On The Existence Of Conditions Of a Classical Solution Of BGK-Poisson's Equation In Finite Time

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Abstract—**In this paper we prove the existence of solution to the periodic Boltzmann BGK model (Bhatnagar-Gross-Krook) coupled with poisson's equation in one space dimension. BGK model is a collision operator for the evolution of gases which satisfies several fundamental properties. Different collision operators for gas evolutions have been introduced earlier but none of them could satisfy all the basic physical properties : conservation, positivity, correct exchange coefficients, entropy inequality. However contrary to Boltzmann model which has a quadratic form, the BGK model, presents a heavy nonlinearity who explains the complexity of this analysis.**

Keywords: Kinetic equations, BGK model, Maxellian, Boltzmann's equations, Plasma's physics, Schauder's fixed point, Poisson's equation.

1 Introduction

We study the initial value problem of BGK model [1] coupled with Poisson's equation, which is a simple relaxation model introduced by Bhatnagar, Gross & Krook to mimic Boltzmann flows, where $f(x, v, t)$ is the density of plasma particles at time t in the space of position x and velocity v, and $\phi(x, t)$ is the potential of electric field of the plasma.
The existence and uniqueness problem of BGK model were proved by B.Perthame and M.Pulvirenti [5] but without coupling with Poisson's equation.
S.Ukay and T.Okabe [6] had proved the existence and uniqueness of (f, ϕ) for the Vlasov-Poisson equation (without collision term).
In this paper we are interested in the existence of a solution of initial value BGK model coupled with Poisson's equation. In periodic case the dimensionless BGK model coupled with Poisson's equation in one space dimension is written as :

$$\begin{cases} L_f f = M[f] - f, & (x,v) \in \Omega \times \mathbb{R},\ \ t \geq 0 \\ f(t=0) = f_0(x,v), & x \in \Omega,\ \ v \in \mathbb{R} \\ -\phi_{xx} = \int_{\mathbb{R}} f(x,v,t)dv, & \phi(0) = \phi(L) = 0. \end{cases} \tag{1}$$

$\Omega =]0, L[$.
Where :

$$L_f = \partial_t + v\partial_x + E(x,t)\partial_v \tag{2}$$

$$E(x,t) = -\frac{1}{2}\phi_x \tag{3}$$

$$M[f] = \frac{\rho}{(2\pi T)^{1/2}} \exp\left(-\frac{|u-v|^2}{2T}\right) \tag{4}$$

$M[f]$ is the Maxwellian associated to f, where :

$$(\rho, \rho u, \rho(u^2+T)) = \int_{\mathbb{R}} (1, v, v^2) f(v) dv \tag{5}$$

Remark 1.1 The notation L_f for the deferential operator in (2) is chosen to see that it depends on f according to (3).

2 Existence of Solution

Let G be the fundamental solution of Δ_x in $\mathbb{R}$ given as :

$$G(x,y) = \begin{cases} x(1 - \frac{y}{L}) & 0 \leq x \leq y \\ (1 - \frac{x}{L})y & y \leq x \leq 1 \end{cases} \tag{6}$$

and the potential of electric field, is given by :

$$\phi(x,t) = \int_0^L G(x,y)\left(\int_{\mathbb{R}} f(y,v,t)dv\right)dy$$

Using ϕ, we can solve the initial value problem of the first order partial differential equation,

$$\begin{cases} L_f + f = M[f], & (x,v) \in \Omega \times \mathbb{R},\ \ t \geq 0 \\ f(t=0) = f_0(x,v), & x \in \Omega,\ \ v \in \mathbb{R} \end{cases} \tag{7}$$

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We can easily solve equation (7) using the characteristic (X, V) solution of :

$$\begin{cases} \dfrac{dX}{dt} = V(t), \quad X(s) = x, \\ \dfrac{dV}{dt} = E(X(t), t), \quad V(s) = v. \end{cases} \tag{8}$$

Then the solution of equation (7) is given implicitly as

$$f(x, v, t) = e^{-t} f_0(X(0, x, v, t), V(0, x, v, t)) + \int_0^t e^{-(t-s)} M[f](X(s), V(s), s) ds$$

In this way we shall have assigned a function f to a given function g which we will denote by $f = \Phi(g)$. So we shall specify a set S of functions g in such a way that the map Φ defined on S which can be shown to be a fixed point , with the aid of Schauder's fixed point theorem and that any fixed point of Φ in S is a classical solution of (1).

