A Subclass of Uniformly Convex Functions Associated with Certain Fractional Calculus Operator

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Abstract—**In this paper, we introduce a new class $K^{\mu,\gamma,\eta}(\alpha,\beta)$ of uniformly convex functions defined by a certain fractional calculus operator. The class has interesting subclasses like β-uniformly starlike, β-uniformly convex and β-uniformly pre-starlike functions. Properties like coefficient estimates, growth and distortion theorems, modified Hadamard product, inclusion property, extreme points, closure theorem and other properties of this class are studied. Lastly, we discuss a class preserving integral operator, radius of starlikeness, convexity and close-to-convexity and integral mean inequality for functions in the class $K^{\mu,\gamma,\eta}(\alpha,\beta)$.**

Keywords and Phrases: Fractional derivative, Univalent function, Uniformly convex function, Fractional integral operator, Incomplete beta function, Modified Hadamard product.

1 Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Also denote by T the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0, z \in U) \tag{1.2}$$

which are analytic and univalent in U.

For $g(z) = z - \sum\limits_{k=2}^{\infty} b_k z^k$ the modified Hadamard product of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

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A function $f(z) \in S$ is said to be β-uniformly starlike of order $\alpha, (-1 \leq \alpha < 1), \beta \geq 0$ and $(z \in U)$, denoted by $UST(\alpha,\beta)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \tag{1.4}$$

A function $f(z) \in S$ is said to be β-uniformly convex of order $\alpha, (-1 \leq \alpha < 1), \beta \geq 0$ and $(z \in U)$, denoted by $UCV(\alpha,\beta)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta\left|\frac{zf''(z)}{f'(z)}\right| \tag{1.5}$$

Notice that, $UST(\alpha,0) = S(\alpha)$ and $UCV(\alpha,0) = K(\alpha)$, where $S(\alpha)$ and $K(\alpha)$ are respectively the popular classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$. The classes $UST(\alpha,\beta)$ and $UCV(\alpha,\beta)$ were introduced and studied by Goodman [4], Rønning [13] and Minda and Ma [8].

Clearly $f \in UCV(\alpha,\beta)$ if and only if $zf' \in UST(\alpha,\beta)$.A function $f(z)$ is said to be close-to-convex of order r, $0 \leq r < 1$ if $Ref'(z) > r$. Let $\phi(a,c;z)$ be the incomplete beta function defined by

$$\phi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \quad (a \neq -1,-2,-3,\cdots \text{ and } c \neq 0,-1,-2,-3,\cdots) \tag{1.6}$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & : \ k = 0 \\ a(a+1)(a+2)\cdots(a+k-1) & : \ k \in I\!N \end{cases}$$

We note that $L(a,c)f(z) = \phi(a,b;z) * f(z)$, for $f \in S$ is the Carlson-Shaffer operator [1], which is a special case of the Dziok-Srivastava operator [2].

Following Saigo [15] the fractional integral and derivative operators involving the Gauss's hypergeometric function ${}_2F_1(a,b;c;z)$ are defined as follows.

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Definition 1 : Let $\mu > 0$ and $\gamma, \eta \in I\!R$. Then the generalized fractional integral operator $I_{0,z}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined by

$$I_{0,}^{\mu,\gamma,\eta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} f(t)\, {}_2F_1(\mu+\gamma, -\eta; \mu; 1-\frac{t}{z})dt$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with order

$$f(z) = 0(|z|^r), \quad z \to 0 \tag{1.7}$$

where $r > \max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$ and is well defined in the unit disc.

Definition 2 : Let $0 \le \mu < 1$ and $\gamma, \eta \in I\!R$. Then the generalized fractional derivative operator $J_{0,z}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz}$$
$$\left\{ z^{\mu-\gamma} \int_0^z (z-t)^{-\mu} f(t)\, {}_2F_1(\gamma-\mu, 1-\eta; 1-\mu; 1-\frac{t}{z})dt \right\}$$

where the function is analytic in the simply-connected region of the z-plane containing the origin, with the order as given in (1.7) and multiplicity of $(z-t)^{-\mu}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$ and is well defined in the unit disc.

