Electromagnetic Energy-Momentum Tensor for Non-Homogeneous Media in the Theory of Relativity

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Abstract — **The tensor calculus, using certain suitable transformations, permits to establish the expression of the energy-momentum tensor, also called energy quantity of motion tensor, for domains submitted to an electromagnetic field in various cases interesting in the Theory of Relativity, and which have not been examined in the known works. In literature, in the works devoted to the Theory of Relativity, this problem has been especially treated for the vacuum medium. Here, the author presents a new approach to the analysis of the case of linear but non-homogeneous electrically and magnetically polarized media. The problem of passing from a system of reference to another one in motion, and the selection of the volume density force formulae which are in accordance with the Theory of Relativity are also examined.**

Index Terms — **Energy-momentum tensor, Tensor calculus, Theory of relativity.**

I. Introduction

In Electrodynamics and in the Theory of Relativity, the energy-momentum tensor has a very important role [1-9]. Besides the widely accepted fact that this tensor allows a compact way of writing the conservation laws of linear momentum and energy in Electromagnetism, it permits to calculate the energy and stress, in any reference frame in terms of another reference frame, and especially in terms of the reference frame in which the substance is at rest.

The developments of the principles of the concerned mathematical methods, started from a relatively long time, are still examined nowadays [10-14].

Many works have been devoted to this subject. However, in the most treated case of empty space as well as in the case of a space filled with substance, the transition from a reference frame to another in motion has not been carefully analysed. In this paper, a new approach to the analysis of the tensor will be presented namely, the construction of the tensor, the case of non-homogeneous electrically and magnetically polarized substances, and the transition from a reference frame to another one, with the involved consideration on the Theory of Relativity.

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II. Volume Density of the Electromagnetic Force

An analysis of the electromagnetic forces in the frame of classical theories can be found in certain works among which [15]. In the works concerning the Theory of Relativity the analysis of electromagnetic forces is achieved from the Lorentz formula of the force, e.g., [5, p. 133]. In the present paper, we shall start from the general formula of the electromagnetic force acting on a substance submitted to an electromagnetic field. It is derived from the principle of conservation of energy and the Theory of Relativity, through certain approximations, [8, p. 157]. The reasoning has led to the following formulae, both also deduced in various other manners and accepted by several authors:

$$\boldsymbol{f} = \rho_v \boldsymbol{E} - \frac{1}{2}\boldsymbol{E}^2 \operatorname{grad}\varepsilon - \frac{1}{2}\boldsymbol{H}^2 \operatorname{grad}\mu + \boldsymbol{J}\times\boldsymbol{B}, \qquad (1\text{ a})$$

and

$$\boldsymbol{f} = \rho_v \boldsymbol{E} - \frac{1}{2}\boldsymbol{E}^2 \operatorname{grad}\varepsilon - \frac{1}{2}\boldsymbol{H}^2 \operatorname{grad}\mu + \boldsymbol{J}\times\boldsymbol{B} + \frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}), \qquad (1\text{ b})$$

where the symbols are the usual ones. In this case, the quantities ε and μ are considered as constant, but strongly depending on the point of the substance, hence varying in space. We shall denote the three axes of a Cartesian system of co-ordinates, by the indices i, j, k. In the further analysis, we shall consider formula (1 a), and we shall mention the modification occurring due to the supplementary term, if using formula (1 b). Relations (1 a) and (1 b) are considered as having, along each axis, three and four terms (components), respectively:

$$f_k = (f_k)_1 + (f_k)_2 + (f_k)_3 + (f_k)_4, \qquad (2)$$

where the index k indicates the axis. The four terms are given by the following expressions:

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$$(f_k)_1 = \rho_v E_k\,;\quad (f_k)_2 = [\boldsymbol{J}\times\boldsymbol{B}]_k\,;$$
$$(f_k)_3 = \left[-\frac{1}{2}\boldsymbol{E}^2\,\mathrm{grad}\,\varepsilon - \frac{1}{2}\boldsymbol{H}^2\,\mathrm{grad}\,\mu\right]_k\,; \qquad (3\text{ a, …, d})$$
$$(f_k)_4 = \left[\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})\right]_k .$$

Henceforth, we shall write the expressions of the electromagnetic field state quantities by using the scalar and vector potentials V and $\boldsymbol{A}$, in the well-known form:

$$E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t};\quad B_k = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j};\quad \forall i, j, k \in [1, 3]. \qquad (4\text{ a, b})$$

The relations (4 a, b) may be written in a new form, using the relations (A.9 a-d) and (A.10 a, b) from Appendices, as follows:

$$F_{ij} = c_j \frac{\partial A_j}{\partial x^i} - c_i \frac{\partial A_i}{\partial x^j},$$
$$E_i = F_{i0} = \frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0},\quad \forall i, j \in [0, 3]; \qquad (5\text{ a-e})$$
$$B_k = B_{ij} = \frac{1}{c}F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j},$$
$$A_0 = -V,\quad x_0 = ct,\quad \forall i, j, k \in [1, 3].$$

Further, we shall have in view the two groups (sets) of equations of the electromagnetic field (in the order used by H. A. Lorentz, which differs from that of J. C. Maxwell) in a four-dimensional continuum space-time, where the symbols are those of [10, 11]. For the sake of facility, we shall recall these symbols, firstly in the case of empty space (vacuum). The equations of the first group (set) are given by the relationships:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i,\quad \forall i, j \in [0, 3],\quad i \neq j;\quad J^0 = c\rho_v, \qquad (6)$$

and

$$E_i = F_{i0};\quad F_{ij} = e_{ii}e_{jj}F^{ij};\quad F_{ij} = -F_{ji};$$
$$G^{0j} = c\varepsilon_0 F^{0j}; \qquad (6\text{ a-f})$$
$$G_{i0} = c\varepsilon_0 F_{i0};\quad \forall i, j \in [0, 3];\quad c^2 = \frac{1}{\varepsilon_0\mu_0};$$

also

$$B_{ij} = \frac{1}{c}F_{ij};\quad G^{ij} = \frac{1}{\mu_0}\cdot\frac{1}{c}F^{ij};$$
$$D^i = \frac{1}{c}G^{0i} = \varepsilon_0\delta^{ii}E_i;\quad D_k = D^k; \qquad (6\text{ g-j})$$
$$\forall i, j, k \in [1, 3];$$

$$H^{ij} = G^{ij};\quad H_k = H^{ij};$$
$$B_k = B_{ij}; \qquad (6\text{ k-m})$$
$$\forall i, j, k \in [1, 3];$$

where the subscript index k in the relations (6 j) and (6 l) refers to the usual three-dimensional vectors, whereas indices i and j refer, as previously, to tensor components. All situations in which the index k has this role will be mentioned. It is to be noted that the components of the form F_{ii} and G^{ii} vanish.