2.1 Class of functions

For any set $\Xi \subset \mathbb{R}^2 \times \mathbb{R}^+$, we denote $B^{l+\sigma}(\Xi)$ the set of all continuous and bounded functions defined on Ξ having continuous and bounded l-th derivatives which are uniformly Holder continuous in Ξ with exponent σ, where l is integer ≥ 0 and $0 \leq \sigma \leq 1$.

2.2 Notations

For $f \in L^1(\mathbb{R}),\ q \geq 0,\ f \geq 0$ we denote :

$N_q(f) = \sup_{v\in\mathbb{R}}(|v|^q f(v))$ and $\mathbb{N}_q(f) = \sup_{v\in\mathbb{R}}((1 + |v|^q)f(v)$ and for $\tau \geq 0$ we introduce :

$\Omega_\tau = \Omega\times]0, \tau[$ and $Q_\tau = \Omega \times \mathbb{R}\times]0, \tau[$.

Lemma 2.1 *For $f \geq 0$ and $f(v) \in L^1(\mathbb{R}, (1 + v^2)dv)$, we have :*

$$i)\ \frac{\rho}{T^{1/2}} \leq N_0(f), \quad q > 3,$$

$$ii)\ \rho(T + u^2)^{\frac{(q-1)}{2}} \leq C_q N_q(f), \quad q > 3, \tag{9}$$

$$iii)\ \sup_{v\in\mathbb{R}}\{|v|^q M[f]\} \leq C_q N_q(f), \quad q > 3.$$

where (ρ, u, T) are given by (5).
Proof. We will prove the first assertion. For (ii) and (iii) the proofs are essentially identical and will not be repeated here. See [5].
For brevity, the variables x and t are considered as parameters. For any $R \geq 0$ we write :

$$\begin{aligned} \rho(x,t) &= \int_{|v-u|>R} f(v)dv + \int_{|v-u|<R} f(v)dv \\ &\leq \frac{1}{R^2}\int_{|v-u|>R} |v - u|^q f(v)dv + \int_{|v-u|<R} f(v)dv \\ &\leq \frac{\rho T}{R^2} + 2RN_0(f). \end{aligned}$$

We choose the value $R = (\dfrac{\rho T}{N_0(f)})^{1/3}$ corresponding to the minimum of the right term. Thus, (i) was proved.

Proposition 2.2 *Suppose that f is solution of (1) Then*

$$\mathbb{N}_q(f) \leq C_q \exp(C_q t) \ \ \textit{for} \ \ q = 0 \ \ \textit{or} \ \ q \geq 3$$

Proof. $i)$ We shall first prove the case $q = 0$,
From (9)(i) we have $M[f] \leq CN_0(f)$ where C is a positive constant. From (1) we have,

$$\begin{aligned} \frac{d}{dt}(e^t f(X(t), V(t), t)) &\leq Ce^t \sup_{x\in\Omega} N_0(f)(t), \\ \Rightarrow e^t f(X(t), V(t), t)) &\leq f_0(x, v) + C\int_0^t e^s \sup_{x\in\Omega} N_0(f)(s)ds, \\ \Rightarrow e^t f(x, v, t)) &\leq f_0(X(0, x, v, t), V(0, x, v, t)) \\ &\quad + C\int_0^t e^s \sup_{x\in\Omega} N_0(f)(s)ds, \\ \Rightarrow e^t \sup_{x\in\Omega} N_0(f)(t) &\leq ||f_0||_\infty + C\int_0^t e^s \sup_{x\in\Omega} N_0(f)(s)ds. \end{aligned}$$

The Gronwall lemma ended the proof.

