Perfect Graphs and Vertex Coloring Problem

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Abstract—**The graph is perfect, if in all its induced subgraphs the size of the largest clique is equal to the chromatic number.In 1960 Berge formulated two conjectures about perfct graphs one stronger than the other, the weak perfect conjecture was proved in 1972 by Lovasz and the strong perfect onjecture was proved in 2003 by Chudnovsky and al.The problem to determine an optimal coloring of a graph is NP-complete in general case.Grötschell and al developed polynomial algorithm to solve this problem for the whole of the perfect graphs. Indeed their algorithms are not practically efficient. Thus, the search of very efficient polynomial algorithms to solve these problem in the case of the perfect graphs continues to have a practical interest.We will review the wealth of results that have appeared on these topics using bichromatic, trichromatic exchange and contraction operation.** ***Keywords: Optimal coloring, Vertex coloring problem, Perfect graphs, Combinatorial Algorithm***

1 Introduction

All graphs considered here are finite, undirected without loops or multiple edges.
The vertex set of a graph G is denoted by $V(G)$ and its edges set by $E(G)$.A stable set is a set of vertices no two of which are adjacent. A clique is a set of vertices every pair of which are adjacent. For $v \in V$, we denoted by $N_G(v)$ the neighborhood of a vertex v in a graph G. An induced subgraph of G is a graph with vertex set $S \subseteq V(G)$ and edge set comprising all the edges of G with both end in S,it is denoted by $G[X]$. The stability number $\alpha(G)$ defined as the largest number of pairwise nonadjacent vertices in G.The clique covering number $\theta(G)$,defined as the least number of cliques which cover all the vertices of G. The cardinality of a largest clique in graph G is denoted by $\omega(G)$. The number $\gamma(G)$ is a minimum partition into stable sets in graph G .The inequality $\alpha(G) \leq \theta(G)$ holds trivially for all complement of a graph G: if k cliques cover all the vertices then no more than k vertices can be pairewise nonadjacent.Graph which satisfy this inequality with the equality sign played an important role in Claude Shannon's (1956) paper [20] concerning the zero error capacity of noisy channel.In this paper, Shannon remarked that the smallest graph with $\alpha(G) < \theta(G)$ is C_5 the cycle of length five. A hole is a chordless cycle of length at least four and an anti hole is the complementary graph of a hole. We say that G is a Berge graph if G contains neither odd hole nor odd anti hole. A graph is called $\gamma - perfect$(respectively $\alpha - perfet$) if $\gamma(H) = \omega(H)$(respectively $\alpha(G) = \theta(G)$) for every induced subgraph H of G.It was Shannon's work which motivated Claude Berge in 1960 [7] to propose two conjectures: the first known as being the weak perfect graph conjecture (WCPG).

1.1 Conjecture[7]

A graph G is $\gamma - perfect$ if and only if G is $\alpha - perfet$.
Fulkerson (1971)[9] was the first to show that WCPG was closely related to the linear programming.

1.2 Theorem (Fulkerson[9]):

The following statements are equivalent:

1. If G is $\gamma - perfect$, then G is perfect.
2. Let A be a (0,1)-matrix such that the covering program yA≥w,$y \geq 0$, min 1-y,has an integer solution vector y whenever w is a (0,1)-vector. Then this program has an integer solution vector y whenever w is a nonnegative integer vector.

Continuing the same idea, the WCPG was proved by L.Lovasz (1972)[15] and is known as the perfect graph theorem.

1.3 Perfect Graph Theorem (Lovasz[15]):

Graph G is perfect if and only if $\overline{G}$ is perfect. The second conjecture is known as the strong perfect conjecture (SCPG).

1.4 Conjecture[7]

A graph is perfect if and only if it is Berge graph.
M.Chudnovsky, N.Robertson, P.Seymour and R.Thomas proved the strong perfect graph conjecture in 2003 [8] and it became the strong perfect graph theorem.
The problem to determine an optimal coloring of a graph is NP-complete in general case.Grötschell, Lovasz and Schrijver[10]developed polynomial algorithm to solve

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this problem for the whole of the perfect graphs. This algorithm use an alternative of the ellipsoids method for the resolution of linear programs.the interest of the result of Grötschell and all is not algorithmic so much.Indeed their algorithms are not practically efficient, undoubtedly because they do not take the combinatorial structure of perfect graphs into account. Thus, the search of very efficient polynomial algorithms to solve these problem in the case of the perfect graphs continues to have a practical interest.

