Study of Steady Subsonic Flow by Decomposition Method

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Abstract— **The present paper produces the subsonic solution of linearized gasdynamic equation by means of decomposition method which is divided into four parts, such as, (i) regular or ordinary decomposition, (ii) double decompoisition, (iii) modified decomposition and (iv) asymptotic decomposition. Out of these four methods, regular and modified decomposition methods have been developed for inhomogeneous partial differential equations and a few examples of inviscid gasdynamics have been considered for clear illustration of the theories. The regular decomposition method is used for linearized steady axisymmetric subsonic flow past a corrugated circular cylinder whereas the modified decomposition method is used for linearized steady plane subsonic flow past a wave shaped wall.**

Keywords: Subsonic flow, decomposition method, wavy wall, corrugated cylinder, wind tunnel.

1 Introduction

Decomposition method is a most powerful method in the modern age for solving any type of differential equations. This method developed by George Adomian [1, 2]. Kaya [5-8] and Mamaloukas et. [9] provides approximate solutions to linear and nonlinear ordinary and partial differential equations. The solutions obtained by this method demands to be parallel to any modern super computer. It is also used in the development of numerical techniques for the solutions of nonlinear partial differential equations. The advantage of this method is to avoid restrictions which are used for simplifying the equations. The details of the method are given in the references [1, 2].

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Decomposition method is divided into four parts, such as, (i) Regular or ordinary decomposition method, (ii) Double decomposition method, (iii) Modified decomposition method and (iv) Asymptotic decomposition method. Here we develop the regular and modified decomposition methods with the help of a few examples of inviscid gasdynamics. Regular decomposition method is developed by considering the linearized steady axisymmetric subsonic flow past a corrugated circular cylinder and the modified decompoisition method is explained with the help of linearized steady plane subsonic flow past a wave shaped wall.

2 Regular Decomposition Method

Decomposition method prepares a single method which is used for multidimensional linear and nonlinear problems. This method has also been applied to different frontier problems in the other disciplines. The method is used in the development of numerical techniques in order to get the solution of nonlinear partial differential equation. For detail analysis of the theory we consider the inhomogeneous linear partial differential equation related to the fluid flow problems.

y-Partial Solution : For the application of regular decomposition method [2] we begin with the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y} = g(x, y) \tag{1}$$

Let $L_x = \dfrac{\partial^2}{\partial x^2}$ and $L_y = \dfrac{\partial^2}{\partial y^2} + \dfrac{1}{y}\dfrac{\partial}{\partial y} = \dfrac{1}{y}\dfrac{\partial}{\partial y}(y\dfrac{\partial}{\partial y})$. Then the equation (1) becomes

$$L_x u + L_y u = g \tag{2}$$

Solving for $L_y u$ we write (2) as

$$L_y u = g - L_x u \tag{3}$$

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Let L_y^{-1} be the inverse operator of L_y and it is defined by $L_y^{-1} = L_1^{-1}\left[\frac{1}{y}(L_1^{-1}y)\right]$ where $L_1^{-1} = \int(\cdot)dy$. Operating with this inverse operator on both sides of (3) we have the y-partial solution as [2]

$$u = \Phi_y - L_y^{-1}(L_x u) \tag{4}$$

where

$$\Phi_y = \xi_0(x) + \xi_1(x)\log y + L_y^{-1}g \tag{5}$$

Here $\xi_0(x)$ and $\xi_1(x)$ are the integration constants to be determined from the boundary conditions.

We now decompose u into the following form

$$u = \sum_{n=0}^{\infty} \alpha^n u_n \tag{6}$$

Here α is not a perturbation parameter; it is used only for grouping the terms. Then we write the parameterized form of (4) as [1]

$$u = \Phi_y - \alpha L_y^{-1}(L_x u) \tag{7}$$

Putting (6) into (7) and equating the co-efficients of like-power terms of λ from both sides of the resulting expression, we obtain

$$\begin{array}{lcl} u_0 & = & \Phi_y \\ u_1 & = & -L_y^{-1}(L_x u_0) \\ u_2 & = & -L_y^{-1}(L_x u_1) \\ \cdots & & \cdots \quad \cdots \\ \cdots & & \cdots \quad \cdots \\ u_{n+1} & = & -L_y^{-1}(L_x u_n) \end{array} \tag{8}$$