$ii)$ case where $q > 3$
We denote $f_q = (1 + |v|^q)f$, writing the equation verified by f_q, we get :

$$L_f f_q = (1 + |v|^q)M[f] - f_q + v|v|^{(q-2)}Ef \tag{10}$$

Where E can be written as :

$$E(x, t) = \int_\Omega K(x, y)(\int_\mathbb{R} g(y, v, t)dv)dy. \tag{11}$$

K is a bounded kernel, can be easily deduced from (3) and (6).
We can easily see that

$$|E(x, y)| \leq \sup_{(x,y)\in\Omega} |K(x, y)|||f_0||_{L^1(\Omega\times\mathbb{R})} \tag{12}$$

for the values of $|v| \geq 1$ we have :

$$L_f f_q + f_q \leq C_q \mathbb{N}_q(f) \tag{13}$$

the Gronwall lemma applied to the map

$t \longrightarrow e^t \sup_{x\in\Omega} \mathbb{N}_q(f)(t)$ gives the proof.
The case $|v| < 1$ is easy to prove from (i).

Lemma 2.3

$i)$ We suppose that f_0 is not depend on x,

$ii)$ There is $V_1 \in \mathbb{R}$ such that f_0 is increasing as $]-\infty, V_1[$ and decreasing as $]V_1, +\infty[$,

$iii)$ There exist $C_0 > 0$ such that :

$$\int_{|v-V_1|>2\tau} f_0(v)dv \geq C_0. \tag{14}$$

Then

$$\rho(t) \geq C_0 e^{-t}.$$

Proof. We take $||f_0||_{L^1} = 1$

$$L_f f + f \geq 0 \quad \Rightarrow \frac{d}{dt}(e^t f(X(t), V(t), t)) \geq 0$$
$$\Rightarrow e^t f(X(t), V(t), t) \geq f_0(v)$$

thus, according to the proprieties of $(X(t,x,y,s), V(t,x,y,s))$, we can write :

$$e^t f(y,w,t)) \geq f_0(V(0,y,w,t))$$

$$\Rightarrow e^t \rho(y,t) \geq \int_{\mathbb{R}} f_0(V(0,y,w,t))dw \qquad (15)$$

We can see from (12) that $-1 \leq E(x,t) \leq 1$ this implies, together with (8), that

$$v - t \leq V(0,y,w,t) \leq v + t$$

We divide the region of integration in (15) as :

$D_1 = \{v, V(0,y,w,t) < V_1\}, D_2 = \{v, V(0,y,w,t) > V_1\}$

$$\Rightarrow \int_{\mathbb{R}} f_0(V(0,y,w,t))dw = \int_{D_1} f_0(V(0,y,w,t))dw + \int_{D_2} f_0(V(0,y,w,t))dw$$
$$\geq \int_{-\infty}^{V_1-t} f_0(w-t)dw + \int_{V_1+t}^{+\infty} f_0(w+t)dw$$
$$\geq \int_{|w-V_1|>2\tau} f_0(w)dw$$

By virtue to (14) (iii) the prove follows.

Proposition 2.4 *Let $f_0 \geq 0$ and $f_0 \in B^1(\Omega \times \mathbb{R})$ we suppose that there exist $A_0 > 0$ verify,*

$\sup_{v\in\mathbb{R}}\{(1+|v|^q) f_0(x,v)\} = A_0 < +\infty,$ *then*

$\forall t \in [0,\tau], \exists A(t) < +\infty, B(t) \in \mathbb{R}_+^*$ *verify*

$$\begin{aligned} &i)\ 0 < B(t) \leq T(t) \leq A(t),\\ &ii)\ u(t) \leq A(t). \end{aligned} \qquad (16)$$

Proof. From $(9)(i)$, we get :

$$T^{1/2} \geq C\frac{\rho}{N_0(f)} \geq C_1(t),$$

and from $(9)(ii)$ we get (16) (i) and (ii).
Indeed

$$\rho(T+u^2)^{\frac{q-1}{2}} \leq C_q N_q(f),$$

Thus,

$$(T+u^2) \leq A(t).$$

Definition 2.5 *We denote S as the class of functions satisfying :*

$S = \{g \in B^\delta(Q_\tau); ||g||_{B^\delta(Q_\tau)} \leq A_1, \sup_v((1+|v|^q)g) \leq A_2, \forall (x,t) \in \Omega\times]0,\tau[\}$, *where A_1 and A_2 are positive constants.*

