Indeed, according to lemma 1, we have*

$$\Gamma^\sigma = \sum_{\nu=0}^{\sigma-1} \lambda_\nu^\sigma \mathcal{L}_\nu, \ \lambda_{\sigma-1}^\sigma \neq 0, \ 1 \leq \sigma \leq d$$

or equivalently

$$\mathcal{L}_\nu = \sum_{\sigma=1}^{\nu+1} \tau_\sigma^\nu \Gamma^\sigma, \ \lambda_\nu^\nu \neq 0, \ 0 \leq \nu \leq d-1$$

Consequently, any sequence $\{B_n\}_{n\geq 0}$ $d-$*orthogonal with respect to* $\Gamma = \left(\Gamma^1, \Gamma^2, ..., \Gamma^d\right)^T$ *is also d-orthogonal with respect to* $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_{d-1})^T$.

Definition 2 [5], [11]. *The fonctional* Γ *is regular if there exists a sequence* $\{B_n\}_{n\geq 0}$ *satisfing* (2.3) *and* (2.4).

Let D be the derivative operator

$$\langle D\mathcal{L}, p\rangle = -\langle \mathcal{L}, p'\rangle \quad , \ \forall \, p \in \mathcal{P}$$

and also we define the left product form by a polynomial

$$\langle f\mathcal{L}, p\rangle = \langle \mathcal{L}, fp\rangle \quad , \ \forall \, p, f \in \mathcal{P}$$

Definition 3 [11] *A sequence* $\{B_n\}_{n\geq 0}$ *is said strictly* $d-$*quasi-orthogonal of order* s *with respect to* $\Gamma = \left(\Gamma^1, \Gamma^2, ..., \Gamma^d\right)^T$ *if it satisfies:*

$$\langle \Gamma^\alpha, x^m B_n(x)\rangle = 0, \quad n \geq (m+s_\alpha)d+\alpha, \quad m \geq 0 \tag{2.5}$$

$$\left\langle \Gamma^\alpha, x^m B_{(m+s_\alpha)d+\alpha-1}(x)\right\rangle \neq 0, \quad m \geq 0 \tag{2.6}$$

for every $1 \leq \alpha \leq d$ *with* $s = \max_{1\leq\alpha\leq d} s_\alpha$.

Theorem 3 [5], [11]. *For each sequence* $\{B_n\}_{n\geq 0}$ *the following propositions are equivalent:*

(a)- The sequence $\{B_n\}_{n\geq 0}$ *is* $d-$*orthogonal with respect to* $\Gamma = \left(\Gamma^1, \Gamma^2, ..., \Gamma^d\right)^T$.

(b)- The sequence $\{B_n\}_{n\geq 0}$ *verifies a recurrence relation of order* $d+1 (d \geq 1)$:

$$\begin{aligned} B_{m+d+1}(x) &= (x-\beta_{m+d})B_{m+d}(x) \\ &- \textstyle\sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} B_{m+d-1-\nu}(x), \quad m \geq 0 \end{aligned} \tag{2.7}$$

with the initial conditions

$$B_0(x) = 1 \quad , \quad B_1(x) = x-\beta_0 \tag{2.8}$$

and if $d \geq 2$

$$B_n(x) = (x-\beta_{n-1})B_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} B_{n-2-\nu}(x) \tag{2.9}$$

$2 \leq n \leq d$

Corollary 4 *Let* $\{B_n\}_{n\geq 0}$ *be a* $d-OPS$ *with respect to* $\Gamma = \left(\Gamma^1, \Gamma^2, ..., \Gamma^d\right)^T$, *then*

$$\begin{aligned} B_{m+d+1}(x) &= (m+d+2)\, Q_{m+d+1}(x) \\ &- (m+d+1)\left(x-\beta_{m+d+1}\right) Q_{m+d}(x) \\ &+ \sum_{\nu=0}^{d-1} (m+d-\nu)\, \gamma_{m+d+1-\nu}^{d-1-\nu} Q_{m+d-1-\nu}(x) \end{aligned} \tag{2.10}$$

Now we give a definition of the $d-$weakly-orthogonality.

Definition 4 *A sequence* $\{B_n\}_{n\geq 0}$ *is said* $d-$*weakly-orthogonal of index* (p, q) *with respect to* $\Gamma = \left(\Gamma^1, \Gamma^2, ..., \Gamma^d\right)^T$ *if satisfies for every* $1 \leq \beta \leq d$

$$\begin{cases} \left\langle \Gamma^\beta, B_n(x)\right\rangle = 0, n \geq p_\beta d+\beta \\ \left\langle \Gamma^\beta, B_{p_\beta d+\beta-1}(x)\right\rangle \neq 0 \end{cases} \tag{2.11}$$

where $p = \max_{1\leq\beta\leq d} p_\beta$, *and*

$$\begin{cases} \left\langle \Gamma^\beta, xB_n(x)\right\rangle = 0, \ n \geq (q_\beta+1)\, d+\beta \\ \left\langle \Gamma^\beta, B_{(q_\beta+1)d+\beta-1}(x)\right\rangle \neq 0 \end{cases} \tag{2.12}$$

where $q = \max_{1\leq\beta\leq d} q_\beta$,

Remark 2 *A strictly* $d-$*quasi-orthogonal sequences of order* p *with respect to* Γ *is d-weakly orthogonal of index* $(p, p+1)$ *with respect to* Γ.