Thus we see that the components of u are determined and the final solution (6) is, therefore, computable remembering that $\alpha = 1$.

x-Partial Solution : For x-partial solution [2] we write the equation (2) in the form

$$L_x u = g - L_y u \tag{9}$$

Operating with L_x^{-1} defined by $L_x^{-1} = \int\int(\cdot)dx\, dx$ we get

$$u = \phi_x - L_x^{-1}(L_y u) \tag{10}$$

where

$$\Phi_x = \eta_0(y) + \eta_1(y)x + L_x^{-1}g \tag{11}$$

can be determined from the boundary conditions.

We now decompose u into

$$u = \sum_{n=0}^{\infty} \alpha^n u_n \tag{12}$$

and the parameterized form [2] of [10] is

$$u = \Phi_x - \alpha L_x^{-1}(L_y u) \tag{13}$$

Putting (12) into (13) and then comparing the like power terms from both sides of the resulting expression, we have

$$\begin{array}{lcl} u_0 & = & \Phi_x \\ u_1 & = & -L_x^{-1}(L_y u_0) \\ u_2 & = & -L_x^{-1}(L_y u_1) \\ \cdots & & \cdots \quad \cdots \\ \cdots & & \cdots \quad \cdots \\ u_{n+1} & = & -L_x^{-1}(L_y u_n) \end{array} \tag{14}$$

Thus the components of u are determined and the final solution (12) is computable. The method is clearly explained with the help of a few examples related to gas-dynamics.

2.1 Example 1. Steady Axisymmetric Subsonic Flow Past a Corrugated Circular Cylinder

The geometry of the corrugated circular cylinder is described by

$$h(x) = y_1 + \tau \sin \lambda x \tag{15}$$

where y_1 is the radius of the cylinder, τ is the roughness parameter and $2\pi/\lambda$ is the wave length.

The axisymmetric subsonic flow past the cylinder (15) is modelled by the partial differential equation

$$\beta^2 \phi_{xx} + \phi_{yy} + \frac{1}{y}\phi_y = 0 \tag{16}$$

subject to certain boundary conditions, where $\beta^2 = 1 - M_\infty^2$ and $\phi(x, y)$ is the perturbation velocity potential

In order to solve the equation (16) we write it in the form

$$\frac{1}{y}\frac{\partial}{\partial y}(y\phi_y) = k^2 \phi_{xx} \tag{17}$$

where $k^2 = (i\beta)^2$.

Let $L = \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right)$. Then the inverse of this operator is $L^{-1} = L_1^{-1}\left[\frac{1}{y}(L_1^{-1}y)\right]$ where $L_1^{-1} = \int(\cdot)dy$. Now operating on both sides of (17) with the inverse operator L^{-1} we have

$$\phi(x,y) = \phi_0(x,y) + k^2 L^{-1}\phi_{xx} \tag{18}$$

where

$$\phi_0(x,y) = a_0(x) + a_1(x)\log y. \tag{19}$$

Here $a_0(x)$ and $a_1(x)$ are integration constants to be determined by the boundary conditions.

Then we decompose $\phi^{(x,y)}$ into

$$\phi(x,y) = \sum_{n=0}^{\infty} \alpha^n \phi_n(x,y) \tag{20}$$

and the parameterized form [2] of (18) is

$$\phi(x,y) = \phi_0(x,y) + \alpha k^2 L^{-1}\phi_{xx} \tag{21}$$

Here α is not a perturbation parameter. It is used only for grouping the terms of different orders.