Introducing the axis coefficients of the Galilean reference frame, e_{ii}, [11], we can write:

$$A_i = e_{ii}A^i = e_{is}A^s;\quad e_{00} = 1;\quad e_{ii} = -1,\quad \forall i \in [1, 3];$$
$$e_{ij} = 0,\quad \forall i \neq j; \qquad (6\text{ n-r})$$
$$A_i A^i = (A_0)^2 - (A_1)^2 - (A_2)^2 - (A_3)^2 .$$

In the case of isotropic, linear, non-homogeneous media using relation (A.19 a) from Appendices, we may write:

$$G^{0j} = c\,\varepsilon_0(1+\chi_{\mathrm{e}})E_j = \varepsilon_0\varepsilon_r F_{j0} = \varepsilon F^{0j} . \qquad (6\text{ s})$$

Similarly, using relation (A.21 b) from Appendices, we may write:

$$G^{ij} = \frac{1}{\mu_0(1+\chi_{\mathrm{m}})}\cdot\frac{1}{c}F^{ij} = \frac{1}{\mu_0\mu_r}\cdot\frac{1}{c}F^{ij} . \qquad (6\text{ t})$$

The equations of the second group (set) are given by the relationship:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0,\quad \forall i, j, k \in [0, 3],\quad i \neq j \neq k. \qquad (7)$$

In order to emphasize the tensors F_{ij} and G_{ij}, equations (3) can be written in the form below, convenient for passing from three to four dimensions. Relations (6), have been considered. For instance, relation (8 c) has been written taking into account relations (6 a) and (6 k, l), respectively. Hence:

$$(f_k)_1 = \frac{1}{c}\cdot\frac{\partial G^{0j}}{\partial x^j}F_{k0};\quad (f_k)_2 = \frac{1}{c}\cdot\frac{\partial G^{uj}}{\partial x^j}F_{ku};$$
$$(f_k)_3 = -\frac{1}{2}F_{u0}F^{0u}\frac{\partial\varepsilon}{\partial x^k} - \frac{1}{2}G_{uv}G^{uv}\frac{\partial\mu}{\partial x^k}; \qquad (8\text{ a-d})$$
$$(f_k)_4 = \frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{0u}F_{ku}\right);$$
$$\forall j, u, v \in [1, 3];\quad u < v;\quad k \in [1, 3],$$

and also, we have:

$$G^{0j} = c\,\varepsilon_0 \varepsilon_r\, F^{0j}\,; \quad G^{uv} = \frac{1}{\mu_0 \mu_r} \cdot \frac{1}{c} F^{uv}\,;$$
$$c^2 = \frac{1}{\varepsilon_0 \mu_0}; \quad \varepsilon = \varepsilon_0 \varepsilon_r\,, \quad \mu = \mu_0 \mu_r\,; \tag{8 e-i}$$
$$\forall\, j \in [1, 3]; \quad u, v \in [1, 3], \quad u < v.$$

Summing up, side by side, relations (8 a) and (8 b), we shall get the following more compact expression:

$$(f_k)_{12} = \frac{1}{c} \cdot \frac{\partial G^{0j}}{\partial x^j} F_{k0} + \frac{1}{c} \cdot \frac{\partial G^{uj}}{\partial x^j} F_{ku}\,, \quad \forall\, j, k \in [0, 3], \quad u \in [1, 3], \tag{9}$$

relation which, as mentioned, is extended for the four-dimensional continuum since indices j and k may take four values. Then, summing up, there follows:

$$(f_k)_{12} = \frac{1}{c} \cdot \frac{\partial G^{ij}}{\partial x^j} F_{ki}\,, \quad \forall i,\, j, k \in [0, 3]. \tag{10}$$

III. Expression of the Force Components and of the Energy-Momentum Tensor

We shall now consider the case of a linear isotropic electric and magnetic polarization of the considered medium, with the relative permittivity ε_r and the relative permeability μ_r, point functions. In order to express the force component as the derivative of an expression, we shall write relation (10) in the form:

$$c(f_k)_{12} = \frac{\partial G^{ij}}{\partial x^j} F_{ki} = \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) - G^{ij} \frac{\partial F_{ki}}{\partial x^j}, \quad \forall i,\, j, k \in [0, 3]. \tag{11}$$

Now, we shall modify the second term of the right-hand side as follows:

$$G^{ij} \frac{\partial F_{ki}}{\partial x^j} = G^{ij} \frac{\partial F_{ki}}{\partial x^j}, \tag{12 a}$$

$$G^{ij} \frac{\partial F_{ki}}{\partial x^j} = G^{ji} \frac{\partial F_{kj}}{\partial x^i} = -G^{ij} \frac{\partial F_{kj}}{\partial x^i} = G^{ij} \frac{\partial F_{jk}}{\partial x^i}. \tag{12 b}$$

Summing up the left-hand and the last right-hand sides of the two expressions (12 a, b), and taking into account (7), we get:

$$2G^{ij} \frac{\partial F_{ki}}{\partial x^j} = G^{ij} \left(\frac{\partial F_{ki}}{\partial x^j} + \frac{\partial F_{jk}}{\partial x^i} \right) = -G^{ij} \frac{\partial F_{ij}}{\partial x^k}. \tag{13}$$

Replacing (13) into (11), we shall obtain:

$$c(f_k)_{12} = \frac{\partial G^{ij}}{\partial x^j} F_{ki} = \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) + \frac{1}{2} G^{ij} \frac{\partial F_{ij}}{\partial x^k}. \tag{14}$$

By expanding the last term of (14), there follows:

$$c(f_k)_{12} = \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) + \frac{1}{2} G^{0v} \frac{\partial F_{0v}}{\partial x^k} + \frac{1}{2} G^{u0} \frac{\partial F_{u0}}{\partial x^k} + \frac{1}{2} G^{uv} \frac{\partial}{\partial x^k}\left(\mu c G_{uv}\right), \forall i,\, j, k \in [0, 3]; u, v \in [1, 3]. \tag{15}$$

Initially we shall assume index k different from zero.