Remark 3 *If* $d = 1$, *we have the definition of the weakly orthogonal sequence of index* (p, q) [10].

Definition 5 ([5], [6], [7]) *A d-orthogonal monic sequence* $\{B_n\}_{n\geq 0}\,(d\geq 1)$ *is said to be classical, or simply d-classical, if it satisfies the Hahn's property, that is to say, the polynomial sequence* $\{Q_n\}_{n\geq 0}$ *is also d-orthogonal .*

Proposition 5 [9] *Let* $\{B_n\}_{n\geq 0}$ *be a d−OPS, then it satisfie the generalised Christoffel-Darboux identities*

$$\left(\prod_{\mu=0}^{n}\gamma_\mu^0\right)^{-1}\begin{vmatrix} B_{n+d}(x_1) & ... & B_n(x_1) \\ ... & ... & ... \\ ... & ... & ... \\ B_{n+d}(x_{d+1}) & ... & B_n(x_{d+1}) \end{vmatrix}$$
$$=\sum_{\nu=0}^{n}(-1)^{(n-\nu)(d-1)+d}\times\left(\prod_{\mu=0}^{\nu}\gamma_\mu^0\right)^{-1} \tag{2.13}$$
$$\times\begin{vmatrix} B_{\nu-1+d}(x_1) & ... & x_1B_{\nu-1+d}(x_1) \\ ... & ... & ... \\ ... & ... & ... \\ B_{\nu-1+d}(x_{d+1}) & ... & x_{d+1}B_{\nu-1+d}(x_{d+1}) \end{vmatrix}$$

with $x_i\neq x_j$ *if* $i\neq j$ *and when* $\gamma_n^0\neq 0, \forall n\geq 0\,(\gamma_0^0=1)$.

3 Kernel Polynomial of 2-Orthogonal Polynomial

Let $\{B_n\}_{n\geq 0}$ be a 2−OPS with respect to the form $\Gamma=(\Gamma_1,\Gamma_2)^T$, then the Christoffel-Darboux identity ($d=2$) [9] can be written as

$$\sum_{k=0}^{n}\frac{(-1)^{k-n}}{\prod_{\mu=0}^{k}\gamma_\mu^0}\begin{vmatrix} B_{k+1}(x) & B_k(x) & xB_{k+1}(x) \\ B_{k+1}(y) & B_k(y) & yB_{k+1}(y) \\ B_{k+1}(z) & B_k(z) & zB_{k+1}(z) \end{vmatrix}$$

$$=\frac{1}{\prod_{\mu=0}^{n}\gamma_n^0}\begin{vmatrix} B_{n+2}(x) & B_{n+1}(x) & B_n(x) \\ B_{n+2}(y) & B_{n+1}(y) & B_n(y) \\ B_{n+2}(z) & B_{n+1}(z) & B_n(z) \end{vmatrix}$$

and that we can put it under the following form

$$\frac{\prod_{\mu=0}^{n}\gamma_n^0}{\theta_n(y,z)}\sum_{k=0}^{n}\frac{(-1)^{k-n}}{\prod_{\mu=0}^{k}\gamma_\mu^0}\begin{bmatrix} M_k(x,y,z)B_{k+1}(x) \\ +N_k(x,y,z)B_k(x) \end{bmatrix}$$
$$=\frac{1}{(x-y)(x-z)}\begin{bmatrix} B_{n+2}(x)-\alpha_n(y,z)B_{n+1}(x) \\ +\delta_n(y,z)B_n(x) \end{bmatrix}$$

where

$$\begin{cases} M_k(x,y,z)=\dfrac{\begin{vmatrix} (x-y)B_{k+1}(y) & B_k(y) \\ (x-z)B_{k+1}(z) & B_k(z) \end{vmatrix}}{(x-y)(x-z)} \\ N_k(x,y,z)=\dfrac{(y-z)}{(x-y)(x-z)}B_{k+1}(y)B_{k+1}(z) \\ \alpha_n(y,z)=\dfrac{\begin{vmatrix} B_{n+2}(y) & B_n(y) \\ B_{n+2}(z) & B_n(z) \end{vmatrix}}{\theta_n(y,z)}, n\geq 0, \\ \delta_n(y,z)=\dfrac{\theta_{n+1}(y,z)}{\theta_n(y,z)}, n\geq 0, \\ \theta_n(y,z)=\begin{vmatrix} B_{n+1}(y) & B_n(y) \\ B_{n+1}(z) & B_n(z) \end{vmatrix}, n\geq 0. \end{cases}$$