For $g \in S$ we consider f a solution of :

$$\begin{cases} L_g f = M[g] - f, & (x,v) \in \Omega\times\mathbb{R},\ t \geq 0\\ f(t=0) = f_0(x,v), & x \in \Omega,\ v \in \mathbb{R}\\ -\phi_{xx} = \int_{\mathbb{R}} f(x,v,t)dv, & \phi(0) = \phi(L) = 0 \end{cases} \qquad (17)$$

and

$$E(x,t) = \int_\Omega K(x,y)(\int_{\mathbb{R}} g(y,v,t)dv)dy.$$

We denote :

$$f = \Phi(g)$$

We have to prove that Φ is a continuous map from S to itself, which will prove the existence of a solution in S.
The Solution of (17) is given by

$$\begin{aligned} f(x,v,t) &= e^{-t} f_0(X(0,x,v,t), V(0,x,v,t))\\ &+ \int_0^t e^{-(t-s)} M[g](X(s), V(s), s)ds \end{aligned} \qquad (18)$$

where we noted :

$$X(s) = X(s,x,v,t), \quad V(s) = V(s,x,v,t)$$

From (9) (i), we get :

$$M[g] \leq \frac{\rho(x,t)}{T^{1/2}} \leq CN_0(g)$$

by virtue of (18) and the condition imposed to f_0 in proposition (2.4), it's easy to see that $f \in S$ if $g \in S$.
We consider a sequence $g^n \in S$ and $g^\infty \in B_0(Q_\tau)$, verify $||g^n - g^\infty||_{B^0(Q_\tau)} \longrightarrow 0$ when $n \longrightarrow +\infty$.

Lemma 2.6 *S is a compact convex subset of $B^0(Q_\tau)$*

Proof. The convexity is easy to see. We prove the compactness.

For $g \in S$, we have, $\sup_v\{(1+|v|^q)g\} \leq A_2$, this imply

$$\forall \epsilon > \exists A(\epsilon) > 0 \Rightarrow \sup_{\{|v|>A(\epsilon)\}} g(v) \leq \frac{\epsilon}{2} \qquad (19)$$

for $\epsilon > 0$, we introduce the compact set Q_τ^ϵ

$$Q_\tau^\epsilon = [0,L] \times [-A(\epsilon), A(\epsilon)] \times [0,\tau]$$

and the corresponding functional space,

$S^\epsilon = \{g^\epsilon \in B^\delta(Q_\tau^\epsilon); ||g^\epsilon||_{B^\delta(Q_\tau^\epsilon)} \leq A_1, \sup_{v\in\mathbb{R}}\{(1+|v|^q)g\} \leq A_2, \forall (x,t) \in \Omega\times]0,\tau[\}.$

Since $||g^\epsilon||_{B^\delta(Q_\tau^\epsilon)} \leq A_1$ for $g^\epsilon \in S^\epsilon$, it's easily seen that S^ε is an equicontinuous subset to $B^0(Q_\tau)$, thus, we can apply the Ascoli theorem to prove that S^ε is a relative

compact subset to $B^0(Q_\tau)$.
We should prove that from any open cover of S we can extract a finite subcover.

Let $(U_i)_{i\in\mathbb{N}}$ a family of open balls centred at g_i with radius $\frac{\varepsilon}{2}$ such that $S \subset \bigcup_{i\in\mathbb{N}} U_i$,
hence $S^\varepsilon \subset S$, $(U_i)_{i\in\mathbb{N}}$ cover S^ε, and from Ascoli theorem there is a finite set I such that $S^\varepsilon \subset \bigcup_{i\in I} U_i$.
Let $g \in S$, we introduce the decomposition $g = g^\varepsilon + h$ where $g^\varepsilon = g\chi_{Q_\tau^\varepsilon}$, χ is the characteristic function.

On one hand, from (19) we have $||h||_{B^0(Q_\tau)} \leq \frac{\varepsilon}{2}$.