Replacing the symbols of (8 e, f) into (15), after having divided both sides with c, we shall get:

$$(f_k)_{12} = \frac{1}{c} \cdot \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) + \frac{1}{2} \varepsilon_0 \varepsilon_r F^{0j} \frac{\partial F_{0v}}{\partial x^k} + \frac{1}{2} \varepsilon_0 \varepsilon_r F^{u0} \frac{\partial F_{u0}}{\partial x^k} + \frac{1}{2} G^{uv} \frac{\partial}{\partial x^k}\left(\mu G_{uv}\right), \quad \forall i,\, j, k \in [0, 3];\; u, v \in [1, 3]. \tag{16}$$

We remark that:

$$\frac{1}{2} G^{rs} \frac{\partial}{\partial x^k}\left(\mu G_{rs}\right) = G^{pq} \frac{\partial}{\partial x^k}\left(\mu G_{pq}\right), \quad \forall\, r, s \in [1, 3]; \quad \forall\, p, q \in [1, 3], \; p < q. \tag{16 a}$$

We are now going to calculate the components of f_k which, according to the types of the included electromagnetic field state quantities, can be of the following types: electric, magnetic, mixed.

In order to facilitate the understanding of the formulae, we shall successively use the tensor notation and the vector notation. We shall use for indices numbers, instead of letters, because it is easier to perform the computation and to avoid the use of the summation convention when not allowed. Then, the indices may be subscripts. We adopt $k = 3$. We shall not write the terms of the form F_{uu} and G^{uu}, being zero.

We shall express the electric component considering expression (16). We take into account the relation:

$$-\varepsilon F \frac{\partial F}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{1}{2} \varepsilon F^2\right) + \frac{1}{2} F^2 \frac{\partial \varepsilon}{\partial x}. \tag{17 a}$$

The electric component will be obtained from the expanded relation (16), having in view (17 a), and that indices u and v take the same values:

$$\begin{aligned}
(f_k)_{\text{elec}} &= \frac{1}{c} \cdot \frac{\partial}{\partial x^v}\left(G^{0v} F_{k0}\right) - \frac{\partial}{\partial x^k}\left(\frac{1}{2} \varepsilon F^{0u} F_{u0}\right) + \frac{1}{2} F^{0u} F_{u0} \frac{\partial \varepsilon}{\partial x^k} \\
&= \frac{1}{c} \cdot \frac{\partial}{\partial x^1}\left(G^{01} F_{30}\right) + \frac{1}{c} \cdot \frac{\partial}{\partial x^2}\left(G^{02} F_{30}\right) \\
&\quad + \frac{1}{c} \cdot \frac{\partial}{\partial x^3}\left(G^{03} F_{30}\right) - \frac{1}{2} \cdot \frac{1}{c} \cdot \frac{\partial}{\partial x^k}\left(G^{0u} F_{u0}\right) \\
&\quad + \frac{1}{2} F^{0u} F_{u0} \frac{\partial \varepsilon}{\partial x^k}, \quad \forall u, v \in [1, 3],
\end{aligned} \tag{17 b}$$

and therefore:

$$\left(f_k\right)_{\text{elec}} = = \frac{\partial}{\partial x_1}\left(\varepsilon E_1 E_3\right) + \frac{\partial}{\partial x_2}\left(\varepsilon E_2 E_3\right) + \frac{\partial}{\partial x_3}\left(\varepsilon E_3 E_3\right) - \frac{1}{2}\frac{\partial}{\partial x_3}\left(\varepsilon E^2\right) + \frac{1}{2}E^2\frac{\partial \varepsilon}{\partial x_3}. \tag{17c}$$

In tensor form, we have:

$$\left(f_k\right)_{\text{elec}} = \frac{1}{c}\cdot\frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right) - \frac{1}{2}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial x^k}\left(G^{0v}F_{v0}\right) + \frac{1}{2}\cdot\frac{1}{c}\left(G^{0v}F_{v0}\right)\frac{\partial \varepsilon}{\partial x_k}, \quad \forall v \in [1,3]. \tag{17 d}$$

Then, we shall express the magnetic component considering also expression (16). We need to calculate expressions of the form:

$$G^{uv}\frac{\partial}{\partial x}\left(\mu G_{uv}\right) = \frac{1}{\mu}B\frac{\partial B}{\partial x}; \quad G^{uv} = G = \frac{1}{\mu}B. \tag{18 a}$$

By differentiating, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}B^2\right) = \frac{1}{\mu}\cdot\frac{\partial}{\partial x}\left(B^2\right) - \frac{1}{\mu^2}B^2\frac{\partial \mu}{\partial x}; \quad \frac{\partial}{\partial x}\left(\mu G^2\right) = 2G\frac{\partial}{\partial x}\left(\mu G\right) - G^2\frac{\partial \mu}{\partial x}. \tag{18 b}$$

Therefore, replacing in the considered term of (16) the expression (18 b) and having in view relations (16 a) we shall obtain for $k = 3$:

$$\begin{aligned}\left(f_k\right)_{\text{mag}} &= \frac{\partial}{\partial x^v}\left(\mu G^{uv}G_{ku}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(\mu G^{uv}G_{uv}\right) + \frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial \mu}{\partial x^k} \\ &= \frac{\partial}{\partial x^1}\left(\mu G^{u1}G_{3u}\right) + \frac{\partial}{\partial x^2}\left(\mu G^{u2}G_{3u}\right) + \frac{\partial}{\partial x^3}\left(\mu G^{u3}G_{3u}\right) \\ &\quad + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(\mu G^{uv}G_{uv}\right) + \frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial \mu}{\partial x^k} \\ &= \frac{\partial}{\partial x_1}\left(\mu H_3 H_1\right) + \frac{\partial}{\partial x_2}\left(\mu H_3 H_2\right) - \frac{\partial}{\partial x_3}\left[\mu\left(H_2 H_2 + H_1 H_1\right)\right] \\ &\quad + \frac{1}{2}\cdot\frac{\partial}{\partial x_3}\left(\mu H^2\right) + \frac{1}{2}\left(H^2\right)\frac{\partial \mu}{\partial x_3},\end{aligned} \tag{18 c}$$

$\forall u, v \in [1,3]$, and $u < v$ in the products of the form $G^{uv}G_{uv}$.