Notice that $J_{0,z}^{\mu,\mu,\eta} f(z) = D_{0,z}^{\mu} f(z)$ which is the well known fractional derivative operator by Owa [10].

The fractional operator $U_{0,z}^{\mu,\gamma,\eta}$ is defined in terms of $J_{0,z}^{\mu,\gamma,\eta}$ for convenience as follows

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{\Gamma(2-\gamma)\Gamma(2-\mu+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta} f(z) \tag{1.8}$$

$(-\infty < \mu < 1; -\infty < \gamma < 1; \eta \in I\!R^+)$.
Thus,

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = z + \sum_{k=2}^{\infty} \frac{(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} a_k z^k.$$

Note that

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \frac{\Gamma(2-\gamma)\Gamma(2-\mu+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta} f(z); & 0 \le \mu < 1 \\ \frac{\Gamma(2-\gamma)\Gamma(2-\mu+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} I_{0,z}^{-\mu,\gamma,\eta}; & -\infty < \mu < 0 \end{cases}$$

for fractional differential operator $J_{0,z}^{\mu,\gamma,\eta}$ and fractional integral operator $I_{0,z}^{-\mu,\gamma,\eta}$.

Let us now consider another operator $M_{0,z}^{\mu,\gamma,\eta}$ defined using the operators $U_{0,z}^{\mu,\gamma,\eta}$ and the incomplete beta function $\phi(a, b; z)$ as follows.

For real numbers $\mu(-\infty < \mu < 1), \gamma(-\infty < \gamma < 1)$, $\eta \in I\!R^+, a \ne -1, -2, \cdots$, and $c \ne 0, -1, -2, \cdots$ we define the operator $M_{0,z}^{\mu,\gamma,\eta} : S \to S$ by

$$M_{0,z}^{\mu,\gamma,\eta} f(z) = \phi(a, b; z) * U_{0,z}^{\mu,\gamma,\eta} f(z) \tag{1.10}$$
$$= z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} a_k z^k$$
$$= z + \sum_{k=2}^{\infty} h(k) a_k z^k \tag{1.11}$$

for

$$h(k) = \frac{(a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} \tag{1.12}$$

Notice that,

$$M_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} f(z) & \text{if } a = c = 1; \quad \mu = \gamma = 0 \\ zf'(z) & \text{if } a = c = 1; \quad \mu = \gamma = 1 \end{cases}$$

Consider the subclass $S_{\mu,\gamma,\eta}(\alpha, \beta)$ consisting of functions $f \in S$ and satisfying

$$Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha \right\} \ge \beta \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| \tag{1.13}$$

$(z \in U, -\infty < \mu < 1; -\infty < \gamma < 1; \eta \in I\!R^+; -1 \le \alpha < 1; \beta \ge 0; a \ne -1, -2, \cdots; c \ne 0, -1, -2, \cdots)$.

Let $K_{\mu,\gamma,\eta}(\alpha, \beta) = S_{\mu,\gamma,\eta}(\alpha, \beta) \cap T$.

It is also interesting to note that the class $K_{\mu,\gamma,\eta}(\alpha, \beta)$ extends to the classes of starlike, convex, β-uniformly starlike, β-uniformly convex and β-prestarlike functions for suitable choice of the parameters $a, c, \mu, \gamma, \eta, \alpha$ and β. For instance;

1. For $a = c = 1; \mu = \gamma = 0$ the class $K_{\mu,\gamma,\eta}(\alpha, \beta)$ reduces to the class of $\beta - S(\alpha)$.
2. For $a = c = 1; \mu = \gamma = 1$ the class reduces to $\beta - K(\alpha)$.
3. For $a = 2 - 2\alpha; c = 1; \mu = \gamma = 0$ the class reduces to β-pre-starlike functions.

