A Weak Approximated Solution For A Subclass Of Wiener-Hopf Integral Equation

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Abstract—**Consider the problem of solving, approximately, a Wiener-Hopf integral equation $\int_0^\infty g(\theta)k(x-\theta)d\theta = f(x),\ x \geq 0$ in g, with certain conditions on k and f (see below). We use the well known and powerful Riemann-Hilbert problem to develop two techniques to solve, approximately, the Wiener-Hopf integral equation. One of the approximations is based upon the Shannon sampling theorem, which provides a sharp approximation, when applicable. Estimation bounds and application in statistics are given.**

Keywords: ***Padé approximant, continued fraction, Shannon sampling theorem, Fourier transform, Hilbert transform.***

1 Introduction

Consider solving a Wiener-Hopf integral equation with form

$$\int_0^\infty g(\theta)k(x-\theta)d\theta \quad = \quad f(x), \qquad (1)$$

where g is to be determined and f and k are two given functions that both go to zero faster than some power (i.e., $f(\omega) = o(|\omega|^{-\alpha})$, $g(\omega) = o(|\omega|^{-\beta})$, for some positive α, β, as $|\omega| \to \infty$), real parts of both functions are bounded by positive value, and their the Fourier transforms satisfy the Hölder condition on R. The Wiener-Hopf integral equation is an integral equation, which in 75 years of its history has been impressed all who use it to almost all branches of engineering, mathematical physics, and applied mathematics. The technique to solve a Wiener-Hopf integral equation, named the Riemann-Hilbert technique, still, remains an extremely important tool for modern scientists, and the areas of application continue to broaden, a good review may be found in [10], among others. The Riemann-Hilbert technique is theoretically well developed. However, to overcome problem caused by slow evaluation and existence singularises near the integral contour some approximate methods have to be considered, which do not receive enough attention from authors. Until 2000, there does not exist any broadly applicable method to produce an approximate solution for a Wiener-Hopf integral equation. Abraham in 2000, [9], considered solving Wiener-Hopf equation $\int_0^\infty g(\theta)k(x-\theta)d\theta = f(x)$, $x \geq 0$ in g. He suggested to replace the Fourier transform of given kernel k by a Padé approximant which uniformly approximates the Fourier transform of original kernel in an infinite strip about R. The approximation conditions suggested by Abraham are quite strong and in many cases cannot be achieved, see [11][1] generalized Abrahams' results by introducing **(i)** a weaker approximate technique, $L^p(R)$, to solve a Wiener-Hopf equation; and **(ii)** a new methods, based upon the Shannon sampling theorem, to solve a Wiener-Hopf equation. This paper deals with theory and numerical treatment to solve, approximately, a Wiener-Hopf integral equation, with even (or odd) kernel, k. We begin by improving results of [11], in $L^2(R)$. Namely, **(i)** using Parseval's identity (see Lemma 2) shaper results regarding to Theorem (4) of [11] are given; **(ii)** estimate bounds for Theorem 5, which in [11] misspecified, are given, **(iii)** and provides a practical method to solve a class of Wiener-Hopf equations, which motivated by a statistical problem.

This paper develops as following. Section 2 collects and establishes some useful elements of the Riemann-Hilbert problem and properties a function in $L^2(R)$ space. Section 3 presents two techniques to solve, approximately a class of Wiener-Hopf integral equation with even (or odd) kernel. An application in statistics is given.

2 Preliminaries

Now, we collect and establish some useful elements about the Riemann-Hilbert problem and some properties of function in $L^2(R)$ space.

Definition 1 Suppose q is a complex-valued smooth function defined on a smooth oriented curve Γ. If $q(\Gamma)$ is closed and compact, then the index exists and is defined to be the winding number of $q(\Gamma)$ about the origin.

Computing the index of a function is usually a *key step* in determining the existence and number of solutions of

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[1]More precisely, [11] considered the Riemann-Hilbert problem which has close relation with the Wiener-Hopf integral equation.

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a Riemann-Hilbert problem. We are primarily interested in the case of index zero, since a positive and continuous function r which goes to zero faster than some power has zero index [7].