Rearranging the terms, there follows:

$$\begin{aligned}\left(f_k\right)_{\text{mag}} &= \frac{\partial}{\partial x_1}\left(\mu H_3 H_1\right) + \frac{\partial}{\partial x_2}\left(\mu H_3 H_2\right) - \frac{\partial}{\partial x_3}\left[\mu\left(H_2^2 + H_1^2\right)\right] + \frac{1}{2}\cdot\frac{\partial}{\partial x_3}\left(\mu H^2\right) + \frac{1}{2}H^2\frac{\partial \mu}{\partial x^3} \\ &= \frac{\partial}{\partial x_1}\left(\mu H_3 H_1\right) + \frac{\partial}{\partial x_2}\left(\mu H_3 H_2\right) + \frac{\partial}{\partial x_3}\left(\mu H_3 H_3\right) - \frac{\partial}{\partial x_3}\left[\mu\left(H_1^2 + H_2^2 + H_3^2\right)\right] \\ &\quad + \frac{1}{2}\cdot\frac{\partial}{\partial x_3}\left(\mu H^2\right) + \frac{1}{2}H^2\frac{\partial \mu}{\partial x_3},\end{aligned} \tag{18 d}$$

and

$$\left(f_k\right)_{\text{mag}} = \frac{\partial}{\partial x_1}\left(\mu H_3 H_1\right) + \frac{\partial}{\partial x_2}\left(\mu H_3 H_2\right) + \frac{\partial}{\partial x_3}\left(\mu H_3 H_3\right) - \frac{1}{2}\cdot\frac{\partial}{\partial x_3}\left(\mu H^2\right) + \frac{1}{2}H^2\frac{\partial \mu}{\partial x_3} \tag{18 e}$$

or in tensor form:

$$\left(f_k\right)_{\text{mag}} = \frac{\partial}{\partial x^v}\left(\mu G^{uv}G_{ku}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(\mu G^{uv}G_{uv}\right) + \frac{1}{2}\left(G^{uv}G_{ku}\right)\frac{\partial \mu}{\partial x^k}, \tag{18 f}$$

$$\forall u, v \in [1,3], \quad u < v.$$

Returning to previous letter indices, and summing side by side relations (17 b), (18 d) or (18 e), and (8 c), we shall get the sum of electric and magnetic terms:

$$\begin{aligned}\left(f_k\right)_{\text{elmag}} &= \frac{\partial}{\partial x_v}\left(\varepsilon E_k E_v\right) - \frac{1}{2}\cdot\frac{\partial}{\partial x_k}\left(\varepsilon E^2\right) + \frac{1}{2}E^2\frac{\partial \varepsilon}{\partial x_k} + \frac{\partial}{\partial x_v}\left(\mu H_k H_v\right) \\ &\quad - \frac{1}{2}\frac{\partial}{\partial x_k}\left(\mu H^2\right) + \frac{1}{2}H^2\frac{\partial \mu}{\partial x_k} - \frac{1}{2}E^2\frac{\partial \varepsilon}{\partial x_k} - \frac{1}{2}H^2\frac{\partial \mu}{\partial x_k}.\end{aligned} \tag{19}$$

We shall now express the mixed components considering the first term of expression (16) and the expression (8 d). The mixed components are given by:

$$\begin{aligned}\left(f_k\right)_{\text{mix1}} &= \frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{u0}F_{ku}\right) \\ &= \frac{1}{c}\left[\frac{\partial}{\partial x^0}\left(G^{10}F_{31}\right) + \frac{\partial}{\partial x^0}\left(G^{20}F_{32}\right)\right] \\ &= \frac{1}{c}\left[\frac{\partial}{\partial x_0}\left(-cD^1 cB_2\right) + \frac{\partial}{\partial x_0}\left(cD^2 cB_1\right)\right],\end{aligned} \tag{20 a}$$

and

$$\begin{aligned}(f_k)_{\text{mix1}} &= -\frac{\partial}{\partial t}(D_1 B_2 - D_2 B_1) \\ &= -\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})_3, \quad \forall u \in [1,3],\end{aligned} \tag{20 b}$$

and similarly:

$$(f_k)_{\text{mix2}} = \frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{0u}F_{ku}\right) = \frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})_3, \quad \forall u \in [1,3]. \tag{20 c}$$

Returning to previous letter indices, we get:

$$(f_k)_{\text{mix2}} = \frac{\partial}{\partial t}\left(D^i B_j - D^j B_i\right). \tag{21}$$

Adding up relations (19) and (20 b), side by side, we obtain:

$$\begin{aligned}f_k &= \frac{\partial}{\partial x_v}(\varepsilon E_k E_v) - \frac{1}{2}\cdot\frac{\partial}{\partial x_k}\left(\varepsilon E^2\right) \\ &+ \frac{\partial}{\partial x_v}(\mu H_k H_v) - \frac{1}{2}\frac{\partial}{\partial x_k}\left(\mu H^2\right) \\ &- c\frac{\partial}{\partial x_0}\left(D_i B_j - D_j B_i\right), \quad \forall i,j,k,v \in [1,3].\end{aligned} \tag{22}$$