On the other hand, hence $g^\varepsilon \in S^\varepsilon$, there is a function $g_i \in U_i$ $(i \in I)$ such that $||g^\varepsilon - g_i||_{B^0(Q_\tau)} \leq \frac{\varepsilon}{2}$
Thus, $||g - g_i||_{B^0(Q_\tau)} \leq ||g^\varepsilon - g_i||_{B^0(Q_\tau)} + ||h||_{B^0(Q_\tau)} \leq \varepsilon$.
This prove that $S \subset \bigcup_{i\in\mathbb{N}} \tilde{U}_i$ where $\tilde{U}_i$ is a finite family of open balls centred as g_i with a radius $\frac{\varepsilon}{2}$.

2.3 Notation

For $s \geq 0$, we introduce this notation :

$$N_n^s(X,V) = max\{|X^n(s) - X^\infty(s)|, |V^n(s) - V^\infty(s)|\}$$

where, $(X^n(s), V^n(s))$ and $(X^\infty(s), V^\infty(s))$ are deduced from (8) using g^n and g^∞ in (11).

Lemma 2.7

$$N_n^s(X,V) \leq (1+\tau)\tau e^{(M\tau|t-s|)}||E^n - E^\infty||_{B^0(\Omega_\tau)}$$

The proof is detailed in [6].

Lemma 2.8 $||E^n - E^\infty||_{B^0(\Omega_\tau)} \longrightarrow 0,$ *when* $n \longrightarrow +\infty$
Proof.

$$\begin{aligned}
|E^n - E^\infty| &= |\int_\Omega\int_\mathbb{R} K(x,y)(g^n - g^\infty)(y,v,t)dvdy| \\
&\leq \int_\Omega\int_\mathbb{R} K(x,y)|(g^n - g^\infty)(y,v,t)|dvdy \\
&\leq (\int_\Omega\int_\mathbb{R}(K(x,y))^2|(g^n - g^\infty)(y,v,t)|)dvdy)^{1/2} \\
&\quad (\int_\Omega\int_\mathbb{R}|(g^n - g^\infty)(y,v,t)|)dvdy)^{1/2} \\
&\leq (\sup_{y\in\Omega} K(x,y)^2)^{1/2}\int_\Omega\int_\mathbb{R}|(g^n - g^\infty)(y,v,t)|)dvdy
\end{aligned}$$

This implies together with the hypotheses given to g^n and g^∞ that,

$$|E^n - E^\infty| \leq A.$$

the prove follows by the dominated convergence theorem.

Theorem 2.9 *With the conditions of proposition (2.4) for f_0, the problem (1) haves one solution (f, ϕ).*

Before the proof, we introduce these notations

2.4 Notations

First we denote :

$$f^n = \Phi(g^n) \qquad \text{and} \qquad f^\infty = \Phi(g^\infty)$$

and for any function $\mathcal{F}$,

$$\Delta\mathcal{F} = \mathcal{F}^n - \mathcal{F}^\infty.$$

finally, we change $(X(s), V(s))$ by (X_s, V_s).

Proof.

$$\begin{aligned}
&e^t\Delta f(x,v,t) = f_0(X_0^n, V_0^n) - f_0(X_0^\infty, V_0^\infty) + \\
&\int_0^t e^s[M[g^n](X_s^n, V_s^n, s) - M[g^\infty](X_s^\infty, V_s^\infty, s)]ds
\end{aligned} \tag{20}$$

On one hand, we have

$$f_0(X_0^n, V_0^n) - f_0(X_0^\infty, V_0^\infty) \leq ||f_0||_{B^1(\Omega\times\mathbb{R})} N_n^s(X,V)$$

On the other hand,

$$\begin{aligned}
&M[g^n](X_s^n, V_s^n, s) - M[g^\infty](X_s^\infty, V_s^\infty, s) \\
&\qquad = (M[g^n] - M[g^\infty])(X_s^n, V_s^n, s) \\
&\qquad + M[g^\infty](X_s^n, V_s^n, s) - M[g^\infty](X_s^\infty, V_s^\infty, s)
\end{aligned}$$