Definition 6 *We define a sequence* $\left\{B^*_{n,y,z}\right\}_{n\geq 0}$ *by*

$$B^*_{n,y,z}(x)=\frac{\begin{bmatrix} B_{n+2}(x)-\alpha_nB_{n+1}(x) \\ +\delta_nB_n(x) \end{bmatrix}}{(x-y)(x-z)} \tag{3.1}$$

with

$$\begin{cases} \alpha_n=\dfrac{\begin{vmatrix} B_{n+2}(y) & B_n(y) \\ B_{n+2}(z) & B_n(z) \end{vmatrix}}{\theta_n}, & n\geq 0, \\ \delta_n=\dfrac{\theta_{n+1}}{\theta_n}, & n\geq 0, \\ \theta_n=\begin{vmatrix} B_{n+1}(y) & B_n(y) \\ B_{n+1}(z) & B_n(z) \end{vmatrix}, & n\geq 0. \end{cases} \tag{3.2}$$

Remark 4 $B^*_{n,y,z}(x)$ *is a monic polynomial of degree* n *because* y *and* z *are zeros of*

$$B_{n+2}(x)-\alpha_nB_{n+1}(x)+\delta_nB_n(x).$$

Definition 7 *A sequence* $\left\{B^*_{n,y,z}\right\}_{n\geq 0}$ *will be called Kernel polynomial of* $\{B_n\}_{n\geq 0}$.

For every real numbers y and z, we consider the new functional Γ^* of which the moments of order n are defined by

$$\Gamma^*(x^n)=\Gamma_n^*=\Gamma_{n+2}-(y+z)\Gamma_{n+1}+yz\Gamma_n$$

where $\Gamma_n=\Gamma(x^n)$ is the moment of order n of Γ.

It is obvious that for any polynomial $\Pi(x)$ of degree n we have

$$\Gamma^*[\Pi(x)]=(x-y)(x-z)\Gamma[\Pi(x)]$$

We now state the main result of our paper.

Theorem 6 *Let $\{B_n\}_{n\geq 0}$ be a 2-OPS with respect to the functional $\Gamma = (\Gamma_1, \Gamma_2)^T$. Then for any real numbers y and z, the functional $\Gamma^* = (x-y)(x-z)\Gamma$ is quasi-defined if and only if*

$$\theta_n = \begin{vmatrix} B_{n+1}(y) & B_n(y) \\ B_{n+1}(z) & B_n(z) \end{vmatrix} \neq 0, \quad n \geq 0. \tag{3.3}$$

In this case, the 2−orthogonal polynomial sequence relating to the functional $\Gamma^* = (\Gamma_1^*, \Gamma_2^*)^T$ is

$$B^*_{n,y,z}(x) = \frac{B_{n+2}(x) - \alpha_n B_{n+1}(x) + \delta_n B_n(x)}{(x-y)(x-z)} \tag{3.4}$$

with

$$\begin{cases} \theta_n = \begin{vmatrix} B_{n+1}(y) & B_n(y) \\ B_{n+1}(z) & B_n(z) \end{vmatrix}, n \geq 0, \\ \alpha_n = \dfrac{\begin{vmatrix} B_{n+2}(y) & B_n(y) \\ B_{n+2}(z) & B_n(z) \end{vmatrix}}{\theta_n}, n \geq 0, \\ \delta_n = \dfrac{\theta_{n+1}}{\theta_n}, n \geq 0. \end{cases} \tag{3.5}$$

Proof. As $\{B_n\}_{n\geq 0}$ is a 2-OPS with respect to the functional Γ , *then it satisfies* (2.3) and (2.4). For every $\alpha = 1, 2$ we have

$$\langle \Gamma^*_\alpha, x^m B^*_{n,y,z}(x) \rangle = \langle \Gamma_\alpha, x^m B_{n+2}(x) \rangle$$
$$-\alpha_n \langle \Gamma_\alpha, x^m B_{n+1}(x) \rangle$$
$$-\delta_n \langle \Gamma_\alpha, x^m B_n(x) \rangle = 0$$

for $n \geq 2m + \alpha$, $m \geq 0$, because

$$\begin{cases} \langle \Gamma_\alpha, x^m B_{n+2}(x) \rangle = 0, & n \geq 2m + \alpha - 2 \\ \langle \Gamma_\alpha, x^m B_{n+1}(x) \rangle = 0, & n \geq 2m + \alpha - 1 \\ \langle \Gamma_\alpha, x^m B_n(x) \rangle = 0, & n \geq 2m + \alpha \end{cases}$$