By summing up, side by side relations (17 b), (18 f), (20 a), we can write:

$$\begin{aligned}(f_k)_{\text{elmag}} &= \frac{1}{c}\left[\frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right) - \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(G^{0v}F_{v0}\right)\right] \\ &+ \frac{1}{2}\cdot\frac{1}{c}\left(G^{0v}F_{v0}\right)\frac{\partial\varepsilon}{\partial x^k} + \frac{1}{c}\cdot\frac{\partial}{\partial x^v}\left(G^{uv}F_{ku}\right) \\ &+ \frac{1}{2}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right) + \frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial\mu}{\partial x^k} \\ &+ \frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{u0}F_{ku}\right), \quad \forall u,v \in [1,3],\end{aligned} \tag{23 a}$$

and after summing up with relation (8 c), we get:

$$\begin{aligned}(f_k)_{\text{sum}} &= \frac{1}{c}\left[\frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(G^{u0}F_{u0}\right)\right] \\ &+ \frac{1}{c}\left[\frac{\partial}{\partial x^v}\left(G^{uv}F_{ku}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right)\right] \\ &+ \frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{v0}F_{kv}\right), \quad \forall u,v \in [1,3],\ u<v.\end{aligned} \tag{23 b}$$

If we started from formula (1 b) we should have also added in the right-hand side of relation (23 b) the expression (20 c), and then, the final relation would differ.

The force expression may be written in a compact and general form as follows:

$$\begin{aligned}f_k &= \frac{1}{c}\cdot\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki}\right) + \frac{1}{2}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right), \\ &\forall i,j,k \in [0,3], \quad \forall u,v \in [0,3],\ u<v\end{aligned} \tag{24 a}$$

or in a more compact form, as follows:

$$\begin{aligned}f_k &= \frac{1}{c}\cdot\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki} + \frac{1}{2}\delta_k^j G^{uv}F_{uv}\right), \\ &\forall i,j,k \in [0,3];\ \forall u,v \in [0,3],\ u<v.\end{aligned} \tag{24 b}$$

Finally, the component of the volume density of the force along the *k*-axis can be expressed as:

$$f_k = \frac{\partial}{\partial x^j}W_k^j, \tag{25}$$

where the expression:

$$\begin{aligned}W_k^j &= \frac{1}{c}\left(G^{ij}F_{ki}\right) + \delta_k^j\frac{1}{2}\cdot\frac{1}{c}\left(G^{uv}F_{uv}\right), \\ &\forall i,j,k \in [0,3];\ \forall u,v \in [0,3],\ u<v,\end{aligned} \tag{26}$$

represents *the energy-momentum tensor*, also called *tensor of energy and quantity of motion*. It is possible to express the last relation in other forms, taking into account the following relation:

$$\begin{aligned}&W_s^j e^{sk} = W^{jk}, \quad e_{sk} = 0, \quad \forall s \neq k; \\ &e_{00} = 1; \quad e_{jj} = -1, \quad \forall j \in [1,3].\end{aligned} \tag{26 a}$$

Also, we get:

$$W_{jk} = e_{jj}e_{kk}W^{jk}. \tag{26 b}$$

Remarks

1° If the media were not assumed isotropic and had not linear electric and magnetic polarization, the previous transformations of relations (17 a) and (18 b), respectively, would be no longer possible.

2° Having established the expression of the tensor in one reference frame, we can obtain its expression in any other one. The calculation is to be performed by using the group of co-ordinate transformations, for instance the Lorentz transformations. We consider useful to make the following remark. The Lorentz transformation group has been established for empty space (vacuum), and the involved light velocity is that *in vacuo*. In the present case, we consider that polarization exists, and in this case, also all transformations of the quantities are like those established by Minkowski. But a doubt appears, namely if the transformations are still valid because in any media the velocity of light is different. *For this reason, the Lorentz transformation group may be considered as an assumption that is so better the smaller will be the space regions filled with substance.*

It is to be noted that we have established a new form of the tensors used for defining the field state quantities, which facilitate the analysis.

IV. Expression of the Components of the Energy-Momentum Tensor

1° Component W_0^0. Using formula (26), and after performing the calculation, passing from tensor notation to vector notation, we have got:

$$\begin{aligned} & W_0^0 = \frac{1}{c}\left(G^{i0} F_{0i} + \frac{1}{2}\delta_0^0 G^{0v} F_{0v} \right) + \frac{1}{2}\delta_0^0 \frac{1}{c} G^{uv} F_{uv} \\ & \quad \forall i, u, v \in [1, 3],\ u < v; \\ & \frac{1}{c} G^{i0} F_{0i} = D_i E_i, \\ & \frac{1}{2}\cdot\frac{1}{c}\delta_0^0 G^{0v} F_{0v} = -\frac{1}{2} E_v D_v, \\ & \frac{1}{2}\cdot\frac{1}{c}\delta_0^0 G^{uv} F_{uv} = \frac{1}{2} H_i B_i, \end{aligned} \tag{27 a}$$

and summing up the calculated terms, we obtain:

$$\begin{aligned} W_0^0 &= D_i E_i - \frac{1}{2} E_i D_i + \frac{1}{2} H_i B_i, \\ &= \frac{1}{2}\left(\varepsilon E^2 + \mu H^2 \right); \quad \forall i \in [1, 3], \end{aligned} \tag{27 b}$$

which represents the volume density of the electromagnetic energy, and E_i, D_i are considered as three-dimensional vector components.

2° Component W_k^j for both cases $k \neq j$ and $k = j$. We use, as above, formula (26), and after performing the calculation, we shall pass from tensor notation to vector notation.

In the first case, remarking that *j* and *k* are different, we should keep only the first term of expression (26). We get:

$$\begin{aligned} & W_k^j = \frac{1}{c} G^{0j} F_{k0} + G^{ij} F_{ki} \\ & \quad = \varepsilon_0 \varepsilon_r F_{j0} F_{k0} + G^{ij} F_{ki}; \\ & \quad \varepsilon_0 \varepsilon_r F_{j0} F_{k0} = E_j D_k = E_k D_j; \\ & \quad G^{ij} F_{ki} = H_k B_j = H_j B_k; \\ & W_k^j = -W^{jk} = E_j D_k + H_j B_k; \\ & \quad \forall i, j, k \in [1, 3]. \end{aligned} \tag{28 a}$$