Substituting (20) in (21) and then equating like-power terms of α from both sides of the resulting expression we obtain

$$\phi_{n+1}(x,y) = k^2 L^{-1}\phi_{n,xx} \tag{22}$$

where $n = 0, 1, 2,$ etc. Putting $n = 0, 1, 2, \ldots$ in (22) successively we have the following set of expressions for the components of velocity potentials:-

$$\begin{aligned}
\phi_1(x,y) &= k^2 L^{-1}\left(\frac{\partial^2\phi_0}{\partial x^2}\right)\\
\phi_2(x,y) &= k^2 L^{-1}\left(\frac{\partial^2\phi_1}{\partial x^2}\right)\\
\ldots & \quad \ldots \quad \ldots\\
\ldots & \quad \ldots \quad \ldots\\
\phi_{n+1}(x,y) &= k^2 L^{-1}\left(\frac{\partial^2\phi_n}{\partial x^2}\right)
\end{aligned} \tag{23}$$

Using (19) in each of the expressions given in (23), we get

$$\begin{aligned}
\phi_1(x,y) &= a_0^{(2)}(x)\frac{(ky)^2}{2^2} + a_1^{(2)}(x)\left[\log y\frac{(ky)^2}{2^2} - \frac{(ky)^2}{2^2}(1)\right]\\
\phi_2(x,y) &= a_0^{(4)}(x)\frac{(ky)^4}{2^2.4^2} + a_1^{(4)}(x)\left[\log y\frac{(ky)^4}{2^2.4^2} - \frac{(ky)^4}{2^2.4^2}(1+\frac{1}{2})\right]
\end{aligned}$$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

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$$\phi_n(x,y) = a_0^{(2n)}(x)\frac{(ky)^{2n}}{2^2.4^2.\ldots.(2n)^2}$$

$$+a_1^{(2n)}(x)\left[\log y\frac{(ky)^{2n}}{2^2.4^2\ldots(2n)^2} - \frac{(ky)^{2n}}{2^2.4^2\ldots(2n)^2}\left(1+\frac{1}{2}\ldots+\frac{1}{n}\right)\right] \tag{24}$$

where $a_0^{(2n)}(x)$ and $a_1^{(2n)}(x)$ denote the derivatives of $a_0(x)$ and $a_1(x)$ with respect to x indicating their different orders for $n = 1, 2, 3,$ etc.

We now add the components $\phi_0(x,y)$, $\phi_1(x,y)$, etc. given in (19) and (24), and then we use the expanded form of (20) remembering that $\alpha = 1$. Finally, we get

$$\begin{aligned}
\phi(x,y) = &\left[a_0(x) + a_0^{(2)}(x)\frac{(ky)^2}{2^2} + a_0^{(4)}(x)\frac{(ky)^4}{2^2.4^2} + \ldots\right]\\
&+\log y\left[a_1(x) + a_1^{(2)}(x)\frac{(ky)^2}{2^2} + a_1^{(4)}(x)\frac{(ky)^4}{2^2.4^2} + \ldots\right]\\
&-\left[a_1^2(x)\frac{(ky)^2}{2^2}(1) + a_1^{(4)}(x)\frac{(ky)^4}{2^2.4^2}(1+\frac{1}{2}) + \ldots\right]
\end{aligned} \tag{25}$$

The solution (25) contains two unknown functions $a_0(x)$ and $a_1(x)$ which are to be determined now. For this purpose we set the two functions in the following forms looking at the boundary conditions :

$$\begin{aligned}
a_0(x) &= k_1\cos\lambda x\\
a_1(x) &= k_2\cos\lambda x
\end{aligned} \tag{26}$$

Then we substitute (26) in (25) and write the expression for $\phi(x,y)$ as

$$\phi(x,y) = k_1\left[1-\frac{(\lambda ky)^2}{2^2}+\frac{(\lambda ky)^4}{2^2.4^2}-\ldots\right]\cos\lambda x$$

$$+k_2\left[\log y\left\{1-\frac{(\lambda ky)^2}{2^2}-\frac{(\lambda ky)^4}{2^2.4^2}+\ldots\right\}\right.$$
$$\left.+\left\{\frac{(\lambda ky)^2}{2^2}(1)-\frac{(\lambda ky)^4}{2^2.4^2}(1+\frac{1}{2})+\ldots\right\}\right]\cos\lambda x$$

$$= k_1 J_0(i\lambda\beta y)\cos\lambda x + k_2\left[\log y.J_0(i\lambda\beta y)\right.$$
$$\left.+\left\{\frac{(i\lambda\beta y)^2}{2^2}(1)-\frac{(i\lambda\beta y)^4}{2^2.4^2}(1+\frac{1}{2})+\ldots\right\}\right]\cos\lambda x$$