In the same way we have

$$\langle \Gamma^*_\alpha, x^m B^*_{2m+\alpha-1,y,z}(x) \rangle =$$
$$\langle \Gamma_\alpha, x^m B_{2m+\alpha+1}(x) \rangle$$
$$-\alpha_{2m+\alpha-1} \langle \Gamma_\alpha, x^m B_{2m+\alpha}(x) \rangle$$
$$-\delta_{2m+\alpha-1} \langle \Gamma_\alpha, x^m B_{2m+\alpha-1}(x) \rangle \neq 0$$

for $\delta_\lambda \neq 0 \ \ \forall m \geq 0$ because

$$\langle \Gamma_\alpha, x^m B_{2m+\alpha+1}(x) \rangle = 0$$
$$\langle \Gamma_\alpha, x^m B_{2m+\alpha}(x) \rangle = 0$$
$$\langle \Gamma_\alpha, x^m B_{2m+\alpha-1}(x) \rangle \neq 0 \ , \quad m \geq 0$$

That is

$$\delta_{2m+\alpha-1} = \frac{\theta_{2m+\alpha}}{\theta_{2m+\alpha-1}} \neq 0, \quad m \geq 0$$

which gives

$$\theta_{2m+\alpha} \neq 0 \ \text{ and } \ \theta_{2m+\alpha-1} \neq 0 \ , \quad m \geq 0$$

and finally we get

$$\theta_m \neq 0 \ , \quad m \geq 0$$

We conclude from

$$\begin{cases} \langle \Gamma^*_\alpha, x^m B^*_{n,y,z}(x) \rangle = 0, \ n \geq 2m + \alpha, & m \geq 0 \\ \langle \Gamma^*_\alpha, x^m B^*_{2m+\alpha-1,y,z}(x) \rangle \neq 0, & m \geq 0 \end{cases}$$

that $\{B^*_{n,y,z}\}_{n\geq 0}$ is a 2-OPS with respect to the functional $\Gamma^* = (x-y)(x-z)\Gamma$ if $\theta_n(y,z) \neq 0$, $n \geq 0$ ■

Proposition 7 *For an monic 2-OPS $\{Q_n\}_{n\geq 0}$, the following properties are equivalent:*

(i)- $\{Q_n\}_{n\geq 0}$ is a monic Kernel polynomial sequence (MKPS) for some other OPS.

(ii)- There exists two complex numbers y and z, and $\alpha_n, \delta_n \neq 0$ and an monic 2-OPS $\{B_n\}_{n\geq 0}$ such that

$$(x-y)(x-z) Q_n(x) = B_{n+2}(x) - \alpha_n B_{n+1}(x) \tag{3.6}$$
$$+\delta_n B_n(x) \tag{1}$$

under the condition $\theta_n(y,z) \neq 0$, $n \geq 0$

Proof. $(i) \Rightarrow (ii)$. Assume that $\{Q_n\}_{n\geq 0} = \{B^*_{n,y,z}\}_{n\geq 0}$. Then we have (3.6) with

$$\theta_n = \begin{vmatrix} B_{n+1}(y) & B_n(y) \\ B_{n+1}(z) & B_n(z) \end{vmatrix}, n \geq 0$$

$$\alpha_n = \frac{\begin{vmatrix} B_{n+2}(y) & B_n(y) \\ B_{n+2}(z) & B_n(z) \end{vmatrix}}{\theta_n}, n \geq 0$$

$$\delta_n = \frac{\theta_{n+1}}{\theta_n}, n \geq 0.$$

$(ii) \Rightarrow (i)$.Assume that (ii) holds. Then

$$\langle (x-y)(x-z)\Gamma_\alpha, x^m Q_n(x) \rangle =$$
$$\langle \Gamma_\alpha, x^m B_{n+2}(x) \rangle - \alpha_n \langle \Gamma_\alpha, x^m B_{n+1}(x) \rangle$$
$$+\delta_n \langle \Gamma_\alpha, x^m B_n(x) \rangle = 0$$

for $n \geq 2m + \alpha$. Furthermore

$$\langle (x-y)(x-z)\Gamma_\alpha, x^m Q_{2m+\alpha-1}(x) \rangle =$$
$$\langle \Gamma_\alpha, x^m B_{2m+\alpha+1}(x) \rangle$$
$$-\alpha_{2m+\alpha-1} \langle \Gamma_\alpha, x^m B_{2m+\alpha}(x) \rangle$$
$$+\delta_{2m+\alpha-1} \langle \Gamma_\alpha, x^m B_{2m+\alpha-1}(x) \rangle \neq 0$$

for

$$\delta_{2m+\alpha-1}(y,z)=\frac{\theta_{2m+\alpha}}{\theta_{2m+\alpha-1}(y,z)}\neq 0,\quad \forall m\geq 0$$

which gives

$$\theta_{2m+\alpha}\neq 0 \text{ and } \theta_{2m+\alpha-1}\neq 0 \text{ for } m\geq 0$$