In the second case, for more clarity, instead of letter indices, we shall use number indices, considering a certain case, namely for $j = k = 2$. There follows:

$$\begin{aligned} W_2^2 &= \frac{1}{c}\left(G^{02} F_{20} + G^{12} F_{21} + G^{32} F_{23} \right) \\ &+ \frac{1}{2}\cdot\frac{1}{c}\left(G^{0v} F_{0v} + G^{uv} F_{uv} \right); \quad \forall u, v \in [1, 3]; \end{aligned} \tag{28 b}$$

$$\begin{aligned} & W_2^2 = \\ & \quad \frac{1}{c}\left(G^{02} F_{20} + G^{12} F_{21} + G^{32} F_{23} \right) = D_2 E_2 \\ & \quad + H_3 (-B_3) + (-H_1) B_1; \\ & \quad \frac{1}{2}\cdot\frac{1}{c}\left(G^{0v} F_{0v} \right) = -\frac{1}{2}\left(D_1 E_1 + D_2 E_2 + D_3 E_3 \right); \\ & \quad \frac{1}{2}\cdot\frac{1}{c}\left(G^{uv} F_{uv} \right) = \frac{1}{2}\left(H_1 B_1 + H_2 B_2 + H_3 B_3 \right); \\ & \quad \forall u, v \in [1, 3]; \\ & W_2^2 = -W^{22} \\ & \quad = E_2 D_2 - H_3 B_3 - H_1 B_1 + H_2 B_2 - H_2 B_2 \\ & \quad - \frac{1}{2}(E_1 D_1 + E_2 D_2 + E_3 D_3) \\ & \quad + \frac{1}{2}(H_1 B_1 + H_2 B_2 + H_3 B_3); \end{aligned} \tag{28 c}$$

and, grouping the terms, we obtain:

$$W_2^2 = -W^{22} = E_2 D_2 + H_2 B_2 - \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B} \right), \tag{28d}$$

and the general form, as expected, is:

$$\begin{aligned} & W_j^j = -W^{jj} = E_j D_j + H_j B_j - \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B} \right), \\ & \quad \forall j \in [1, 3]. \end{aligned} \tag{28e}$$

The results above, expressed by relations (28 a) and (28 e), represent the Maxwell stress tensors.

3° Component W_0^j. As previously, we shall use formula (26), and after performing the calculation, we pass from tensor notation to vector notation. We begin with one example for $j = 2$, $k = 0$, and then express the general form. There follows:

$$\begin{aligned} & W_0^2 = \frac{1}{c} G^{i2} F_{0i} = \frac{1}{c} G^{12} F_{01} + \frac{1}{c} G^{32} F_{03} \\ & \quad = \frac{1}{c} H^{12}(-E_1) + \frac{1}{c}\left(-H^{23} \right)(-E_3); \\ & W_0^2 = \frac{1}{c}(E_3 H_1 - E_1 H_3); \quad W_0^j = W^{j0} \\ & \quad = \frac{1}{c}(E_k H_i - E_i H_k), \quad \forall i, k \in [1, 3]; \end{aligned} \tag{29}$$

and the general form is:

$$W_0^j = W^{j0} = \frac{1}{c}\left(E_k H_i - E_i H_k\right) = \frac{1}{c}\left(\boldsymbol{E}\times\boldsymbol{H}\right)_j , \tag{30}$$

which apart from the denominator c, represents the j-component of Poynting vector, i.e., the rate of the radiated flux of energy per unit of surface and unit of time.

4° The force along the time axis. We shall use formula (24 a) or (24 b), putting $k=0$, and after performing the calculations, we shall pass from tensor notation to vector notation. There follows:

$$\begin{aligned} & f_0 = \frac{1}{c}\cdot\frac{\partial}{\partial x^j}\left(G^{ij}F_{0i}\right) + \frac{1}{2}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{uv}F_{uv}\right), \\ & \quad \forall i,\, j\in[0,3], \quad \forall u,v\in[0,3], \quad u<v, \end{aligned} \tag{31}$$

and:

$$\begin{aligned} c\, f_0 &= \frac{\partial}{\partial x^0}\left(G^{i0}F_{0i}\right) + \frac{\partial}{\partial x^1}\left(G^{i1}F_{0i}\right) \\ & + \frac{\partial}{\partial x^2}\left(G^{i2}F_{02}\right) + \frac{\partial}{\partial x^3}\left(G^{i3}F_{0i}\right) \\ & + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right), \\ & \forall i\in[0,3],\ \forall u,v\in[0,3], u<v, \quad k=0. \end{aligned} \tag{32}$$

Considering the first and second parentheses, we shall get:

$$\begin{aligned} G^{i0}F_{0i} &= c\,\varepsilon_0\varepsilon_r\, F^{i0}F_{0i} \\ &= -c\,\varepsilon_0\varepsilon_r\left(-F_{i0}\,F_{i0}\right) = \boldsymbol{E}\cdot\boldsymbol{D}, \end{aligned} \tag{33 a}$$

and

$$\begin{aligned} G^{i1}\,F_{0i} &= G^{21}\,F_{02} + G^{31}\,F_{03} \\ &= -H^{12}\left(-E_2\right) + H^{31}\left(-E_3\right) \\ &= \frac{1}{c}\left(E_2\,H_3 - E_3\,H_2\right) = \left(\boldsymbol{E}\times\boldsymbol{H}\right)_1 , \end{aligned} \tag{33 b}$$

where, for $i=2$, $G^{i1}=G^{21}=H^{21}$, the vector $\boldsymbol{H}$ being contravariant, and according to relation (6 l), $H^{21}=-H^{12}=-H_3$, since $H_1=H^{23}$. Calculating the first two parentheses, we get:

$$\begin{aligned} & \frac{\partial}{\partial x^0}\left(G^{i1}F_{0i}\right) = c\,\frac{\partial}{\partial x^0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right); \\ & \frac{\partial}{\partial x^1}\left(G^{i1}F_{0i}\right) = \frac{\partial}{\partial x^1}\left(\boldsymbol{E}\times\boldsymbol{H}\right)_1 . \end{aligned} \tag{33 c}$$

Handling similarly the next two parentheses and summing up all terms, there follows:

$$\begin{aligned} f_0 &= \frac{1}{c}\mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \frac{\partial}{\partial x^0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right) \\ & + \frac{1}{2}\cdot\frac{\partial}{\partial x^0}\left(G^{uv}F_{uv}\right), \\ & \forall u,v\in[0,3],\ u<v. \end{aligned} \tag{34}$$