Several other classes studied can be derived from $K_{\mu,\gamma,\eta}(\alpha, \beta)$.

Definition 3. For two functions f and g analytic in U, we say that the function f is subordinate to g in U, denoted by $f \prec g$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < |z| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$.

2 Coefficient Estimates

Theorem 2.1. *A function $f(z)$ defined by (1.2) is in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$, if and only if*

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k \le 1-\alpha \tag{2.1}$$

where $0 \le \alpha < 1; \beta \ge 0, -\infty < \mu < 1, -\infty < \gamma < 1$, $\eta \in I\!R^+, a \ne -1,-2,\cdots$ and $c \ne 0,-1,-2,\cdots$.

Proof. Assume (1.2) holds, then we show that $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Thus, it is suffices to show that

$$\beta\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right|-Re\left\{\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-\alpha\right\}\le 0$$

that is,

$$\beta\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right|-Re\left\{\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right\}\le 1-\alpha.$$

We have

$$\beta\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right|-Re\left\{\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right\}$$
$$\le(1+\beta)\left|\frac{z(M_{0,z}^{\mu,\gamma,\eta}f(z))'}{M_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right|$$
$$\le\frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)h(k)a_k}{1-\sum\limits_{k=2}^{\infty}h(k)a_k}.$$

This expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k \le 1-\alpha \tag{2.2}$$

Conversely, we show that a function $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ satisfies inequality (2.1).

Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ and z be real, then by relation (1.11) and (1.13), we have

$$\frac{1-\sum\limits_{k=2}^{\infty}kh(k)a_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}h(k)a_kz^{k-1}}-\alpha\ge\beta\left|\frac{\sum\limits_{k=2}^{\infty}(k-1)h(k)a_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}h(k)a_kz^{k-1}}\right|.$$

Allowing $z\to 1$ along real axis, we obtain the desired inequality (2.2).

The equality in (2.2) is attained for the extremal function

$$f(z)=z-\frac{(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]h(k)}z^k\quad(k\ge 2). \tag{2.3}$$

Corollary 2.2. *Let a function f defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then*

$$a_k\le\frac{(1-\alpha)(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}}{[k(1+\beta)-(\alpha+\beta)](a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}},\quad k\ge 2.$$

Next, we give the growth and distortion theorem for the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$.

Theorem 2.3. *Let the function $f(z)$ defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then*

$$||M_{0,z}^{\mu,\gamma,\eta}f(z)|-|z||\le\frac{c(1-\alpha)(2-\gamma)(2-\mu+\eta)}{2a(\beta-\alpha+2)(2-\gamma+\eta)}|z|^2 \tag{2.4}$$

$$||(M_{0,z}^{\mu,\gamma,\eta}f(z))'|-1|\le\frac{c(1-\alpha)(2-\gamma)(2-\mu+\eta)}{a(\beta-\alpha+2)(2-\gamma+\eta)}|z| \tag{2.5}$$

Note that for $a=c=1;\beta=1$, we get the result obtained by G. Murugusundaramoorthy, T. Rosy and M. Darus in [9]. The bounds in (2.4) and (2.5), are attained for the function

$$f(z)=z-\frac{c(1-\alpha)(2-\gamma)(2-\mu+\eta)}{2a(\beta-\alpha+2)(2-\gamma+\eta)}z^2$$

3 Characterization Property

Theorem 3.1. *Let $\mu,\gamma,\eta\in I\!R$ such that $\mu(-\infty<\mu<1),\gamma(-\infty<\gamma<1),\eta\in I\!R^+,a\ne-1,-2,\cdots$ and $c\ne 0,-1,-2,\cdots$. Also let the function $f(z)$ given by (1.2) satisfy*

$$\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]}{1-\alpha}h(k)a_k\le\frac{1}{h(2)} \tag{3.1}$$

for $-1\le\alpha<1,\beta\ge 0$. Then $M_{0,z}^{\mu,\gamma,\eta}f(z)\in K_{\mu,\gamma,\eta}(\alpha,\beta)$, where $h(k)$ is given by (1.12).