The Sokhotskyi-Plemelj integral of a function s is defined by a principal value integral, as follows.

$$\phi_s(\lambda) := \frac{1}{2\pi i} \fint_R \frac{s(\omega)}{\omega - \lambda} d\omega, \quad \text{for } \lambda \in C. \tag{2}$$

In the boundary-value literature, it is well known that the radial limit $\phi_s^{\pm}(\omega) = \lim_{\lambda \to \omega + i0^{\pm}} \phi_s(\lambda)$ has the property (*jump formula*) that $\phi_s^{\pm}(\omega) = \pm s(\omega)/2 + \phi_s(\omega)$, where $\omega \in R$. Moreover, the radial limits $\phi_s^{\pm}$ may be understood as $\phi_s^{\pm}(\omega) = \pm s(\omega)/2 + H(s,\omega)/(2i)$, where $H(s)$ is the Hilbert transform of s and $\omega \in R$ [7].

The Riemann-Hilbert problem is the function-theoretical problem of finding a single function which is analytic separately in upper and lower half-plane (called sectionally analytic), bounded, and having a prescribed jump discontinuity on the real line. (One can replace the real line by other curves, but the case of the real line is the one studied here; more general versions of the Riemann-Hilbert problem can be found in [7].

Definition 2 The Riemann-Hilbert problem with index v is to find a sectionally analytic and bounded function Φ such that the upper and lower radial limits $\Phi^{\pm}$ satisfy

$$\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega), \quad \text{for } w \in R, \tag{3}$$

where *kernel* and *nonhomogeneous* parts r and s respectively, are given, bounded, and continuous functions such that

i) s and r satisfy a Hölder condition on R, and both go to zero faster than some power,

ii) r does not vanish on R and has index v.

In case of r vanishes at α, with order 1, [7] suggests to set up a new Riemann-Hilbert problem $\Phi^+(\omega)/(\omega-\alpha) = r(\omega)/(\omega-\alpha)\Phi^-(\omega) - s(\omega)/(\omega-\alpha)$, for $w \in R$, which meets all desire condition of a Riemann-Hilbert problem. For convenient in presentation hereafter, we call above suggestion as Gakhov's suggestion in [7]. Moreover, we denote the upper and lower half-plane respectively with D^+ and D^-.

To solve the Riemann-Hilbert problem (3), it is usual to define auxiliary functions

$$\begin{aligned} X^+(\omega) &= \lim_{\lambda \to \omega + i0^+} \exp\{\phi_f(\lambda)\}, \\ X^-(\omega) &= \lim_{\lambda \to \omega + i0^-} \lambda^{-v} \exp\{\phi_f(\lambda)\}, \\ \psi^{\pm}(\omega) &= \lim_{\lambda \to \omega + i0^{\pm}} \phi_g(\lambda), \end{aligned}$$

where $f(\omega) = \ln(\omega^{-v} r(\omega))$ and $g(\omega) = -s(\omega)/X^+(\omega)$, for $\omega \in R$ and $\lambda \in C$.

A Riemann-Hilbert problem always has a family of solutions. But, unique solutions can be obtained with further restrictions. Solutions vanishing at infinity is the most practical restriction considered in mathematical physics and in engineering applications. With this restriction, the solution may be found with the following Sokhotskyi-Plemelj method, proof may be found in [7].

Lemma 1 Suppose $\Phi^{\pm}(\omega)$ vanish at infinity, the Riemann-Hilbert problem (3) with index v has

i) v linear independent solutions, whenever $v > 0$,

ii) a unique solution if $v = 0$,

iii) a unique solution, whenever $v < 0$, and $\fint_R \frac{s(\omega)\omega^{n-1}}{X^+(\omega)} d\omega = 0$, for $n = 1, 2, ..., -v$; and no solution otherwise.

If a Riemann-Hilbert problem has solutions, they can be found by

$$\Phi^{\pm}(\omega) = X^{\pm}(\omega)(\psi^{\pm}(\omega) + P_v(\omega)/\omega^v), \quad \omega \in R,$$

where $P_v(\omega)$ is a polynomial of degree v, with arbitrary coefficients, and $P_0(\omega) = 0$.