For the last parenthesis we shall obtain:

$$\begin{aligned} G^{uv}F_{uv} &= G^{0r}F_{0r} + G^{rs}F_{rs} = -D_r\,E_r + H_q\,B_q \\ & -\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{H}\cdot\boldsymbol{B}, \\ & \forall u,v\in[0,3],\ u<v, \quad r,s,q\in[1,3],\ r<s. \end{aligned} \tag{35}$$

Replacing (35) into (34), we get:

$$\begin{aligned} f_0 &= \frac{1}{c}\mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \frac{\partial}{\partial x^0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right) \\ & - \frac{1}{2}\cdot\frac{\partial}{\partial x^0}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^0}\left(\boldsymbol{H}\cdot\boldsymbol{B}\right), \end{aligned} \tag{35 a}$$

hence

$$\begin{aligned} f_0 &= \frac{1}{c}\left[\mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial t}\left(\boldsymbol{E}\cdot\boldsymbol{D}\right)\right] \\ & + \frac{1}{c}\cdot\frac{1}{2}\cdot\frac{\partial}{\partial t}\left(\boldsymbol{H}\cdot\boldsymbol{B}\right). \end{aligned} \tag{35 b}$$

The calculation of the derivatives yields the relation:

$$f_0 = \frac{1}{c}\left[\mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\right], \tag{36}$$

where we shall replace the derivatives with respect to time, with the known Maxwell relations, as follows:

$$f_0 = \frac{1}{c}\left[\mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \boldsymbol{E}\cdot\left(\mathrm{curl}\,\boldsymbol{H} - \boldsymbol{J}\right) - \boldsymbol{H}\cdot\mathrm{curl}\,\boldsymbol{E}\right]. \tag{37}$$

Grouping the terms, we obtain:

$$\begin{aligned} f_0 &= \frac{1}{c}\left[\mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) - \mathrm{div}\left(\boldsymbol{E}\times\boldsymbol{H}\right) - \boldsymbol{J}\cdot\boldsymbol{E}\right] \\ &= -\frac{1}{c}\boldsymbol{J}\cdot\boldsymbol{E}, \end{aligned} \tag{38}$$

which represents the component of the force along the time axis. The same result can be also obtained from formula (10) by putting $k=0$.

The set f_k represents a four-vector, according to formulae (25), and can also result from (1 a) and (38), indeed the product of the four force components and the four-vector velocity yields a scalar.

V. Conclusion

The aim of this paper has been to establish the expression of the energy-momentum tensor within the frame of the Theory of Relativity, starting from the general formula of the electromagnetic force acting on a substance submitted to an electromagnetic field. The case of linear non-homogeneous media has been examined.

This subject has not been treated in the known papers or works published so far. Meanwhile, the analysis carried out has shown that no all-general known formulae are in agreement with the tensor energy-momentum expression when passing from a system of reference to another one. If the media were not assumed as isotropic and had not linear electric and magnetic polarization the deduction carried out for obtaining the tensor would not be possible.

The expression of the tensor established in one system of reference can be obtained in any other system of reference owing to the group of Lorentz transformation and the Minkowski transformation formulae using this group. However, a doubt appears because the velocity of light in any media is different, and the Lorentz transformation has been established for this case.

APPENDICES

A. The Equations of the Electromagnetic Field in the Theory of Relativity

In this Appendix, we shall present the equations of the electromagnetic field in the Special and General Theory of Relativity in a form permitting to deduce the energy-momentum tensor. We shall begin with the covariant forms of the equations, so that the passage from Special to the General Theory of Relativity could be performed without difficulties. In the known literature, various methods are used. We shall have in view the works [10, 11], which present certain advantages for the purpose of this Appendix.

From the formulae of the general theory of the electromagnetic field [16, pages 129, 134, 142, 182], in the usual vector form, we have:

$$\begin{aligned}
&\operatorname{curl}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}; \quad \boldsymbol{B} = \operatorname{curl}\boldsymbol{A};\\
&\boldsymbol{E} = -\operatorname{grad}V - \frac{\partial \boldsymbol{A}}{\partial t};\\
&\operatorname{curl}\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}; \quad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P};\\
&\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}.
\end{aligned} \tag{A.1 a, …, f}$$

We rewrite the equations (A.1 a-f) considering the general case of non-homogeneous media. We have:

$$\begin{aligned}
&\operatorname{curl}(\mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}) = \mu_0 \boldsymbol{J} + \varepsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t}\\
&\quad + \mu_0 \frac{\partial \boldsymbol{P}}{\partial t} + \mu_0 \operatorname{curl}\boldsymbol{M};
\end{aligned} \tag{A.2 a}$$

$$\begin{aligned}
&\operatorname{curl}\boldsymbol{B} = \operatorname{curl}\operatorname{curl}\boldsymbol{A} = \mu_0 \boldsymbol{J}\\
&\quad + \varepsilon_0\mu_0 \frac{\partial}{\partial t}\left(-\operatorname{grad}V - \frac{\partial \boldsymbol{A}}{\partial t} \right)\\
&\quad + \mu_0 \left(\frac{\partial \boldsymbol{P}}{\partial t} + \operatorname{curl}\boldsymbol{M} \right);
\end{aligned} \tag{A.2 b}$$

and by expanding the double curl, there follows:

$$\begin{aligned}
&\operatorname{grad}\operatorname{div}\boldsymbol{A} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}\\
&\quad - \varepsilon_0\mu_0 \left(\operatorname{grad}\frac{\partial V}{\partial t} + \frac{\partial^2 \boldsymbol{A}}{\partial t^2} \right) +\\
&\quad + \mu_0 \left(\frac{\partial \boldsymbol{P}}{\partial t} + \operatorname{curl}\boldsymbol{M} \right).
\end{aligned} \tag{A.3}$$