$$= k_1 J_0(i\lambda\beta y)\cos\lambda x + k_2\left[\log y.J_0(i\lambda\beta y)\right.$$
$$\left.-\sum_{r=1}^{\infty}(-1)^r\frac{1}{r^2}.\left(\frac{i\lambda\beta y}{2}\right)^{2r}(1+\frac{1}{2}+\ldots+\frac{1}{r})\right]\cos\lambda x$$
$$= [k_1 J_0(i\lambda\beta y) + k_2 Y_0(i\lambda\beta y)]\cos\lambda x$$
$$= [k_1 I_0(\lambda\beta y) + k_2 K_0(\lambda\beta y)]\cos\lambda x \qquad (27)$$

where I_0 and K_0 are modified Bessel functions of order zero and k_1, k_2 are arbitrary constants to be determined according to the boundary conditions of the problems. This solution is used in the following three different cases :

Case I : Cylinder in an Unlimited Air Stream [10]

Consider flow past an infinitely long corrugated circular cylinder in an unlimited air stream. The boundary conditions in this case imposed on the perturbation velocity potential $\phi(x,y) are$

$$\phi(x,y) \;=\; \text{finite} \quad \text{at } y \;=\; \infty \qquad (28)$$

and

$$\phi_y(x,y) = U_\infty\frac{dh}{dx} = U_\infty\tau\lambda\cos\lambda x \quad \text{at} \quad y = y_1 \qquad (29)$$

The boundary condition (28) shows that the velocity potential $\phi(x,y)$ is finite at infinity. This means that the asymptotic behavior of the function I_0 requires that k_1 should be zero and the relation (27) reduces to

$$\phi(x,y) \;=\; k_2 K_0(\lambda\beta y)\cos\lambda x \qquad (30)$$

Then we satisfy the condition (29) by (30) to get $k_2 = -\frac{\tau U_\infty}{\beta K_1(\lambda\beta y_1)}$ and the required expression for $\phi(x,y)$ is given by

$$\phi(x,y) \;=\; -\frac{\tau U_\infty}{\sqrt{l-M_\infty^2}}\frac{K_0(\lambda\beta y)}{K_1(\lambda\beta y_1)}\cos\lambda x \qquad (31)$$

which is a closed form solution of the problem

Case II : Cylinder in an Open Throat Wind Tunnel[3]

If we consider flow past a corrugated circular cylinder in symmetrical subsonic flow in an open throat wind tunnel, then the boundary conditions on $\phi(x,y)$ are

$$\phi_x(x,y) \;=\; 0 \;\; \text{at} \;\; y \;=\; H \qquad (32)$$

and

$$\phi_y(x,y) \;=\; U_\infty\frac{dh}{dx} \;=\; U_\infty\tau\lambda\cos\lambda x \;\; \text{at} \;\; y \;=\; y_1 \qquad (33)$$

where H is the distance of the wind tunnel wall from the axis of the cylinder. Satisfying the boundary conditions (32) and (33) by (27) we get

$$k_1 I_0(\lambda\beta H) + k_2 K_0(\lambda\beta H) \;=\; 0 \qquad (34)$$

and

$$k_1 I_1(\lambda\beta y_1) - k_2 K_1(\lambda\beta y_1) \;=\; \frac{\tau U_\infty}{\beta} \qquad (35)$$

which, on solving, give

$$\left.\begin{aligned} k_1 &= \frac{\tau U_\infty}{\beta}\,\frac{K_0(\lambda\beta H)}{I_0(\lambda\beta H)K_1(\lambda\beta y_1)+I_1(\lambda\beta y_1)K_0(\lambda\beta H)} \\ k_2 &= -\frac{\tau U_\infty}{\beta}\,\frac{I_0(\lambda\beta H)}{I_0(\lambda\beta H)K_1(\lambda\beta y_1)+I_1(\lambda\beta y_1)K_0(\lambda\beta H)} \end{aligned}\right\} \qquad (36)$$

And hence the expression for $\phi(x,y)$ is given by

$$\phi(x,y) = \frac{\tau U_\infty}{\beta}\frac{K_0(\lambda\beta H)I_0(\lambda\beta y)-I_0(\lambda\beta H)K_0(\lambda\beta y)}{K_1(\lambda\beta y_1)I_0(\lambda\beta H)+I_1(\lambda\beta y_1)K_0(\lambda\beta H)}\cos\lambda x \qquad (37)$$

which is the exact solution of the problem.