Hereafter, we study the Riemann-Hilbert problem with index zero.[2]

Due to numerical problems caused by singularities near the integral contour and slow evaluation, the implementation of the above lemma is very difficult. There is an alternative method to solve a Riemann-Hilbert problem named Carlemann's method, which amounts to solution by inspection. Carlemann's method in two steps provides solutions of a Riemann-Hilbert problem. In the first step, the kernel r must be decomposed into a product of two sectionally analytic and bounded functions r^+ and r^- in D^+ and D^- respectively. In the second step, ratio s/r^+ decomposes into a summation of two sectionally analytic and bounded function s^+ and s^- in D^+ and D^- respectively. Then, the solutions of above Riemann-Hilbert problem are obtained.

The key step to solve a Riemann-Hilbert problem is determining terms r^+, r^-, s^+, and s^-, satisfying the properties given earlier (Carleman's method). In most cases, these functions cannot find explicitly and the original functions have to be approximated by simpler functions.

[2] Conditions on the kernels of the Riemann-Hilbert problem that are considered in this paper (positivity, continuity, and goes to zero faster than some power) force the associated Riemann-Hilbert problem to have zero index, see Ghakhov (1990) p. 86.

Abrahams (2000) considered the homogenous Riemann-Hilbert problem $\Phi^-(\omega) = r(\omega)\Phi^+(\omega)$ with index zero. He assumed that one could replace the given kernel r with a simple kernel, say r_0, which in a strip about the real line, uniformly approximates r. Then, he showed that solutions of the approximate Riemann-Hilbert problem $\Phi_0^-(\omega) = r_0(\omega)\Phi_0^+(\omega)$ uniformly approximate solutions of the original Riemann-Hilbert problem. Since such uniform approximations cannot always be found (see [11] Abrahams' result is sometimes inapplicable. Moreover, Abrahams' result does not apply to the nonhomogeneous Riemann-Hilbert problem.

$$\begin{aligned}\Phi^+(\omega) &= -r^+(\omega)s^+(\omega),\\ \Phi^-(\omega) &= s^-(\omega)/r^-(\omega).\end{aligned}$$

The most favorable situation for the Carlemann's method is probably the case where r and s are rational functions. It would be worth to recall that [3] established the following class of functions as rational Fourier transform.

$$\mathcal{P} = \{p(x) : p(x) = \sum_{k=1}^{N} a_k[\sin(b_k x) + \cos(c_k x)]e^{-d_k x}\}, \quad (4)$$

where N and M are some positive and integer numbers, c_k and g_k are two positive real value numbers, and a_k, b_k, d_k, and e_k are in R.

In most cases exact solutions cannot be found and solutions have to approximate. Probably, [2] was the first author who attempted to solve the homogenous Riemann-Hilbert problem $\Phi^+(\omega) = r(\omega)\Phi^-(\omega)$, approximately. He suggested two methods, both are based on an approximation of the kernel of a homogenous Riemann-Hilbert problem by a rational function. In the first method the kernel r is approximated by the function $r_n(\omega) = \sum_{k=-n}^{n} a_k\omega^k$. In the second method, by taking logarithms, the homogenous problem is reduced to the jump problem (i.e., $\ln \Phi^+(\omega) - \ln \Phi^-(\omega) = \ln r(\omega)$); $\ln r(\omega)$ is then approximated by the same rational function $r_n(\omega)$. [1,4] proposed replacing a complicated kernel, r, in a homogenous Riemann-Hilbert problem with a simple one, say, r_0, that uniformly approximates r in a strip about the real line. Until 2000 there did not exist any broadly applicable method to produce an approximating kernel r_0. Abrahams in 2000 introduced a new scheme for generating a rational approximation for a homogenous Riemann-Hilbert problem with zero index (named the Wiener-Hopf problem). He suggested to replace the given kernel r by a Padé approximant of r that uniformly approximates the original kernel in an infinite strip. The conditions on the approximate kernel that are suggested by Abrahams are quite strong and in many cases cannot be achieved, (see [11]). Moreover, Abrahams' result does not apply to the nonhomogeneous Riemann-Hilbert problem. Recently, [11] provide two techniques to solve a subclass of nonhomogeneous Riemann-Hilbert problem.

An useful result is the Parseval's identity, proof may be found in [8].

Lemma 2 (Parseval's identity) Suppose $f \in L^1(R) \cap L^2(R)$. Then $||\hat{f}||_2 = ||f||_2$.