Proof. We have from (1.11)

$$M_{0,z}^{\mu,\gamma,\eta}f(z)=z-\sum_{k=2}^{\infty}h(k)a_kz^k. \tag{3.2}$$

Under the condition stated in the hypothesis of this theorem, we observe that the function $h(k)$ is a non-increasing function of k for $k\ge 2$, and thus

$$0<h(k)\le h(2)=\frac{2a(2-\gamma+\eta)}{c(2-\gamma)(2-\mu+\eta)}. \tag{3.3}$$

Therefore, (3.1) and (3.3) yields

$$\sum_{k=2}^{\infty}\frac{k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}h(k)a_k\le h(2)$$
$$\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}h(k)a_k\le 1.$$

Hence by Theorem 1, we conclude that

$$M_{0,z}^{\mu,\gamma,\eta} f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta).$$

Remark. *The inequality in (3.1) is attained for the function $f(z)$ defined by*

$$f(z) = z - \frac{c^2(1-\alpha)(2-\gamma)^2(2-\mu+\eta)^2}{4a^2(\beta-\alpha+2)(2-\gamma+\eta)^2} z^2. \qquad (3.5)$$

4 Results on Modified Hadamard Product

Theorem 4.1. *For functions $f(z)$ and $g(z)$ defined by (1.2), let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ and $g(z) \in K_{\mu,\gamma,\eta}(\xi,\beta)$. Then*

$$(f * g)(z) \in K_{\mu,\gamma,\eta}(\delta,\beta)$$

where

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)(1-\xi)}{(\beta-\alpha+2)(\beta-\xi+2)h(2) - (1-\alpha)(1-\xi)} \qquad (4.1)$$

for $h(2)$ defined by (3.3).

The result is sharp for

$$f(z) = z - \frac{(1-\alpha)}{(\beta-\alpha+2)h(2)} z^2$$

and

$$g(z) = z - \frac{(1-\alpha)}{(\beta-\xi+2)h(2)} z^2$$

Proof. In view of Theorem 2.1 it is sufficient to show that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\delta+\beta)]h(k)}{1-\delta} a_k b_k \le 1 \qquad (4.2)$$

for δ defined by (4.1).

Now, $f(z)$ and $g(z)$ belong to $K_{\mu,\gamma,\eta}(\alpha,\beta)$ and $K_{\mu,\gamma,\eta}(\xi,\beta)$, respectively and so, we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{1-\alpha} a_k \le 1 \qquad (4.3)$$

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\xi+\beta)]h(k)}{1-\xi} b_k \le 1 \qquad (4.4)$$

By applying Cauchy-Schwarz inequality to (4.3) and (4.4), we get

$$\sum_{k=2}^{\infty} \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}}{\sqrt{(1-\alpha)(1-\xi)}} h(k)\sqrt{a_k b_k} \le 1. \qquad (4.5)$$

In view of (4.2) it suffices to show that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\delta+\beta)]h(k)}{1-\delta} a_k b_k \le \sum_{k=2}^{\infty} \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}}{\sqrt{(1-\alpha)(1-\xi)}} h(k)\sqrt{a_k b_k}$$

or equivalently

$$\sqrt{a_k b_k} \le \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}}{\sqrt{(1-\alpha)(1-\xi)}} \frac{1-\delta}{[k(1+\beta)-(\delta+\beta)]} \quad \text{for} \quad k \ge 2. \qquad (4.6)$$