Case III : Cylinder in a Closed throat Wind Tunnel [4]

If we again consider flow past a corrugated circular cylinder in a closed throat wind tunnel, then the boundary conditions in this case are

$$\phi_y(x,y) \ = \ 0 \ \text{ at } \ y \ = \ H \tag{38}$$

and

$$\phi_y(x,y) = U_\infty \frac{dh}{dx} \ = \ U_\infty \tau\lambda\cos\lambda x \ \text{ at } \ y \ = \ y_1 \tag{39}$$

where H has its usual meaning. Satisfying the boundary conditions (38) and (39) by (27) we have

$$k_1 I_1(\lambda\beta H) - k_2 K_1(\lambda\beta H) \ = \ 0 \tag{40}$$

and

$$k_1 I_1(\lambda\beta y_1) - k_2 K_1(\lambda\beta y_1) \ = \ \frac{\tau U_\infty}{\beta} \tag{41}$$

Then we solve (40) and (41) for k_1 and k_2 and get

$$\left.\begin{aligned} k_1 &= \frac{\tau U_\infty}{\beta} \frac{K_1(\lambda\beta H)}{I_1(\lambda\beta y_1)K_1(\lambda\beta H) - I_1(\lambda\beta H)K_1(\lambda\beta y_1)} \\ k_2 &= \frac{\tau U_\infty}{\beta} \frac{I_1(\lambda\beta H)}{I_1(\lambda\beta y_1)K_1(\lambda\beta H) - I_1(\lambda\beta H)K_1(\lambda\beta y_1)} \end{aligned}\right\} \tag{42}$$

Using (42) in (27) we write the expression for $\phi(x,y)$ as

$$\phi = \frac{\tau U_\infty}{\beta} \frac{K_1(\lambda\beta H)I_0(\lambda\beta y) + I_1(\lambda\beta H)K_0(\lambda\beta y)}{I_1(\lambda\beta y_1)K_1(\lambda\beta H) - I_1(\lambda\beta H)K_1(\lambda\beta y_1)} \cos\lambda x \tag{43}$$

which is the closed form solution of the problem.

3 Modified Decomposition Method

The method which produces a slight variation in the solution of regular decomposition method [1] is called modified decomposition method [2]. This method can be applied to the gasdynamic equations of physical porblems. We shall now discuss this method in details for the equation which is relevant to the real problems of gasdynamics.

***y*-Partial Solution [2]** : For the application of modified decomposition method we consider the following inhomogeneous partial differential equation :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y) \tag{44}$$

Let $L_x = \dfrac{\partial^2}{\partial x^2}$ and $L_y = \dfrac{\partial^2}{\partial y^2}$ be two linear operators. Then the equation (44) becomes

$$L_x u + L_y u = g(x,y) \tag{45}$$

which, on solving for $L_y u$, gives

$$L_y u = g(x,y) - L_x u \tag{46}$$

Let L_y^{-1} be the inverse operator of L_y and it is defined by $L_y^{-1} = \int\int(\cdot)dy\, dy$. Operating with L_y^{-1} on both sides of (46) we get the y-partial solution as [2]

$$u(x,y) = u_0(x,y) + L_y^{-1}[g(x,y) - L_x u] \tag{47}$$

which is called regular decomposition solution.

Here

$$u_0(x,y) = \xi_0(x) + \xi_1(x)y, \tag{48}$$

$\xi_0(x)$ and $\xi_1(x)$ being the integration constants to be evaluated by the boundary conditions.