From the Parseval's identity, one can observe that, if $\{s_n\}$ is a sequence of functions converging in $L^2(R)$, to s, then the Fourier transforms of s_n converge in $L^2(R)$ to the Fourier transform of s. Another immediate conclusion of the Parseval's identity can be the following.

Lemma 3 Suppose $f \in L^1(R) \cap L^2(R)$. Then $||H(f)||_2 = ||f||_2$, where H stands for the Hilbert transform.

Proof. This readily can be observed from the fact that the Hilbert transform of f can be rewritten as $H(f, \lambda) = i\hat{f}(sgn(\cdot)\hat{f}^{-1}(\cdot), \lambda)$. The Parseval's identity completes the desired proof.

The following recalls some further useful elements, from [11], for the next section. Their result using Lemma (3) can be improved for L^2-norm as the following.

Theorem 1 Suppose r and r_n are positive and continuous functions which go to zero faster than some power and have zero index; s and s_n satisfy the Hölder condition and go to zero faster than some power. Then solutions of Riemann-Hilbert problems $\Phi_n^+(\omega) = r_n(\omega)\Phi_n^-(\omega) - s_n(\omega)$ and $\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega)$, satisfy

$$\begin{aligned}||\Phi_n^+ - \Phi^+||_2 &\leq \frac{B}{2b^2}\left[1 + \frac{A\sqrt{b}}{\sqrt{B}}\right]||r_n - r||_2\\ &\quad + \frac{\sqrt{B}}{\sqrt{b}}||s_n - s||_2,\\ ||\Phi_n^- - \Phi^-||_2 &\leq \frac{3A}{2b^2}||r_n - r||_2 + \frac{1}{\sqrt{Bb}}||s_n - s||_2,\end{aligned}$$

b, B, and A are positive values such that $b := \inf\{r, r_n\}$, $B := \sup\{r, r_n\}$, and $A := \sup\{|s|, |s_n|\}$.

Remark 1 It is worth recalling that the above Theorem apply only to Riemann-Hilbert problems with zero index and a real-valued kernel. But one can extend it to handle Riemann-Hilbert problems with a scalar multiple of real-valued kernel (i.e., given kernel r in Riemann-Hilbert problem (2) satisfies $r(\omega) = (a + ib)r^*(\omega)$, where r^* is a real-valued function and a and b are real-valued constants). This can be readily observed by dividing both sides of corresponding Riemann-Hilbert problem by imaginary constant $a + ib$ to obtain a Riemann-Hilbert problem with real-valued kernel.

To use the pervious Theorem, we suggest to replace r and s with two sequence of rational functions, say respectively

r_n and s_n, which produced by the Padé approximant or a continued fraction expansion and satisfy condition of Theorem. Also, form the above theorem, one can observe that solutions of Riemann-Hilbert problems $\Phi_n^+(\omega) = r_n(\omega)\Phi_n^-(\omega) - s_n(\omega)$ converge in $L^2(R)$ to solutions of Riemann-Hilbert problem $\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega)$, whenever $r_n - r$ and $s_n - s$ both converge to zero in $L^2(R)$ sense.

As pointed out above, due to numerical problems caused by singularities near the integral contour and slow evaluation, a Riemann-Hilbert problem usually cannot solve explicitly and has to be approximated. To approximate solutions of a Riemann-Hilbert problem, we suggest to replace two complicated given functions r and s in (2) by two rational functions r_0 and s_0 which generate using the Padé approximant or continued fraction techniques and that approximate original functions in $L^2(R)$, see [6] for more detail. Result of Theorem (1) warrants that solutions of a Riemann-Hilbert problem $\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega)$ approximated, in $L^2(R)$ sense, by solutions of a Riemann-Hilbert problem $\Phi_0^+(\omega) = r_0(\omega)\Phi_0^-(\omega) - s_0(\omega)$.

Also, [11] used the Shnnon sampling theorem and provided exact solutions of a Riemann-Hilbert problem, see Theorem (6) of [11]. Their condition on given functions, r and s, are strong and difficult to achieve. To overcome this difficulty, one has to use approximate technique. The following theorem provides an approximate version of [11].