2 Coloring Techniques:

In a graph $G = (V, E)$, a k-coloring is a mapping $C : V \to \{1, 2, \cdots, k\}$ such that $C(u) \neq C(v)$ for every edge uv.Note that each color class is a stable set ,hence a k-coloring can be thought of a partition of the vertices of a graph into stable sets $S_1, S_2, ..., S_k$.The chromatic number $\gamma(G)$ is the smallest k such that G admits a k-coloring.A $\gamma(G)$-coloring of G is called optimal or minimum coloring.It exist three technique of coloring using a vertex coloring problem:

The first sequential method was proposed by Ait Haddadene and Maffray in [1], let v be a vertex of a graph G with $\omega(G) \geq 4$ and assume $G - v$ has been $\omega(G)$-colored. Suppose that the following property holds: there exists a triple of distinct colors $i, j, k \in \{1, 2, \cdots, \omega(G)\}$ such that $G[S_i \cup S_j \cup S_k \cup \{v\}]$ is a K_4-free Berge subgraph. We can apply Tucker's [21] algorithm to 3-color the component of this subgraph containing v and get in this way an ω-coloring of G in time $O(n^3)$. We call this operation a trichromatic exchange. We say that v is a Tucker vertex if the previous property holds for every ω-coloring of $G - v$.

The second sequential method was proposed by Meyniel in [18],For a graph G, if there is a k-coloring of G and a vertex v of G-v such as either a color i misses in $N(v)$, or it exists a pair (i, j) of colors such as all vertices of color j in $N(v)$ belong to a connected component of the subgraph of G induced by vertices colored by i and j, not containing any vertex of $N(v)$ colored i (in particular a vertex colored j in $N(v)$ is not adjacent to a vertex of $N(v)$ colored i), then G is k-colorable. To obtain this k-coloring , it's enough when all the colors are present in N(v), to exchange, on the connected component described previously, the color j with color ii. if G is colored i and j , i.e. G_{ij}=$G[S_i \cup S_j]$. An (i, j) interchanges the colors i and j in the component of G_{ij} containing x.This operation is carried out in polynomial time and is called bichromatic exchange.

The third sequential method was proposed by Hayward in [12], two non adjacent vertices u and v in a graph G form an even pair (respectively a 2-pair) if every induced path between them has an even number of edges (respectively two edges). For a given pair $\{u, v\}$ in a graph G, we denote by G_{uv} the graph obtained by deleting u and v and adding a new vertex uv adjacent to precisely those vertices of $G - u - v$ which were adjacent to at least one of u or v in G. We say that G_{uv} is obtained by contracting on $\{u, v\}$. The importance of this contraction operation is that $\{u, v\}$ is a 2-pair in G then the chromatic number of G is equal to the chromatic number of G_{uv}, this fact yields a simple procedure which gives a k-coloring of G_{uv} yields a k-coloring of G. As we shall see, these procedures can be used to develop fast algorithms for finding an optimal coloring.

3 Results:

We try to summaries some results about coloring some perfect graphs by a sequential algorithms using bichromatic exchange, trichromatic exchange and contraction operation.

3.1 Some results about coloring using contraction vertex:

We discuss here various classes of graphs G such as there is a sequence $G_0 = G, G_1, ..., G_j$ such that G_j is a clique for $i \leq j - 1$, G_{i+1} is obtained from G_i via the contraction of an even pair of G_i (G is called even contractile). The classes discussed include triangulated graphs, comparability graphs, parity graphs and clique separable graphs.Proving that a class of graphs contains a graph which each of its induced subgraph is even contractile not only means that every graph in the class is perfect it also suggests a natural algorithm which will find optimal coloring. In the next the principle sequential algorithm is the following:

Algorithm Data: let $G = (V, E)$ be a perfect graph satisfying property P, with $|V| = n$
Result: the optimal coloring of G

1. Find an even pair (respectively a 2-pair) in G
2. Contract all even pair in G into a clique G_i of size at most $\omega(G)$
3. Find an optimal coloring of G_i.