In view of (4.5) and (4.6) it is sufficient to show that

$$\frac{\sqrt{(1-\alpha)(1-\xi)}}{h(k)\sqrt{[k(1+\beta)-(\alpha+\beta)[k(1+\beta)-(\xi+\beta)]}} \le \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}(1-\delta)}{\sqrt{(1-\alpha)(1-\xi)}[k(1+\beta)-(\delta+\beta)]}$$

for $k \ge 2$ which simplifies to

$$\delta \le 1 - \{(1+\beta)(k-1)(1-\alpha)(1-\xi)\}/ \{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]h(k) (1-\alpha)(1-\xi)\} \qquad (4.7)$$

where

$$h(k) = \frac{(a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} \quad \text{for} \quad k \ge 2.$$

Notice that $h(k)$ is a decreasing function of k ($k \ge 2$), and thus δ can be chosen as below.

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)(1-\xi)}{(\beta-\alpha+2)(\beta-\xi+2)h(2) - (1-\alpha)(1-\xi)}$$

for $h(2)$ defined by (3.3). This completes the proof.

Theorem 4.2. *Let the function $f(z)$ and $g(z)$ be defined by (2.1) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $(f * g)(z) \in K_{\mu,\gamma,\eta}(\delta,\beta)$, where*

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)^2}{(\beta-\alpha+2)^2 h(2) - (1-\alpha)^2}$$

for $h(2)$ given by (3.3).

Proof. Substituting $\alpha = \xi$ in the Theorem 4.1 above, the result follows.

Theorem 4.3. *Let the function $f(z)$ defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Consider*

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{for} \quad |b_k| \le 1.$$

Then $(f*g)(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$.

Proof. Notice that

$$\begin{aligned}&\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)|a_k b_k|\\ &=\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k|b_k|\\ &\le\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k\\ &\le 1-\alpha \qquad \text{using Theorem 2.1.}\end{aligned}$$

Hence $(f*g)(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$.

Corollary 4.4. *Let the function $f(z)$ defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Also let $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ for $0 \le b_k \le 1$. Then $(f*g)(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$.*

Next we prove the following inclusion property for functions in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$.

Theorem 4.5. *Let the functions $f(z)$ and $g(z)$ defined by (2.1) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then the function $h(z)$ defined by*

$$h(z) = z - \sum_{k=2}^{\infty}(a_k^2+b_k^2)z^k$$

is in the class $K_{\mu,\gamma,\eta}(\theta,\beta)$ where

$$\theta = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(\beta-\alpha+2)^2h(2)-2(1-\alpha)^2}$$

with $h(2)$ given by (3.3).

Proof. In view of Theorem 2.1 it is sufficient to show that

$$\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\theta+\beta)]h(k)}{1-\theta}(a_k^2+b_k^2) \le 1. \qquad (4.8)$$

Notice that, $f(z)$ and $g(z)$ belong to $K_{\mu,\gamma,\eta}(\alpha,\beta)$ and so

$$\sum_{k=2}^{\infty}\left[\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}\right]^2 a_k^2 \le \left[\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}a_k\right]^2 \le 1 \qquad (4.9)$$

$$\sum_{k=2}^{\infty}\left[\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}\right]^2 b_k^2 \le \left[\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}b_k\right]^2 \le 1. \qquad (4.10)$$

Adding (4.9) and (4.10), we get

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}\right]^2(a_k^2+b_k^2) \le 1. \qquad (4.11)$$

Thus (4.8) will hold if

$$\frac{[k(1+\beta)-(\theta+\beta)]}{1-\theta} \le \frac{1}{2}\frac{h(k)[k(1+\beta)-(\alpha+\beta)]^2}{(1-\alpha)^2}.$$

That is, if

$$\theta \le 1 - \frac{2(1+\beta)(k-1)(1-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2h(k)-2(1-\alpha)^2} \qquad (4.12)$$

Notice that, θ can be further improved by using the fact that $h(k) \le h(2)$ for $k \ge 2$. Therefore,

$$\theta = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(\beta-\alpha+2)^2h(2)-2(1-\alpha)^2}$$

where $h(2)$ is given by (3.3).