Finally we get

$$\theta_m(y,z)\neq 0 \ , \ m\geq 0$$

So that $\{Q_n\}_{n\geq 0}$ is an MOPS relative to $(x-y)(x-z)\Gamma$. Hence $\{Q_n\}_{n\geq 0} = \{B^*_{n,y,z}\}_{n\geq 0}$ by theorem 2. ■

The Kernel polynomial $\{B^*_{n,y,z}\}_{n\geq 0}$ of the 2-orthogonal sequence $\{B_n\}_{n\geq 0}$verifies the following properties

Proposition 8 *If $\{B_n\}_{n\geq 0}$ is a strictly $2-$quasi-orthogonal sequence of order s with respect to the linear form Γ, then its Kernel polynomial $\{B^*_{n,y,z}\}_{n\geq 0}$ is also a strictly $2-$quasi-orthogonal of order s with respect to the linear form $\Gamma^*=(x-y)(x-z)\Gamma$, under the condition*

$$\theta_{2(s_\alpha+m)+\alpha}(y,z)\neq 0 \quad \textit{and} \quad \theta_{2(s_\alpha+m)+\alpha-1}(y,z)\neq 0$$

where $s=\max\limits_{1\leq\alpha\leq 2} s_\alpha$

Proof. As $\{B_n\}_{n\geq 0}$ is strictly $2-$quasi-orthogonal of order s with respect to the linear form Γ, then it satisfy (2.4) and (2.5). For every $\alpha=1,2$ we have

$$\begin{aligned}\left\langle\Gamma^*_\alpha, x^m B^*_{n,y,z}(x)\right\rangle &= \langle\Gamma_\alpha, x^m B_{n+2}(x)\rangle\\ &\quad -\alpha_n\langle\Gamma_\alpha, x^m B_{n+1}(x)\rangle\\ &\quad +\delta_n\langle\Gamma_\alpha, x^m B_n(x)\rangle\\ &= 0\end{aligned}$$

for $n\geq 2(s_\alpha+m)+\alpha$ and $m\geq 0$, because

$$\begin{cases}\langle\Gamma_\alpha, x^m B_{n+2}(x)\rangle=0, & n\geq 2(m+s_\alpha)+\alpha-2,\\ \langle\Gamma_\alpha, x^m B_{n+1}(x)\rangle=0, & n\geq 2(m+s_\alpha)+\alpha-1,\\ \langle\Gamma_\alpha, x^m B_n(x)\rangle=0, & n\geq 2(m+s_\alpha)+\alpha.\end{cases}$$

Furthermore (let $\gamma_\alpha=2(s_\alpha+m)+\alpha-1$)

$$\left\langle\Gamma^*_\alpha, x^m B^*_{\gamma_\alpha,y,z}(x)\right\rangle=\left\langle\Gamma_\alpha, x^m B_{\gamma_\alpha+2}(x)\right\rangle$$

$$-\alpha_{\gamma_\alpha}(y,z)\left\langle\Gamma_\alpha, x^m B_{\gamma_\alpha+1}(x)\right\rangle$$

$$+\delta_{\gamma_\alpha}\left\langle\Gamma_\alpha, x^m B_{\gamma_\alpha}(x)\right\rangle\neq 0$$

for $\delta_{\gamma_\alpha}\neq 0$, $m\geq 0$, because

$$\begin{aligned}&\left\langle\Gamma_\alpha, x^m B_{\gamma_\alpha+2}(x)\right\rangle=0\\ &\left\langle\Gamma_\alpha, x^m B_{\gamma_\alpha+1}(x)\right\rangle=0\\ &\left\langle\Gamma_\alpha, x^m B_{\gamma_\alpha}(x)\right\rangle\neq 0, \qquad m\geq 0.\end{aligned}$$

That is

$$\delta_{2(s_\alpha+m)+\alpha-1}(y,z)=\frac{\theta_{2(s_\alpha+m)+\alpha}(y,z)}{\theta_{2(s_\alpha+m)+\alpha-1}(y,z)}\neq 0.$$

Then

$$\theta_n(y,z)\neq 0$$

for

$$n=2(s_\alpha+m)+\alpha, 2(s_\alpha+m)+\alpha-1.$$

It follows that $\{B^*_{n,y,z}\}_{n\geq 0}$ is also a strictly $2-$quasi-orthogonal of order s with respect to the linear form $\Gamma^*=(x-y)(x-z)\Gamma$, if $\theta_{2(s_\alpha+m)+\alpha}(y,z)\neq 0$ and $\theta_{2(s_\alpha+m)+\alpha-1}(y,z)\neq 0$. ■