By virtue of (16) (i), it is easily seen that $M[g]$ has at last the same regularity as $g \in S$ then

$$\begin{aligned}
&|M[g^\infty](X_s^n, V_s^n, s) - M[g^\infty](X_s^\infty, V_s^\infty, s)| \leq \\
&\quad ||M[g^\infty]||_{B^\delta(Q_\tau)} N_n^s(X,V)^\delta
\end{aligned}$$

It remains then to estimate the term :

$$(M[g^n] - M[g^\infty])(X_s^n, V_s^n, s).$$

We pose for $\theta \in [0,1]$,

$$(\rho_\theta^n, u_\theta^n, T_\theta^n) = \theta(\rho^\infty, u^\infty, T^\infty) + (1-\theta)(\rho^n, u^n, T^n)$$

We denote M_θ^n the maxwellian associated to $(\rho_\theta^n, u_\theta^n, T_\theta^n)$.
We have :

$$\begin{aligned}
&|M[g^n] - M[g^\infty]|(X_s^n, V_s^n, s) \leq |\Delta\rho(X_s^n, s)\frac{\partial M_\theta^n}{\partial\rho}| \\
&+|\Delta u(X_s^n, s)\frac{\partial M_\theta^n}{\partial u}| + |\Delta T(X_s^n, s)\frac{\partial M_\theta^n}{\partial T}|
\end{aligned} \tag{21}$$

the derivatives of M_θ^n verify :

$$\begin{aligned}
|\frac{\partial M_\theta^n}{\partial\rho}| &\leq C(T_\theta^n)^{-1/2} \\
|\frac{\partial M_\theta^n}{\partial u}| &\leq C\rho_\theta^n(T_\theta^n)^{-1} \\
|\frac{\partial M_\theta^n}{\partial T}| &\leq C\rho_\theta^n(T_\theta^n)^{-3/2}.
\end{aligned}$$

To conclude from (21) we shall need the estimates :

$$(i) : |\Delta\rho(X_s^n, s), (ii) : |\Delta u(X_s^n, s)|, (iii) : |\Delta T(X_s^n, s)|$$

which can be estimated by :

$(i) : |\Delta\rho(X_s^n, s)| = |\int_{\mathbb{R}} (g^n - g^\infty)(X_s^n, w, s)dw|$

$(ii) : |\Delta u(X_s^n, s)| \leq C|\Delta u(X_s^n, s)\rho^\infty(X_s^n, s)|$

$\leq C|(\Delta(\rho u)u^\infty)(X_s^n, s)| + |u^n \Delta\rho(X_s^n, s)|$

$(iii) : |\Delta T(X_s^n, s)| \leq C|\Delta T(X_s^n, s)\rho^\infty(X_s^n, s)|$

$\leq C|\Delta(\rho T))(X_s^n, s)| + |T^n(\Delta\rho)(X_s^n, s)|.$

Hence $q > 3$, the dominated convergence theorem applied to (20) ended the proof.

We conclude that Φ is a continuous map in S, so it has a fixed point in S, which is a solution of BGK-Poisson's equations (1).
The schauder's point fixed theorem does not allow to show the uniqueness of the solution, this question remains open. Numerical tests were implemented to fill this fault, and they gave reassuring results.

Remark 2.10 The conditions (ii) and (iii) imposed to f_0 in lemma (14) can be generalized as :

$(ii)'$ There is a finite sequence $(V_n)_{n\in\mathbb{N}}$ such that f_0 is increasing as $]-\infty, V_0[$ and decreasing as $]V_n, +\infty[$.

$(iii)'$ There exist $C_0 > 0$ such that :
$$\int_{-\infty}^{V_0-2\tau} f_0(v)dv + \int_{V_n+2\tau}^{+\infty} f_0(v)dv \geq C_0$$

Remark 2.11 It is interesting to know that the condition (ii) imposed to f_0 is not excessive, the distributions of particles has generally this shape, like Gaussian curves for example.

3 Conclusion

We proved the existence of a solution of the complete model, with the collision's term and coupled with the Poisson's equation. It is a nonlinear problem, where the nonlinearity appears twice, first in the electric field term E and secondly in the collision term. BGK-collision's term, although it have an attractive shape, it presents a special nonlinearity which gives the complicated calculations.

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