5 Integral Transform of the Class $K_{\mu,\gamma,\eta}(\alpha,\beta)$

For $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ we define the integral transform

$$L_\lambda(f)(z) = \int_0^1 \frac{\lambda(t)f(tz)}{t}dt$$

where $\lambda(t)$ is real valued, non-negative weight function normalized such that
$\int_0^1 \lambda(t)dt = 1$. Note that, $\lambda(t)$ have several special interesting definitions. For instance,
$\lambda(t) = (1+c)t^c, c > -1$, for which L_λ is known as the Bernardi operator. For

$$\lambda(t) = \frac{2^\delta}{\Gamma(\delta)}t(\log\frac{1}{t})^{\delta-1}, \quad \delta \ge 0 \qquad (5.1)$$

we get the integral operator introduced by Jung, Kim and Srivastava [6].

Let us consider the function

$$\lambda(t) = \frac{(c+1)^\delta}{\Gamma(\delta)}t^c(\log\frac{1}{t})^{\delta-1}, \quad c > -1, \quad \delta \ge 0. \qquad (5.2)$$

Notice that for $c = 1$ we get the integral operator introduced by Jung, Kim and Srivastava.

We next show that the class is closed under $L_\lambda(f)$ for $\lambda(t)$ given by (5.2).

Theorem 5.1. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$.*

Proof. By using the definition of $L_\lambda(f)$, we have

$$L_\lambda(f) = \frac{(c+1)^\delta}{\Gamma(\delta)}\int_0^1 \frac{t^c(\log\frac{1}{t})^{\delta-1}f(tz)}{t}dt \qquad (5.3)$$

$$= \frac{(c+1)^{\delta}}{\Gamma(\delta)} \int_0^1 (\log \frac{1}{t})^{\delta-1} t^c \left(z - \sum_{k=2}^{\infty} a_k t^{k-1} z^k \right) dt.$$

Simplifying by using the definition of gamma function, we get

$$L_{\lambda}(f) = z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k} \right)^{\delta} a_k z^k. \qquad (5.4)$$

Now $L_{\lambda}(f) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$ if

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta) - (\alpha+\beta)]h(k)}{(1-\alpha)} \left(\frac{c+1}{c+k} \right)^{\delta} a_k \leq 1. \qquad (5.5)$$

Also by Theorem 2.1 we have $f \in K_{\mu,\gamma,\eta}(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta) - (\alpha+\beta)]h(k)}{(1-\alpha)} a_k \leq 1. \qquad (5.6)$$

Thus, in view of (5.5) and (5.6) and the fact that $\left(\frac{c+1}{c+k} \right) < 1$ for $k \geq 2$, (5.5) holds true. Therefore, $L_{\lambda}(f) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$ and the proof is complete.

6 Extreme Points of $K_{\mu,\gamma,\eta}(\alpha, \beta)$

Theorem 6.1. *Let*

$$f_1(z) = z \qquad (6.1)$$

and

$$f_k(z) = z - \frac{(1-\alpha)}{[k(1+\beta) - (\alpha+\beta)]h(k)} z^k, \quad (k \geq 2). \quad (6.2)$$

Then $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$, if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Let $f(z)$ be expressible in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

Then

$$f(z) = z - \sum_{k=2}^{\infty} \frac{(1-\alpha)}{[k(1+\beta) - (\alpha+\beta)]h(k)} \lambda_k z^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{(1-\alpha)\lambda_k}{[k(1+\beta) - (\alpha+\beta)]h(k)} \frac{[k(1+\beta) - (\alpha+\beta)]h(k)}{(1-\alpha)} = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1.$$

Therefore, $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$.

Conversely, suppose that $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$. Thus,

$$a_k \leq \frac{(1-\alpha)}{[k(1+\beta) - (\alpha+\beta)]h(k)} \quad (k \geq 2).$$

Setting

$$\lambda_k = \frac{[k(1+\beta) - (\alpha+\beta)]h(k)}{(1-\alpha)} a_k \quad (k \geq 2)$$

and $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$, we get

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

This completes the proof.