We now follow the modified decomposition procedure [2] and write for this purpose

$$u(x,y) = \sum_{m=0}^{\infty} a_m(x)y^m \tag{49}$$

$$g(x,y) = \sum_{m=0}^{\infty} g_m(x)y^m \tag{50}$$

Putting (49) and (50) into (47), we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(x)y^m &= \xi_0(x) + \xi_1(x)y \\ &\quad + \int\int \left[\sum_{m=0}^{\infty} g_m(x)y^m - \frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty} a_m(x)y^m\right] dxdy \\ &= \xi_0(x) + \xi_1(x)y \\ &\quad + \left[\sum_{m=0}^{\infty} \frac{g_m(x)y^{m+2}}{(m+1)(m+2)} - \sum_{m=0}^{\infty} \frac{\partial^2}{\partial x^2} a_m(x) \frac{y^{m+2}}{(m+1)(m+2)}\right] \end{aligned}$$

Replacing m by $(m-2)$ on the right hand side of the above relation, we get

$$\begin{aligned} \sum_{m=0}^{\infty} a_m(x)y^m &= \xi_0(x) + \xi_1(x)y \\ &\quad + \sum_{m=2}^{\infty}\left[g_{m-2}(x) - \frac{\partial^2}{\partial x^2} a_{m-2}(x)\right]\frac{y^m}{m(m-1)} \end{aligned} \tag{51}$$

The co-efficients are identified by

$$\begin{aligned}\xi_0(x) &= a_0(x)\\ \xi_1(x) &= a_1(x)\end{aligned} \quad (52)$$

and for $m \geq 2$ the recurrence relation is

$$a_m(x) = \frac{g_{m-2}(x) - \frac{\partial^2}{\partial x^2} a_{m-2}(x)}{m(m-1)} \quad (53)$$

The relation (49) together with the relation (53) gives the complete solution of the problem.

x**-Partial Solution [2]** : For x-partial solution we write the equation (45) in the form

$$L_x u = g(x, y) - L_y u \quad (54)$$

If L_x^{-1} be the inverse operator of L_x defined by $L_x^{-1} = \int\int(\cdot)dxdx$ and if we operate both sides of (54) with L_x^{-1}, then we have the x-partial solution as

$$u(x, y) = \eta_0(y) + \eta_1(y)x + L_x^{-1}[g(x, y) - L_y u] \quad (55)$$

where $\eta_0(y)$ and $\eta_1(y)$ are to be evaluated from the prescribed boundary conditions.

We now follow the modified decomposition procedure [2] and assume that

$$u(x, y) = \sum_{m=0}^{\infty} a_m(y)x^m \quad (56)$$

and

$$g(x, y) = \sum_{m=0}^{\infty} g_m(y)x^m \quad (57)$$

Putting (56) and (57) into (55) and then integrating we get after somewhat straight forward calculations

$$\begin{aligned}\sum_{m=0}^{\infty} a_m(y)x^m &= \eta_0(y) + \eta_1(y)x\\ &+ \sum_{m=2}^{\infty} \frac{g_{m-2}(y) - \frac{\partial^2}{\partial y^2} a_{m-2}(y)}{m(m-1)}\end{aligned} \quad (58)$$

Then we equate the co-efficients of like-power terms from both sides of (58) and get

$$\left.\begin{aligned}\eta_0(y) &= a_0(y)\\ \eta_1(y) &= a_1(y)\end{aligned}\right\} \quad (59)$$

and the recurrence relation for $m \geq 2$

$$a_m(y) = \frac{g_{m-2}(y) - \frac{\partial^2}{\partial y^2} a_{m-2}(y)}{m(m-1)} \quad (60)$$

Thus the relation (56) together with the relation (60) constitutes the solution of the problem.

We now consider a few examples of fluid flow problems from gasdynamics in order to make the procedure clear as far as possible.

3.1 Example 2. Steady Plane Subsonic Flow Past a Wave Shaped Wall

The wavy wall is described by

$$h(x) = \tau \sin \lambda x \quad (61)$$

where τ is the roughness parameter and $2\pi/\lambda$ is the wave length.

The steady plane subsonic flow past a wavy wall is formulated by the partial differential equation

$$\beta^2 \phi_{xx} + \phi_{yy} = 0, \quad (62)$$

where $\beta^2 = 1 - M_\infty^2$ together with certain boundary conditions.