Proposition 9 *If $\{B_n\}_{n\geq 0}$ is a $2-$weakly-orthogonal of index (p,q) with respect to the linear form Γ, then its associated sequence $\{B^*_{n,y,z}\}_{n\geq 0}$ is also a $2-$weakly-orthogonal of index(p,q) with respect to the linear form $\Gamma^*=(x-y)(x-z)\Gamma$, under the condition*

$$\theta_n(y,z)\neq 0$$

for

$$n=2p_\alpha+\alpha,\ 2(q_\alpha+1)+\alpha$$

or

$$n=2p_\alpha+\alpha-1,\ 2(q_\alpha+1)+\alpha-1$$

where $p=\max\limits_{\alpha} p_\alpha$ *and* $q=\max\limits_{\alpha} q_\alpha$

Proof. As $\{B_n\}_{n\geq 0}$ is a $2-$weakly-orthogonal of index (p,q) with respect to the linear form Γ, then it satisfies (2.13) , and throughout

$$\begin{aligned}\left\langle\Gamma^*_\alpha, B^*_{n,y,z}(x)\right\rangle &= \langle\Gamma_\alpha, B_{n+2}(x)\rangle\\ &\quad -\alpha_n\langle\Gamma_\alpha, B_{n+1}(x)\rangle\\ &\quad +\delta_n\langle\Gamma_\alpha, B_n(x)\rangle\\ &= 0\end{aligned}$$

for $n\geq 2p_\alpha+\alpha$, because

$$\begin{cases}\langle\Gamma_\alpha, B_{n+2}(x)\rangle=0, & n\geq 2p_\alpha+\alpha-2,\\ \langle\Gamma_\alpha, B_{n+1}(x)\rangle=0, & n\geq 2p_\alpha+\alpha-1,\\ \langle\Gamma_\alpha, B_n(x)\rangle=0, & n\geq 2p_\alpha+\alpha.\end{cases}$$

Furthermore (let $\eta_\alpha=2p_\alpha+\alpha-1$)

$$\begin{aligned}\left\langle\Gamma^*_\alpha, B^*_{\eta_\alpha,y,z}(x)\right\rangle &= \left\langle\Gamma_\alpha, B_{\eta_\alpha+2}(x)\right\rangle\\ &\quad -\alpha_{\eta_\alpha}\left\langle\Gamma_\alpha, B_{\eta_\alpha+1}(x)\right\rangle\\ &\quad +\delta_{\eta_\alpha}\left\langle\Gamma_\alpha, B_{\eta_\alpha}(x)\right\rangle\\ &\neq 0\end{aligned}$$

for $\delta_{\eta_\alpha}\neq 0$, because

$$\begin{aligned}&\left\langle\Gamma_\alpha, B_{\eta_\alpha+2}(x)\right\rangle=0\\ &\left\langle\Gamma_\alpha, B_{\eta_\alpha+1}(x)\right\rangle=0\\ &\left\langle\Gamma_\alpha, B_{\eta_\alpha}(x)\right\rangle\neq 0,\end{aligned}$$

Then

$$\delta_{2p_\alpha+\alpha-1} = \frac{\theta_{2p_\alpha+\alpha}}{\theta_{2p_\alpha+\alpha-1}} \neq 0$$

hence

$$\theta_{2p_\alpha+\alpha} \neq 0 \quad \text{and } \theta_{2p_\alpha+\alpha-1} \neq 0$$

$\{B_n\}_{n\geq 0}$satisfies also (2.14), throughout

$$\begin{aligned} \left\langle \Gamma^*_\alpha, xB^*_{n,y,z}(x)\right\rangle &= \langle \Gamma_\alpha, xB_{n+2}(x)\rangle \\ &\quad -\alpha_n \langle \Gamma_\alpha, xB_{n+1}(x)\rangle \\ &\quad +\delta_n \langle \Gamma_\alpha, xB_n(x)\rangle \\ &= 0 \end{aligned}$$

for $n \geq 2(q_\alpha+1)+\alpha$, because

$$\begin{array}{ll} \langle \Gamma^\alpha, xB_{n+2}(x)\rangle = 0, & n \geq 2(q_\alpha+1)+\alpha-2, \\ \langle \Gamma^\alpha, xB_{n+1}(x)\rangle = 0, & n \geq 2(q_\alpha+1)d+\alpha-1, \\ \langle \Gamma^\alpha, xB_n(x)\rangle = 0, & n \geq 2(q_\alpha+1)+\alpha. \end{array}$$