7 Closure Theorem

Theorem 7.1. *Let the function $f_j(z)$ defined by (2.1) be in the class $K_{\mu,\gamma,\eta}(\alpha, \beta)$. Then the function $h(z)$ defined by*

$$h(z) = z - \sum_{k=2}^{\infty} e_k z^k \text{ belongs to } K_{\mu,\gamma,\eta}(\alpha, \beta)$$

where $f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k, \quad j = 1, 2, \cdots, \ell,$ and

$$e_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \quad (a_{k,j} \geq 0).$$

Proof. Since $f_j(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$, in view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta) - (\alpha+\beta)]h(k)}{(1-\alpha)} a_{k,j} \leq 1. \qquad (7.1)$$

Now,

$$\frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} \left(z - \sum_{k=2}^{\infty} a_{k,j} z^k \right) = z - \sum_{k=2}^{\infty} e_k z^k$$

where $e_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j}$.

Notice that,

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta) - (\alpha+\beta)]h(k)}{(1-\alpha)} \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \leq 1, \quad \text{using (7.1)}.$$

Thus, $h(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$.

8 Radius of Starlikeness, Convexity and Close-to-Convexity

Theorem 8.1. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is starlike of order $s, 0 \le s < 1$ in $|z| < R_1$ where*

$$R_1 = \inf_k \left[\frac{(1-s)[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)(k-s)} \right]^{\frac{1}{(k-1)}}. \quad (8.1)$$

Proof. $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is said to be starlike of order $s, 0 \le s < 1$, if and only if

$$Re\left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} \right\} > s \quad (8.2)$$

or equivalently

$$\left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| < 1 - s.$$

With fairly straight forward calculations, we get

$$|z|^{k-1} \le \frac{(1-s)[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)(k-s)}, \quad k \ge 2.$$

Setting $R_1 = |z|$, the result follows.

Next, we state the radius of convexity using the fact that f is convex, if and only if zf' is starlike. We omit the proof of the following theorems as the results can be easily derived.

Theorem 8.2. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is convex of order $c, 0 \le c < 1$ in $|z| < R_2$ where*

$$R_2 = \inf_k \left[\frac{(1-c)[k(1+\beta)-(\alpha+\beta)]}{k(1-\alpha)(k-c)} \right]^{\frac{1}{(k-1)}}.$$

Theorem 8.3. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is close-to-convex of order $r, 0 \le r < 1$ in $|z| < R_3$ where*

$$R_3 = \inf_k \left[\frac{(1-r)[k(1+\beta)-(\alpha+\beta)]}{k(1-\alpha)} \right]^{\frac{1}{(k-1)}}.$$

Thoerem 8.4. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)$ is starlike of order $p, 0 \le p < 1$ in $|z| < R_4$ where*

$$R_4 = \inf_k \left[\frac{(1-p)[k(1+\beta)-(\alpha+\beta)]h(k)(c+k)^\delta}{(1-\alpha)(k-p)(c+1)^\delta} \right]^{\frac{1}{(k-1)}}.$$

Theorem 8.5. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)$ is convex of order $q, 0 \le q < 1$ in $|z| < R_5$ where*

$$R_5 = \inf_k \left[\frac{(1-q)[k(1+\beta)-(\alpha+\beta)]h(k)(c+k)^\delta}{k(1-\alpha)(k-q)(c+1)^\delta} \right]^{\frac{1}{(k-1)}}.$$

Theorem 8.6. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)$ is close-to-convex of order $m, 0 \le m < 1$ in $|z| < R_6$ where*

$$R_6 = \inf_k \left[\frac{(1-m)[k(1+\beta)-(\alpha+\beta)]h(k)(c+k)^\delta}{k(1-\alpha)(c+1)^\delta} \right]^{\frac{1}{(k-1)}}.$$

9 Integral Mean Inequalities for the Fractional Calculus Operator

Lemma 9.1. *Let f and g be analytic in the unit disc, and suppose $g \prec f$. Then for $0 < p < \infty$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \le \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \quad (0 \le r < 1, p > 0).$$

Strict inequality holds for $0 < r < 1$ unless f is constant or $w(z) = \alpha z, \quad |\alpha| = 1$.