3.1.1 Weakly triangulated graphs:

This class of graph was introduced by Hayward in 1985 [12].A graph is weakly triangulated if it contains no hole ($C_k \geq 5$) and no antihole ($\overline{C_k} \geq 5$).

Lemma(Hayward and al[13]): Contracting a 2-pair in a weakly triangulated graph yields a weakly triangulated graph.

Theorem (Hayward and al[13]): Every weakly triangulated graph which is not a clique contains a 2-pair.

One can take advantage of the even-pair contraction sequences to optimally color weakly triangulated graphs.In [13], Hayward, Hoàng and Maffray formally described $O(n^4m)$ algorithms which solve the minimum coloring problem on weakly triangulated graphs.

3.1.2 Meyniel graphs:

A Meyniel graph is any graph in which every odd cycle of length at least five has at least two chords. This class of graph was introduced by Meyniel in 1984 [17].Later [19], Meyniel showed that every such graph is either a clique or contains an even pair.However, there are non-complete Meyniel graphs which do not contain an even pair whose contration yields a Meyniel graph. Hertz [14], sidestepped this problem by defining a slightly larger class of graphs; the quasi-Meyniel graphs. The graphs is called quasi-Meyniel if

1. it contains no C_{2k+1}
2. for some vertex x of G, every edge which is the chord of some D_{2k+1}(where D_p denotes a cycle of length p with exactly one chord which forms a triangle with two consecutive edges of the cycle) of G has x as an endpoint.

Given a Quasi Meyniel G we call tip of G any vertex x that is endpoint of every chord of a D_{2k+1}(Every vertex of a Meyniel graph G is a tip of G)
Lemma(Hertz [14]): Contracting a 2-pair in a quasi-Meyniel graph yields a quasi Meyniel graph.

Even-pair contraction can be used as an optimization tool in Meyniel and quasi Meyniel graphs.We can quickly find an optimum coloring and maximum clique of G, this yields an $O(mn)$ algorithm for these two optimization problems given a quasi-Meyniel graph with a specified tip.Since every vertex of a Meyniel graph G is a tip,this yields an $O(mn)$ algorithm for the two optimization problems on Meyniel graphs [14].

3.2 Some results about coloring using bichromatic or trichromatic exchange:

We are interested in class of graphs where every induced subgraph has a vertex v satisfying a property P. Note that , an elimination ordering satisfying property P is a labelling $[v_1, v_2, ..., v_n]$ of vertices of a graph G such that each v_i has property P in the subgraph G_i=$G[v_i, v_{i+1}, ..., v_n]$.In the next, the principle of the sequential algorithm is the following:

Algorithm Data: let $G = (V, E)$ be a perfect graph satisfying property P, with $|V| = n$
Result: the optimal coloring of G

1. Search an elimination ordering $[v_1, v_2, ..., v_n]$ satisfying the property P;
2. For i from n-1 down to 1, compute the value of $\omega(G_i)$
3. For i from n-1 down to 1, from an optimally coloring of G_{i+1},find an optimally coloring of G_i by using trichromatic exchange technique,bichromatic exchange or coloring contraction.

3.2.1 Coloring Perfect $(K_4 - e)$-free:

This class of graph was introduced by Tucker in 1987 [22].The main result in [22]is the following:
Theorem (Tucker[22]) An n-vertex perfect $(K_4 - e)$-free graph G can be $\omega(G)$-colored in $O(n^3)$.
The proof of this theorem suggests a polynomial sequential algorithm using a elimination ordering [22] and bichromatic exchange for ω- coloring any $(K_4 - e)$-free graph.Tucker propose an algorithm in [21] to color a K_4-free perfect graph which is based on a reduction procedure which consist to contract such a graph into (K_4-e)-free perfect graph, which can be colored using Tucker's algorithm [22].