Furthermore (let $\theta_\alpha = 2(q_\alpha+1)+\alpha-1$)

$$\begin{aligned} \left\langle \Gamma^*_\alpha, xB^*_{\theta_\alpha,y,z}(x)\right\rangle &= \langle \Gamma_\alpha, xB_{\theta_\alpha+2}(x)\rangle \\ &\quad -\alpha_{\theta_\alpha+1} \langle \Gamma_\alpha, xB_{\theta_\alpha+1}(x)\rangle \\ &\quad +\delta_{\theta_\alpha} \langle \Gamma_\alpha, xB_{\theta_\alpha}(x)\rangle \\ &\neq 0 \end{aligned}$$

for $\delta_{\theta_\alpha} \neq 0$, because

$$\begin{array}{l} \langle \Gamma^\alpha, xB_{2\theta_\alpha+2}(x)\rangle = 0 \\ \langle \Gamma^\alpha, xB_{\theta_\alpha+1}(x)\rangle = 0 \\ \langle \Gamma^\alpha, xB_{\theta_\alpha}(x)\rangle \neq 0 \end{array}$$

That is

$$\delta_{\theta_\alpha} = \frac{\theta_{2(q_\alpha+1)+\alpha}}{\theta_{2(q_\alpha+1)+\alpha-1}} \neq 0$$

Then

$$\theta_{2(q_\alpha+1)+\alpha}(y,z) \neq 0 \quad , \quad \theta_{2(q_\alpha+1)+\alpha-1}(y,z) \neq 0$$

It follows that $\left\{B^*_{n,y,z}\right\}_{n\geq 0}$ is a $2-$weakly-orthogonal of index (p,q) with respect to the linear form $\Gamma^* = (x-y)(x-z)\Gamma$, under the condition

$$\theta_n \neq 0$$

for

$$n = 2p_\alpha+\alpha, 2p_\alpha+\alpha-1,\ 2(q_\alpha+1)+\alpha$$

or

$$n = 2(q_\alpha+1)+\alpha-1$$

■

Denoting by $\{\mathcal{L}_n\}_{n\geq 0}$ the dual sequence of $\{B_n\}_{n\geq 0}$, then the sequence $\{B_n\}_{n\geq 0}$ is said d-orthogonal if and only if

$$\left\{ \begin{array}{ll} \langle \mathcal{L}_\alpha, x^m B_n(x)\rangle = 0, & n \geq md+\alpha+1, \quad m \geq 0 \\ \langle \mathcal{L}_\alpha, x^m B_{md+\alpha}(x)\rangle \neq 0, & m \geq 0 \end{array}\right.$$

Proposition 10 *Let $\{B_n\}_{n\geq 0}$ be a $2-$classical OPS with respect to the form $\Lambda = (\mathcal{L}_0, \mathcal{L}_1)^T$, then the sequence $\left\{B^*_{n,y,z}\right\}_{n\geq 0}$ is also a $2-$classical OPS with respect to the form $\Lambda^* = (x-y)(x-z)\Lambda$ under the condition*

$$\delta_{2m+\alpha-1} \neq 0$$

and

$$\gamma^0_{2m+\alpha-2} \neq \frac{(2m+\alpha)}{(2m+\alpha-1)} \frac{\left\langle \breve{\mathcal{L}}^*_\alpha, x^m Q_{2m+\alpha}(x)\right\rangle}{\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} Q_{2m+\alpha-2}(x)\right\rangle}$$

for $m \geq 0$. Here we take $\breve{\Lambda}^* = (x-y)(x-z)\breve{\Lambda}$.

Proof. Let $\{B_n\}_{n\geq 0}$ be a 2-OPS with respect to the form Λ, then $\left\{B^*_{n,y,z}\right\}_{n\geq 0}$ is also 2-OPS with respect to the form Γ^* (according to the proposition ...) . it remains to be shown that $\left\{Q^*_{n,y,z}\right\}_{n\geq 0}$ is a $2-$OPS with respect to $\breve{\Lambda}^*$.

Indeed

$$\begin{aligned} &\left\langle \breve{\mathcal{L}}^*_\alpha, x^m Q^*_{n,y,z}(x)\right\rangle \\ &= \frac{1}{n+1}\left\langle \breve{\mathcal{L}}^*_\alpha, x^m DB^*_{n+1,y,z}(x)\right\rangle \\ &= \frac{1}{n+1}\left\langle \breve{\mathcal{L}}^*_\alpha, D\left(x^m B^*_{n+1,y,z}(x)\right)\right\rangle \\ &\quad -\frac{m}{n+1}\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{n+1,y,z}(x)\right\rangle \\ &= -\frac{1}{n+1}\left\langle D\breve{\mathcal{L}}^*_\alpha, x^m B^*_{n+1,y,z}(x)\right\rangle \\ &\quad -\frac{m}{n+1}\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{n+1,y,z}(x)\right\rangle \end{aligned}$$