Theorem 9.2. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ and suppose that*

$$\sum_{k=2}^{\infty} |a_k| \le \frac{(1-\alpha)}{h(2)[j(1+\beta)-(\alpha+\beta)]} \quad (9.1)$$

Also let the function

$$f_j(z) = z + \frac{(1-\alpha)}{h(j)[j(1+\beta)-(\alpha+\beta)]} z^j \quad (j \ge 2). \quad (9.2)$$

If there exists an analytic function $w(z)$ given by

$$w(z)^{j-1} = \frac{[j(1+\beta)-(\alpha+\beta)]}{(1-\alpha)} \sum_{k=2}^{\infty} h(k) a_k z^{k-1}$$

then for $z = re^{i\theta}$ with $0 < r < 1$,

$$\int_0^{2\pi} |M_{0,z}^{\mu,\gamma,\eta} f(z)|^p d\theta$$
$$\le \int_0^{2\pi} |M_{0,z}^{\mu,\gamma,\eta} f_j(z)|^p d\theta \quad (0 \le \lambda \le 1, p > 0).$$

Proof. By virtue of relation (1.11) and (9.2), we have

$$M_{0,z}^{\mu,\gamma,\eta} f(z) = z + \sum_{k=2}^{\infty} h(k) a_k z^k. \quad (9.3)$$

and

$$M_{0,z}^{\mu,\gamma,\eta} f_j(z) = z + \frac{(1-\alpha)}{[j(1+\beta)-(\alpha+\beta)]} z^j. \quad (9.4)$$

For $z = re^{i\theta}, 0 < r < 1$, we need to show that

$$\int_0^{2\pi} \left| z + \sum_{k=2}^{\infty} h(k) a_k z^k \right|^p d\theta$$
$$\le \int_0^{2\pi} \left| z + \frac{(1-\alpha)}{[j(1+\beta)-(\alpha+\beta)]} z^j \right|^p d\theta \quad (p > 0). \quad (9.5)$$

By applying Littlewood's subordination theorem, it would be sufficient to show that

$$1 + \sum_{k=2}^{\infty} h(k) a_k z^{k-1} \prec 1 + \frac{(1-\alpha)}{[j(1+\beta)-(\alpha+\beta)]} z^{j-1}. \quad (9.6)$$

Setting

$$1+\sum_{k=2}^{\infty}h(k)a_kz^{k-1}=1+\frac{(1-\alpha)}{[j(1+\beta)-(\alpha+\beta)]}w(z)^{j-1}.$$

We note that

$$(w(z))^{j-1}=\frac{[j(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\sum_{k=2}^{\infty}h(k)a_kz^{k-1}, \quad (9.7)$$

and $w(0) = 0$. Moreover, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1, z \in U$

$$\begin{aligned}|w(z)|^{j-1} &\leq \left|\frac{[j(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\sum_{k=2}^{\infty}h(k)a_kz^{k-1}\right|\\ &\leq \frac{[j(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\sum_{k=2}^{\infty}h(k)|a_k||z|^{k-1}\\ &\leq |z|\frac{[j(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\ h(2)\sum_{k=2}^{\infty}|a_k|\\ &\leq |z|<1 \quad \text{by hypothesis (9.1).}\end{aligned}$$

This completes the proof of Theorem 9.2.

As a particular case of Theorem 9.2 we can derive the result for the function $f(z)$ by taking $a = c = 1$ and $\mu = \gamma = 0$ and thus $M_{0,z}^{\mu,\gamma,\eta}f(z) = f(z)$.

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