Theorem 2 Suppose r and r_n are positive and continuous functions which go to zero faster than some power, have zero index, and $r_n \equiv r_n^+ r_n^-$, where r_n^+ and r_n^-, respectively, are analytic and bounded in D^+ and D^-. Moreover, suppose that s satisfy the Hölder condition, go to zero faster than some power (i.e., $s(\omega) = o(|\omega|^\alpha)$, whenever $\alpha < -1$ as $|\omega| \to \infty$), and the Fourier transform of s/r_n^+ vanishes outside of the interval $[-T/2, T/2]$. Then, solutions of Riemann-Hilbert problem $\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega)$

i) can be approximated, in $L^2(R)$, by $\Phi_n^+(\omega) = -r_n^+(\omega)t_N^+(\omega)$ and $\Phi_n^-(\omega) = -t_N^-(\omega)/r_n^-(\omega)$;

ii) approximated solutions Φ_n^+ and Φ_n^- satisfy

$$\begin{aligned} ||\Phi_n^+ - \Phi^+||_2 &\leq \frac{B}{2b^2}\left[1 + \frac{A\sqrt{b}}{\sqrt{B}}\right]||r_n - r||_2 \\ &+ \frac{2\sqrt{B}}{T^\alpha\sqrt{b}}\sum_{j=N+1}^{\infty} j^\alpha, \\ ||\Phi_n^- - \Phi^-||_2 &\leq \frac{3A}{2b^2}||r_n - r||_2 + \frac{2}{T^\alpha\sqrt{Bb}}\sum_{j=N+1}^{\infty} j^\alpha; \end{aligned}$$

where $t_N^-(\omega) = \sum_{j=-N}^{+N} t_n(\frac{j}{T})\frac{\exp\{-i\pi(T\omega - j)\} - 1}{2i\pi(T\omega - j)}$, $t_N^+(\omega) = \sum_{j=-N}^{+N} t_n(\frac{j}{T})\frac{\exp\{i\pi(T\omega - j)\} - 1}{2i\pi(T\omega - j)}$, and $t_n = s/r_n$ and N, b, B, and A are positive values which for all $\varepsilon > 0$, $|t_n - (t_N^+ - t^- N)| < \varepsilon$, $b := \inf\{r, r_n\}$, $B := \sup\{r, r_n\}$, and $A := \sup\{|s|\}$.

Proof. Solutions of Riemann-Hilbert problem $\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega)$ can be approximated by solutions of Riemann-Hilbert problems $\Phi_n^+(\omega) = r_n(\omega)\Phi_n^-(\omega) - r_n^+ t_N(\omega)$, where $t_N = t_N^+ - t_N^-$. Results of [11] finishes proof of part (i). To establish part (ii), suppose firstly one approximates solutions of the desired Riemann-Hilbert problem by solution of the Riemann-Hilbert problems $\Phi_n^{+*}(\omega) = r_n(\omega)\Phi_n^{-*}(\omega) - s(\omega)$ and secondly by solutions of Riemann-Hilbert problems $\Phi_n^+(\omega) = r_n(\omega)\Phi_n^-(\omega) - r_n^+ t_N(\omega)$. Now, using the Hölder inequality, one can estimate error bounds of two approximation steps as the following

$$\begin{aligned} ||\Phi^+ - \Phi_n^+||_2 &\leq ||\Phi^+ - \Phi_n^{+*}||_2 + ||\Phi_n^{+*} - \Phi_n^+||_2 \\ &\leq \frac{B}{2b^2}\left[1 + \frac{A\sqrt{b}}{\sqrt{B}}\right]||r_n - r||_2 \\ &+ \frac{|r_n^+|\sqrt{B}}{\sqrt{b}}||\frac{s}{r_n^+} - t_N||_2 \\ &\leq \frac{B}{2b^2}\left[1 + \frac{A\sqrt{b}}{\sqrt{B}}\right]||r_n - r||_2 \\ &+ \frac{|r_n^+|\sqrt{B}}{\sqrt{b}}\sum_{|j|>N}||t_n(\frac{j}{T})\frac{\sin(\pi(T\omega - j))}{\pi(T\omega - j)}||_2 \\ &\leq \frac{B}{2b^2}\left[1 + \frac{A\sqrt{b}}{\sqrt{B}}\right]||r_n - r||_2 \\ &+ \frac{\sqrt{B}}{\sqrt{b}}\sum_{|j|>N}|s(\frac{j}{T})| \\ &\leq \frac{B}{2b^2}\left[1 + \frac{A\sqrt{b}}{\sqrt{B}}\right]||r_n - r||_2 \\ &+ \frac{2\sqrt{B}}{T^\alpha\sqrt{b}}\sum_{j=N+1}^{\infty} j^\alpha. \end{aligned}$$