3.2.2 Coloring perfect degenerate graphs :

This class of graph was introduced by Ait Haddedene and Marray in 1997 [1]. A graph is usually called k-degenerate if every induced subgraph of G(including G itself) has a vertex of degree at most k.Obviously graph is k-degenerate for sufficiently large k;on the other hand, any such k must be greater than or equal to $\omega(G) + 1$.A graph G is called degenerate if every induced subgraph H has a vertex of degree at most$\omega(H) + 1$.This class of graph contain a chordal graph. A chordal graph is defined as any graph containing no induced hole.The main result in [1]is the following:
Theorem (Ait Haddedene and al [1]): Let G be a degenerate Berge graph and v a vertex of degree at most $\omega(H) + 1$. Then from any $\omega(G)$- coloring of $G - v$ one can obtain in polynomial time an $\omega(G)$- coloring of G.
The proof of this theorem suggests a polynomial sequential algorithm using a degenerate elimination ordering [1] and trichromatic exchange ,for ω- coloring any degenerate Berge graph; for fixed ω.This yields an O($n^{max\{\omega+1,5\}}$) algorithm for ω- coloring[1].

3.2.3 Coloring perfect weakly diamond-free graphs :

This class of graph was introduced by Ait Haddadene and Gravier in 1996 [3].A vertex v in a graph G is called a weakly diamond-free (noted WDF) vertex if its degree is at most $2\omega(G)-1$ and its neighborhood in G induced a diamond-free subgraph,where a diamond means K_4 minus an edge. A graph G is called WDF if every induced subgraph of G has a WDF vertex.This class of graph contain a chordal graphs, a (K_4-e)-free graphs and line graphs.The main result in [3]is the following:
Theorem(Ait Haddedene and al [3]): Let $G=(V,E)$ be a diamond-free graph with $\omega(G)\le p-1$ $(p\ge 3)$. If there exists a p-coloring of G for which each possible triple exists, then $|V|>2p-1$.

Using theorem6 we can find a polynomial sequential algorithm for any $\omega(G)$- coloring any WDF Berge graph.
Theorem (Ait Haddedene and al [3]): Let G be a WDF Berge graph and v be a WDF vertex of G where $\omega(G)\ge 3$. Then from any $\omega(G)$- coloring of $G-v$ one can obtain in polynomial time an $\omega(G)$- coloring of G. The algorithmic proof of the previous result suggest a combinatorial algorithm in $O(n^6)$using a WDF elimination ordering [3] and trichromatic exchange,for $\omega(G)$- coloring any WDF Berge graph.Here $\omega(G)$ can be recursively computed using Tucker result [21]in $O(n^6)$.

3.2.4 Coloring perfect MAG graphs :

This class of graph was introduced by Ait Haddadene, Gravier and Maffray in 1998 [2].A vertex v is called MAG vertex if the union of any four triangle in $N(v)$ contain a K_4.A graph is called MAG graph if any induced subgraph contain a MAG vertex.This class of graph contain a chordal graphs, a (K_4-e)-free graphs, a K_4-free and line graphs of bipartite graphs. The main result in [3]is the following:
Theorem(Ait Haddedene and al [2]): Let G be a MAG Berge graph and v be a MAG vertex of G. Then from any $\omega(G)$- coloring of $G-v$ one can obtain in polynomial time an $\omega(G)$- coloring of G. The algorithmic proof of the previous result suggest a combinatorial algorithm in $O(n^4)$ when the appropriate vertex ordering was given [3].

3.2.5 Coloring perfect Quasi locally paw-free graphs :

This class of graph was introduced by Ait Haddadene and Mechebek in 2003 [4]. A paw is a graph induced by the vertices a, b, c, d and the edges ab, ac, ad, cd. A Quasi locally paw-free (QLP) vertex in a graph G is a vertex whose neighborhood induced a paw-free subgraph. We say that a graph G is quasi locally paw-free(QLP-free) if every induced subgraph of G contains a QLP vertex.A vertex v is called QLP_0 vertex it is a QLP vertex such that $G[N(v)]$ admits at most one complete multipartite component.So, we say that G is a QLP_0 graph if every induced subgraph of G contains a QLP_0 vertex.This class of QLP graph contain a chordal graphs and K_4-free . The main result in [4]is the following:
Theorem (Ait Haddedene and al [4]): Let G be a QLP_0 Berge graph and v be a QLP_0 vertex of G. Then from any $\omega(G)$- coloring of $G-v$ one can obtain in polynomial time an $\omega(G)$- coloring of G. The previous result suggests a sequential algorithm using a QLP_0 elimination ordering [4] and trichromatic exchange for $\omega(G)$- coloring in polynomial time any QLP_0 Berge graph G.This yields an O(n^4)[4]. Note that $\omega(G)$ can be compute recursively in polynomial time[4].