Let $L_x = \dfrac{\partial^2}{\partial x^2}$ and $L_y = \dfrac{\partial^2}{\partial y^2}$ be two linear differential operators. Then the equation (62) takes the form

$$\beta L_x \phi + L_y \phi \;=\; 0 \quad (63)$$

which, on solving for $L_y \phi$, gives

$$L_y \phi = -\beta^2 L_x \phi \quad (64)$$

If L_y^{-1} is the inverse operator of L_y and if it is defined by $L_y^{-1} = \int\int(\cdot)dy\ dy$, then operating on both sides of (64) with the operator L_y^{-1} we have

$$\phi(x, y) = a_0(x) + a_1(x)y - L_y^{-1}(\beta^2 L_x \phi), \quad (65)$$

$a_0(x)$ and $a_1(x)$ being the integration constants to be determined from the boundary conditions.

We now proceed for modified decomposition procedure and write $\phi(x,y)$ in the following form :

$$\phi(x,y) = \sum_{m=0}^{\infty} a_m(x) y^m \tag{66}$$

Substituting (66) in (65) we get

$$\begin{aligned}
\sum_{m=0}^{\infty} a_m(x) y^m &= a_0(x) + a_1(x) y \\
&- \int\int \left[\beta^2 \frac{\partial^2}{\partial x^2} \sum_{m=0}^{\infty} a_m(x) y^m \right] dxdy \\
&= a_0(x) + a_1(x) y \\
&- \sum_{m=0}^{\infty} \frac{\partial^2}{\partial x^2} a_m(x) \frac{\beta^2}{(m+1)(m+2)} y^{m+2} \\
&= a_0(x) + a_1(x) y \\
&- \sum_{m=2}^{\infty} \frac{\partial^2}{\partial x^2} a_{m-2}(x) \frac{\beta^2}{m(m-1)} y^m
\end{aligned} \tag{67}$$

which gives the recurrence relation for $m \geq 2$ as

$$a_m(x) = -\frac{\partial^2}{\partial x^2} a_{m-2}(x) \frac{\beta^2}{m(m-1)} \tag{68}$$

The relation (68) gives the other co-efficient for different values of m. Putting $m = 2, 3, 4$, etc. in (68), we get

$$\left.\begin{aligned}
a_2(x) &= -a_0^{(2)}(x) \frac{\beta^2}{2!} \\
a_3(x) &= -a_1^{(2)}(x) \frac{\beta^2}{3!} \\
a_4(x) &= a_0^{(4)}(x) \frac{\beta^4}{4!} \\
a_5(x) &= a_1^{(4)}(x) \frac{\beta^4}{5!}, \quad \text{etc.}
\end{aligned}\right\} \tag{69}$$

where the numbers in the first bracket at the heads of $a_0(x)$ and $a_1(x)$ indicate the orders of differentiation of the functions with respect to x. By virtue of (69), the solution (66) becomes

$$\begin{aligned}
\phi(x,y) &= \left[a_0(x) - a_0^{(2)}(x) \frac{(\beta y)^2}{2!} + a_0^{(4)}(x) \frac{(\beta y)^4}{4!} \ldots \right] \\
&+ \frac{1}{\beta} \left[a_1(x) \frac{(\beta y)}{1!} - a_1^{(2)}(x) \frac{(\beta y)^3}{3!} + a_1^{(4)}(x) \frac{(\beta y)^5}{5!} + \ldots \right]
\end{aligned} \tag{70}$$

We now proceed to find out the unknown functions $a_0(x)$ and $a_1(x)$ in order to complete the solution of the problem. For this purpose we set the functions $a_0(x)$ and $a_1(x)$ in the following forms :

$$\left.\begin{aligned}
a_0(x) &= \gamma \cos \lambda x \\
a_1(x) &= \delta \cos \lambda x
\end{aligned}\right\} \tag{71}$$

Using (71) in (70), we obtain

$$\phi(x,y) = \frac{1}{2} \left[(\gamma + \frac{\delta}{\lambda\beta}) \, e^{\lambda\beta y} + (\gamma - \frac{\delta}{\lambda\beta}) \, e^{-\lambda\beta y} \right] \cos \lambda x \tag{72}$$

The relation (72) is the general solution of (62) involving the integration constants γ and δ. The determination of these constants will depend upon the boundary conditions and give the complete solution of the problem. We shall use this solution in the following three cases :