The second, third, last inequalities, respectively, follow from Theorem (1), the Shannon sampling theorem, and fact that s goes to zero faster than some power, i.e., $s(\omega) = o(|\omega|^\alpha)$, whenever $\alpha < 1$ as $|\omega| \to \infty$. Proof of the second part is quite similar.

3 Application to the Wiener-Hopf integral equation

This section introduces two new schemes to approximate solution(s) of a class of Wiener-Hopf integral equation with odd or even kernel, in $L^2(R)$ sense.

To state a Wiener-Hopf integral question as a Riemann-Hilbert problem, we need the following lemma from [5].

Lemma 4 Suppose f and k are two given functions and g is to determine from the Wiener-Hopf equation $\int_0^\infty g(\theta)k(x-\theta)d\theta = f(x),\ x \geq 0$. If the Fourier transform of f and k satisfy the Höler condition and go to zero faster than some power. Moreover, the Fourier transform of k does not vanish on R. Then, the unknown function g is the inverse Fourier of Φ^- in the Riemann-Hilbert problem $\Phi^-(\omega)\hat{k}(\omega) = \hat{f}(\omega) + \Phi^+(\omega)$, where Φ^+ is the Fourier transform of the unknown function of h determined by $\int_0^\infty g(\theta)k(x-\theta)d\theta = h(x),\ x \leq 0$.

The following theorem provides an $L^2(R)$ approximated solution for the Wiener-Hopf integral equation with an odd or even kernel.

Theorem 3 Suppose $k,\ k_n,\ f$, and f_n are given and piecewise continuous functions in $L^1(R) \cap L^2(R)$. And, g and g_n are unknown functions where are to be determined from Wiener-Hopf equations $\int_0^\infty g(\theta)k(x-\theta)d\theta = f(x),\ x \geq 0$ and $\int_0^\infty g_n(\theta)k_n(x-\theta)d\theta = f_n(x),\ x \geq 0$. Then

a) solution of g and g_n satisfy, whenever k and k_n, are odd functions, $\hat{k}(\omega)/\omega$ and $\hat{k}_n(\omega)/\omega$ have zero indexes

$$\begin{aligned} ||g - g_n||_2 \ &\leq \ \frac{3A_1}{2b_1^2}||\frac{\hat{k}(\omega) - \hat{k}_n(\omega)}{\omega}||_2 \\ &+ \frac{1}{\sqrt{B_1 b_1}}||\frac{\hat{f}(\omega) - \hat{f}_n(\omega)}{\omega}||_2, \end{aligned}$$

b) solution of g and g_n satisfy, whenever k and k_n, are even functions and $\hat{k}(\omega)$ and $\hat{k}_n(\omega)$ have zero indexes

$$||g - g_n||_2 \leq \frac{3A_2}{2b_2^2}||k - k_n||_2 + \frac{1}{\sqrt{B_2 b_2}}||f - f_n||_2,$$

where $B_1 := \sup\{\hat{k}(\omega)/\omega, \hat{k}_n(\omega)/\omega\}$, $b_1 := \inf\{\hat{k}(\omega)/\omega, \hat{k}_n(\omega)/\omega\}$, $A_1 := \sup\{|\hat{f}(\omega)/\omega|, |\hat{f}_n(\omega)/\omega|\}$, $B_2 := \sup\{\hat{k}, \hat{k}_n\}$, $b_2 := \inf\{\hat{k}, \hat{k}_n\}$, and $A_2 := \sup\{|\hat{f}|, |\hat{f}_n|\}$,

Proof. (a) Using Lemma (4) and the fact that $e^{ix\omega} - e^{-ix\omega} = 2i\sin(x\omega)$, one can observe that the kernel corresponding Riemann-Hilbert problems vanish at origin. Using the Gakhov's suggestion along result of Theorem (1), Remark (1), and Lemma (2) the desire result obtain. (b) Using Lemma (4) and the fact that $e^{ix\omega} + e^{-ix\omega} = 2i\cos(x\omega)$, along result of Theorem (1) and Lemma (2) the desire result obtained

Remark 2 From the Parseval's identity and properties of the Fourier transform. It is easy to observe that, $||\hat{k}_1(\omega)/\omega - \hat{k}_2(\omega)/\omega||_2$ corresponds to an L^2-norm of an antiderivative for $k_1 - k_2$.