3.2.6 Coloring perfect quasi-locally $P^*(\omega)$:

This class of graph was introduced by Ait Haddadene and Zenia in 2003 [5]. A quasi locally $P^*(\omega)$(QL$P^*(\omega)$) vertex in a graph G is a vertex whose neighborhood contain a number of triangles does not exceed $P^*(\omega)$ where $P^*(\omega)$=$1/6(\omega+3)(\omega^2+2)$,$\forall\omega\ge 3$.We say that a graph is called QL$P^*(\omega)$graph if every induced subgraph of G contain a QL$P^*(\omega)$vertex.This class of graph contains WDF graphs.The main result in [5]is the following:
Theorem(Ait Haddedene and al [5]): Let G be a QL$P^*(\omega)$ Berge graph and v be a QL$P^*(\omega)$ vertex of G. Then from any $\omega(G)$- coloring of $G-v$ one can obtain in polynomial time an $\omega(G)$- coloring of G. The coloring algorithm for a perfect graph QLP$P^*(\omega)$ proposed in [5] is polynomial. Note that $\omega(G)$ can be compute recursively in polynomial time[5].

3.2.7 Coloring perfect $P(\omega)$Graphs:

This class of graph was introduced by Gravier in 1999 [11].We consider the following property $P(\omega)$of a vertex v: the neighborhood of v does not induce a diamond and the number of neighbors of v belonging to a triangle does not exceed $3\omega-5$ if $\omega\le 6$ and $3\omega-6$ otherwise.This class of graph contain a chordal graphs, K_4-free,(K_4-e)-free and WFD-graph.The main result in [11]is the following:
Theorem(Gravier [11]): if a vertex v of a graph G satisfy $P(\omega(G))$ then v is a Tucker vertex.

The previous result suggests a sequential algorithm using an elimination ordering [11] and trichromatic exchange for $\omega(G)$- coloring in polynomial time any perfect $P(\omega)$Graph G.This yields an O(n^6)[11]. Note that $\omega(G)$ can be compute recursively in polynomial time[11].

3.2.8 Coloring Perfect Split Neighborhood Graphs:

This class of graph was introduced by Maffray and preissman in 1997 [16].A vertex v in a graph G is called a split vertex if its neighborhood in G can be partitioned into a clique and a stable set in G. Now, we say that a graph G is called perfect split-neighborhood graphs (SNAP) if G is perfect and every induced subgraph of G contains a split vertex.This class of graph contain a triangulated graphs and bipartite graph.The polynomial combinatorial algorithm for ω-coloring any graph of SNAP graph is proposed by Ait Haddadene and Issaadi in [6],the main result in [6] is the following:
Theorem(Ait Haddedene and al [6]): Let $G = (V, E)$ be a SNAP graph and v a split vertex. Then from any $\omega(G)$-coloring of $G - v$ one can possible obtain in polynomial time an $\omega(G)$-coloring of G. The proof of the previous result suggests a sequential algorithm using an elimination ordering [16] ,the trichromatic exchange and the contraction coloring for $\omega(G)$- coloring in polynomial time $O(n^3(n + m))$[6] any SNAP Graph. Note that $\omega(G)$ can be compute recursively in polynomial time $O(n^3(n + m))$[16]. In [1] an $O(n^5)$combinatorial algorithm which produce an $\omega(G)$- coloring for any degenerate SNAP graph was proposed.

4 Prospect and open problem:

The application of the coloring techniques presented, in the resolution of significant problems relating to the perfect graphs, increases the motivation for its development and leads to an interesting research, especially that it provides a way of improving the efficiency (on the practical level) of the algorithm of Grötschel and al. It also makes it possible to show by an algorithmic approach the validity of the famous strong perfect graph conjecture.It also makes it possible to show by an algorithmic approach the validity of the famous strong perfect graph conjecture. We develop the idea of (the trichromatic exchange, bichromatic exchange, contracting exchange) by an exchange which we propose to call sequential exchange. This development will make it possible to enrich knowledge of the structure of perfect graphs.For further researcher it would be interesting investigate the following open problem which is defined as follows:For any Berge graph G of Ω_P class (P selected property) containing a vertex v which satisfies the property P then, from any $\omega(G)$-coloring of G-v, we can obtain in polynomial time an $\omega(G)$-coloring of G.

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