Case I : Wavy Wall in an Unlimited Air Stream

Consider steady subsonic flow over an infinitely long wavy wall given by (61) in an unlimited fluid. The corresponding boundary conditions for this flow field are

$$\phi(x,y) = \text{finite} \quad \text{at} \quad y = \infty \tag{73}$$

and

$$\phi_y(x,y) = U_\infty \frac{dh}{dx} = U_\infty \tau \lambda \, \cos \lambda x \quad \text{at } y = 0 \tag{74}$$

By virtue of (73) we put

$$\gamma + \frac{\delta}{\lambda\beta} = 0 \tag{75}$$

and the expression (72) for $\phi(x,y)$ reduces to

$$\phi(x,y) = \frac{1}{2} (\gamma - \frac{\delta}{\lambda\beta}) \, e^{-\lambda\beta y} \cos \lambda x \tag{76}$$

Again, satisfying the boundary condition (74), we get $\frac{1}{2}(\gamma - \frac{\delta}{\lambda\beta}) = -U_\infty\tau/\beta$ and (76) becomes

$$\phi(x,y) = -\frac{U_\infty\tau}{\beta} e^{-\lambda\beta y}\cos\lambda x \qquad (77)$$

which is the exact solution of the mathematical model described by the partial differential equation (62) subject to the boundary conditions (73) and (74).

Case II : Wavy Wall in an Open Throat Wind Tunnel

If we consider steady subsonic flow past a wave shaped wall in an open throat wind tunnel, then the boundary conditions in this case are

$$\phi_x(x,y) = 0 \text{ at } y = H \qquad (78)$$

and

$$\phi_y(x,y) = U_\infty\frac{dh}{dx} = U_\infty\tau\lambda \cos\lambda x \text{ at } y = 0 \qquad (79)$$

where H is the distance between the wavy wall and the wall of the wind tunnel.

By virtue of the boundary condition (78), we have from (72)

$$(\gamma + \frac{\delta}{\lambda\beta})\, e^{\lambda\beta H} + (\gamma - \frac{\delta}{\lambda\beta})\, e^{-\lambda\beta H} = 0 \qquad (80)$$

Again, satisfying the condition (79) by (72), we get

$$\delta = U_\infty\tau\lambda \qquad (81)$$

Solving (80) and (81), we have

$$\gamma = -\frac{U_\infty\tau}{\beta}. \frac{e^{\lambda\beta H} - e^{-\lambda\beta H}}{e^{\lambda\beta H} + e^{-\lambda\beta H}} \qquad (82)$$

Using (81) and (82) in ([72] we obtain the velocity potential as

$$\phi(x,y) = \frac{U_\infty\tau}{\beta}. \frac{\sinh[\lambda\beta(y-H)]}{\cosh[\lambda\beta H]} \qquad (83)$$

which represents the closed form solution of the problem.

Case III : Wavy Wall in a Closed Throat Wind Tunnel

For steady subsonic flow over the wavy wall in a closed throat wind tunnel, we write the boundary conditions as

$$\phi_y(x,y) = 0 \text{ at } y = H \qquad (84)$$

and

$$\phi_y(x,y) = U_\infty\frac{dh}{dx} = U_\infty\tau\lambda \cos\lambda x \text{ at } y = 0 \qquad (85)$$

Satisfying the conditions (84) and (85) by (72) we have

$$\left.\begin{array}{r} (\gamma + \frac{\delta}{\lambda\beta})\, e^{\lambda\beta H} - (\gamma - \frac{\delta}{\lambda\beta})\, e^{-\lambda\beta H} = 0 \\ \delta = U_\infty\tau\lambda \end{array}\right\} \qquad (86)$$

which, on solving for γ, give

$$\gamma = -\frac{U_\infty\tau}{\beta}. \frac{e^{\lambda\beta H} + e^{-\lambda\beta H}}{e^{\lambda\beta H} - e^{-\lambda\beta H}} \qquad (87)$$

Putting the values of δ and γ from (86) and (87) in (72) we get $\phi(x,y)$ as

$$\phi(x,y) = \frac{U_\infty\tau}{\beta} \frac{\cosh[\lambda\beta(y-H)]}{\sinh[\lambda\beta H]} \qquad (88)$$

which is an exact solution.

The example 2 can also be treated by means of regular decomposition method and this is left to the readers.

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