The significance of the above theorem lies in the fact that it gives a continuity result for the solution process for Wiener-Hopf equations of certain class. In general, one dues not expect such result for integral equations as can for example be seen from the simplest of all integral equation $\int_0^x s(\theta)d\theta = f(x)$, where f is given, which a small change (in $L^2(R)$) of f leads to large changes in s.

Theorem 4 Suppose $k,\ k_n$, and f are given and piecewise continuous functions in $L^1(R) \cap L^2(R)$. And, g and g_n are unknown functions where are to be determined from Wiener-Hopf equations $\int_0^\infty g(\theta)k(x-\theta)d\theta = f(x),\ x \geq 0$ and $\int_0^\infty g_n(\theta)k_n(x-\theta)d\theta = f(x),\ x \geq 0$. Moreover, suppose that the Fourier transform $\hat{k}_n$ can be written as $\hat{k}_n \equiv r_n^+ r_n^-$, where r_n^+ and r_n^-, respectively, are analytic and bounded in D^+ and D^-, $\hat{f}$ goes to zero faster than some power (i.e., $\hat{f}(\omega) = o(|\omega|^\alpha)$, whenever $\alpha < -1$ as $|\omega| \to \infty$), and the Fourier transform of $\hat{f}/r_n^+$ vanishes outside of the interval $[-T/2, T/2]$. Then

a) solution of g and g_n satisfy, whenever k and k_n, are odd functions, $\hat{k}(\omega)/\omega$ and $\hat{k}_n(\omega)/\omega$ have zero indexes

$$\begin{aligned} ||g - g_n||_2 \ &\leq \ \frac{3A_1}{2b_1^2}||\frac{\hat{k}(\omega) - \hat{k}_n(\omega)}{\omega}||_2 \\ &+ \frac{2T^{1-\alpha}}{\sqrt{B_1 b_1}} \sum_{j=N+1}^{\infty} j^{\alpha-1}, \end{aligned}$$

b) solution of g and g_n satisfy, whenever k and k_n, are even functions and $\hat{k}(\omega)$ and $\hat{k}_n(\omega)$ have zero indexes

$$||g - g_n||_2 \leq \frac{3A_2}{2b_2^2}||\frac{\hat{k}(\omega) - \hat{k}_n(\omega)}{\omega}||_2 + \frac{2T^{-\alpha}}{\sqrt{B_2 b_2}} \sum_{j=N+1}^{\infty} j^{\alpha},$$

where $B_1 := \sup\{\hat{k}(\omega)/\omega, \hat{k}_n(\omega)/\omega\}$, $b_1 := \inf\{\hat{k}(\omega)/\omega, \hat{k}_n(\omega)/\omega\}$, $A_1 := \sup\{|\hat{f}(\omega)/\omega|\}$, $B_2 := \sup\{\hat{k}, \hat{k}_n\}$, $b_2 := \inf\{\hat{k}, \hat{k}_n\}$, and $A_2 := \sup\{|\hat{f}|\}$,

[12] considered a statistical problem which can be restated as the following Wiener-Hopf integral equation

$$\int_0^\infty g(\theta)k(x-\theta)d\theta \ = \ f_0(x), \text{ for } x \geq 0, \qquad (5)$$

where $f_0(x)$ is a given log-concave and even density function defined on $x \geq 0$, $k(x) = -\text{sgn}(x)f_0(x)$, and g is a real-valued and positive function to be determined. Kucerovsky et al. (2009) established that the above Wiener-Hopf equations have real-valued, positive, and unique solution.
As pointed out before, in many case the Wiener-Hopf equation cannot be solved explicitly, where a approximated technique has to be employed. Using the above

fact along the Carlemann's method (see above), to solve integral equation (5), we suggest: **(i)** approximate f_0 in the original Wiener-Hopf integral equation by a function in class of $\mathcal{P}$, given by (4), which has rational Fourier transform; **(ii)** solve its corresponding Riemann-Hilbert problem using the Carlemann's method; **(iii)** take inverse Fourier transform of Φ^-. Since, integral equation (5) has unique solution which has to be positive, continuous, and real-valued function, we believe that its solution should be in class $\mathcal{P}$, given by (4).
Now, we solve an example.