While using (2.2) we have

$$\begin{aligned} &\left\langle \breve{\mathcal{L}}^*_\alpha, x^m Q^*_{n,y,z}(x)\right\rangle \\ &= \frac{\alpha+1}{n+1}\left\langle \mathcal{L}^*_\alpha, x^m B^*_{n+1,y,z}(x)\right\rangle \\ &\quad -\frac{m}{n+1}\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{n+1,y,z}(x)\right\rangle \end{aligned}$$

according to the definition of the 2-orthogonality, we have

$$\left\langle \mathcal{L}^*_\alpha, x^m B^*_{n+1,y,z}(x)\right\rangle = 0, \quad 2m+\alpha, \quad m \geq 0$$

and

$$\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{n+1,y,z}(x) \right\rangle = \left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} B_{n+3}(x) \right\rangle - \alpha_{n+1} \left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} B_{n+2}(x) \right\rangle + \delta_{n+1} \left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} B_{n+1}(x) \right\rangle$$

By using the result (2.12), i.e.

$$B_n(x) = (n+1) Q_n(x) + n\beta_n Q_{n-1}(x) - (n-1)\gamma^1_n Q_{n-2}(x) - (n-2)\gamma^0_{n-1} Q_{n-3}(x) - nxQ_{n-1}(x)$$

we obtain for $m \geq 0$

$$\left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} B_{n+3}(x) \right\rangle = 0, \quad n \geq 2m+\alpha-1,$$
$$\left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} B_{n+2}(x) \right\rangle = 0, \quad n \geq 2m+\alpha,$$
$$\left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} B_{n+1}(x) \right\rangle = 0, \quad n \geq 2m+\alpha+1$$

Finally we have

$$\left\langle \breve{\mathcal{L}}^*_\alpha, x^m Q^*_{n,y,z}(x) \right\rangle = 0, \quad n \geq 2m+\alpha+1, \quad m \geq 0$$

Let us study the regularity now, let $\varrho_\alpha = 2m+\alpha$

$$\left\langle \breve{\mathcal{L}}^*_\alpha, x^m Q^*_{\varrho_\alpha,y,z}(x) \right\rangle$$
$$= \frac{1}{\varrho_\alpha+1} \left\langle \breve{\mathcal{L}}^*_\alpha, x^m D B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle$$
$$= \frac{1}{\varrho_\alpha+1} \left\langle \breve{\mathcal{L}}^*_\alpha, D\left(x^m B^*_{\varrho_\alpha+1,y,z}(x)\right) \right\rangle - \frac{m}{\varrho_\alpha+1} \left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle$$
$$= -\frac{1}{\varrho_\alpha+1} \left\langle D\breve{\mathcal{L}}^*_\alpha, x^m B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle - \frac{m}{\varrho_\alpha+1} \left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle$$
$$= \frac{\alpha+1}{\varrho_\alpha+1} \left\langle \mathcal{L}^*_\alpha, x^m B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle - \frac{m}{\varrho_\alpha+1} \left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle$$

according to the definition of the 2-orthogonality, we have

$$\left\langle \mathcal{L}^*_\alpha, x^m B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle = 0$$

and

$$\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle$$
$$= \left\langle (x-y)(x-z)\breve{\mathcal{L}}_\alpha, x^{m-1} B^*_{\varrho_\alpha+1,y,z}(x) \right\rangle$$
$$= \left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} \begin{pmatrix} B_{\varrho_\alpha+3}(x) \\ -\alpha_{\varrho_\alpha+1} B_{\varrho_\alpha+2}(x) \\ +\delta_{\varrho_\alpha+1} B_{\varrho_\alpha+1}(x) \end{pmatrix} \right\rangle$$
$$= \delta_{\varrho_\alpha+1} \left[\begin{array}{c} -(\varrho_\alpha-1)\gamma^0_{2m+\alpha} \left\langle \breve{\mathcal{L}}_\alpha, x^{m-1} Q_{\varrho_\alpha-2}(x) \right\rangle \\ -(\varrho_\alpha+1) \left\langle \breve{\mathcal{L}}_\alpha, x^m Q_{\varrho_\alpha}(x) \right\rangle \end{array} \right]$$
$$\neq 0$$

Thus we have the regularity if for $m \geq 0$

$$\delta_{\varrho_\alpha-1} \neq 0$$

and

$$\gamma^0_{\varrho_\alpha-2} \neq \frac{\varrho_\alpha}{\varrho_\alpha-1} \frac{\left\langle \breve{\mathcal{L}}^*_\alpha, x^m Q_{\varrho_\alpha}(x) \right\rangle}{\left\langle \breve{\mathcal{L}}^*_\alpha, x^{m-1} Q_{\varrho_\alpha-2}(x) \right\rangle}$$

■

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