Example 1 (Normal density function) Suppose f_0 is integral equation (5) be the Normal density function $f_0(x) = \exp\{-x^2/2\}/\sqrt{2\pi}$. Using the above steps, approximated solution of integral equation (5) is

$$\begin{aligned} g(\theta) &= 0.9971 + \sum_{i=1}^{15} a_i \cos(b_i\theta)\exp\{-c_i\theta\} \\ &\quad + \sum_{i=1}^{15} d_i \sin(e_i\theta)\exp\{-f_i\theta\}, \end{aligned}$$

where a=(0.0004, 0.0389, -2.9991, -18.3763, -0.0027, -1.0988, 7.7404, 0.0382, -1.1011, 0.0001, -0.0001, 1.8629, -13.0964, 7.6762, 18.7692), b=(1.9950, 2.2450, 0.4110, 0.4895, 0.6200, 2.2453, 1.2451, 2.2450, 2.2447, 1.9949, 0.6196, 0.2970, 0.6038, 1.2374, 0.0067), c=(1.6455, 1.7615, 3.1395, 2.9146, 1.7313, 2.6837, 2.9117, 1.7615, 2.6839, 1.6456, 1.7310, 3.2354, 2.8876, 2.9247, 2.6393), d=(0.0006, 14.5099, 3.0866, 0.0030, 0.7965, -5.8484, 0.3457, 0.8019, -0.0005, 5.1804, 16.7046, -6.5426, -0.0033, -3.5832, 0.3461), e=(1.9950, 0.4110, 0.4895, 0.6200, 2.2453, 1.2451, 2.2451, 2.2447, 1.9950, 0.6038, 0.2970, 1.2374, 0.6196, 0.0067, 2.2448), and f=(1.6455, 3.1395, 2.9146, 1.7313, 2.6838, 2.9117, 1.7615, 2.6840, 1.6457, 2.8876, 3.2354, 2.9247, 1.7310, 2.6393, 1.7615). One could observe that $\lim_{x\to+\infty} g_0(x) = 0.9971$. Moreover, the error of approximated solution may be measured by replacing g_0 in $|\int_0^\infty g(\theta)k(x-\theta)d\theta - f_0(x)|$, i.e., $\text{Error}(x) := |\int_0^\infty g(\theta)k(x-\theta)d\theta - f_0(x)|$, where $\sup_x\{\text{Error(x)}\} = 0.007$.

Example 2 (Hyperbolic secant density function) Suppose f_0 is integral equation (5) be the Hyperbolic secant density function $f_0(x) = 1/(\pi\cosh(x))$. Using the above steps, approximated solution of integral equation (5) is

$$\begin{aligned} g(\theta) &= 0.8574 + \sum_{i=1}^{15} a_i \cos(b_i\theta)\exp\{-c_i\theta\} \\ &\quad + \sum_{i=1}^{15} d_i \sin(e_i\theta)\exp\{-f_i\theta\}, \end{aligned}$$

Above example can readily extent as following. Consider the integral equation

$$\int_0^\infty s(\theta)k(\theta - x)d\theta = f_0(x), \ x \geq 0, \qquad (6)$$

where $f_0(x) = e^{|x|}/(1+e^{|x|})^2$, $k(x) = -\text{sgn}(x)f_0(x)$, and g is to de determined. As Example 1 pointed out, the Fourier transform $\hat{k}$ cannot uniformly be approximated by a rational function. One can replace given f_0 by the Gaussian function $f_G = \exp\{-\pi x^2/16\}/4$, which f_G approximates f_0 in L^2, i.e., $\int_R |f_G(x) - f_0(x)|^2 dx \leq 0.000417$. Now, one can employ result of previous example to approximate solution of (6), say g_G, in $L^2(R)$, i.e., $\int_0^\infty |\int_0^\infty g_G(\theta)k(x-\theta)d\theta - f_0(x)|^2 dx \leq 0.001062$.

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