By rearranging the terms, we get:

$$\begin{aligned}
&\nabla^2 \boldsymbol{A} - \varepsilon_0\mu_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2} =\\
&-\mu_0 \left(\boldsymbol{J} + \frac{\partial \boldsymbol{P}}{\partial t} + \operatorname{curl}\boldsymbol{M} \right) + \operatorname{grad}\left(\operatorname{div}\boldsymbol{A} + \varepsilon_0\mu_0 \frac{\partial V}{\partial t} \right).
\end{aligned} \tag{A.4}$$

Since only the curl of vector $\boldsymbol{A}$ is imposed, the divergence can be chosen by using the L. V. Lorenz (do not confuse with H. A. Lorentz) gauge condition:

$$\operatorname{div}\boldsymbol{A} + \varepsilon_0\mu_0 \frac{\partial V}{\partial t} = 0. \tag{A.5}$$

The last relation may be written in the form:

$$\begin{aligned}
&\frac{\partial A_i}{\partial x^i} + \frac{1}{c^2} \cdot \frac{\partial V}{\partial x^0} = 0; \quad x^0 = ct;\\
&c^2 = \frac{1}{\varepsilon_0\mu_0}.
\end{aligned} \tag{A.6 a, b, c}$$

The components along any axis of a Cartesian system of co-ordinates will be:

$$\begin{aligned}
&E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t}; \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k};\\
&\forall i, j, k \in [1, 3].
\end{aligned} \tag{A.7 a, b}$$

The relations (A.1 a, b) may be written using a set of four quantities A_i as follows:

$$\begin{aligned}
&E_i = \frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial t}, \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k},\\
&A_0 = -V, \quad \forall i, j, k \in [1, 3].
\end{aligned} \tag{A.8 a, b, c}$$

A new, more general and convenient form may be the following antisymmetric form with respect to indices i and j:

$$F_{ij} = c_i \frac{\partial A_j}{\partial x^i} - c_j \frac{\partial A_i}{\partial x^j}, \quad A_0 = -V,$$
$$c_k = c, \quad \forall k = 0;$$
$$c_k = 1, \quad \forall k \neq 0; \quad \forall i, j, k \in [0, 3]. \qquad \text{(A.9 a, …, d)}$$

Therefore:

$$E_i = F_{i0} = \frac{\partial A_0}{\partial x^i} - c\frac{\partial A_i}{\partial x^0}, \quad A_0 = -V;$$
$$B_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}, \quad \forall i, j \in [1, 3]. \qquad \text{(A.10 a, b, c)}$$

It follows that the quantities E_i and B_{ij} can be expressed as follows:

$$E_i = F_{i0}; \quad B_{ij} = \frac{1}{c} F_{ij}; \quad \forall i, j \in [1, 3]. \qquad \text{(A.11 a, b)}$$

For expressing the other field state quantities, we shall introduce the following symbols for the case of empty space (vacuum):

$$G^{0j} = c\,\varepsilon_0 F^{0j}; \quad G_{i0} = c\,\varepsilon_0 F_{i0};$$
$$\forall i, j \in [1, 3];$$
$$c^2 = \frac{1}{\varepsilon_0 \mu_0};$$
$$B_{ij} = \frac{1}{c} F_{ij}; \quad G^{ij} = \frac{1}{c} \cdot \frac{1}{\mu_0} F^{ij}; \qquad \text{(A.12 a, …, j)}$$
$$D^i = \frac{1}{c} G^{0i} = \varepsilon_0 \delta^{ii} E_i;$$
$$D_k = D^k; \quad H^{ij} = G^{ij}; \quad H_k = H^{ij};$$
$$B_k = B_{ij}; \quad \forall i, j, k \in [1, 3].$$

B. The Equations for Any Polarized Medium

For a polarized medium, the substance state quantities of electric and magnetic polarization, respectively, have to be introduced by symbols P and M_{J}. In this Sub-section, we shall consider only the temporary polarization. In addition, we shall indicate in this Section, the components of the usual three-dimensional vectors by index k denoting one of the three axes. The respective quantities, can be, in this case, introduced by the following relations:

$$P_k = P^k; \quad M_k = M^{ij}; \quad H_k = H^{ij};$$
$$P_k = \varepsilon_0 \chi_{\mathrm{e},kj} E_j; \quad M_k = \chi_{\mathrm{m},kj} H_j; \qquad \text{(A.13 a, …, f)}$$
$$M_{\mathrm{J},k} = \mu_0 M_k \quad \forall i, j, k \in [1, 3];$$

where in relations from *a* to *c*, in the left-hand side, the usual vector components are written, but in the right-hand side the tensor components are written; while in relations from *d* to *f*, in both sides, only usual vector components are written. In this manner, the components of tensors may be easier expressed. It follows:

$$D^i = \varepsilon_0 \delta^{ij} E_j + \varepsilon_0 \chi_{\mathrm{e}}^{ij} E_j;$$
$$B_k = B_{ij} = \mu_0 H_k + M_{\mathrm{J},k}; \; \forall i, j, k \in [1, 3], \qquad \text{(A.14 g, h)}$$

where E_j may represent the usual vector component as well as the tensor component, whereas B_{ij} and B_k represent the tensor component, and the corresponding vector component, respectively.

This subject has been thoroughly analysed in work [6, p. 156, 268]. However, further on, we have used, to some extent, another way, in order to allow for including, apart from the temporary polarization, also, the permanent polarization.

C. The Maxwell Equations for Empty Space

With the symbols introduced above, we can write the Maxwell equations for empty space (vacuum) as follows. We shall consider two groups of equations. For the first group, we shall use the tensor G^{ij}, and for the second one, the tensor F_{ij}.

The equations of the first group, using the symbols of the *List of symbols* and of (A.13 a-f) are given by the following formula:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i, \quad \forall i, j \in [0, 3],$$
$$i \neq j; \quad J^0 = c\rho_v. \qquad \text{(A.15)}$$

The relations (A.15) yield the local (differential) form of the of the electric flux law (after simplifying *c* of the numerator), and of the magnetic circuital law, for each axis, as $i = 0$ or $i \neq 0$, respectively.

The equations of the second group are:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0,$$
$$\forall i, j, k \in [0, 3], \quad i \neq j \neq k. \qquad \text{(A.16)}$$

The relation (A.16) yields the local (differential) form of the magnetic flux law (after simplifying *c* of the numerator), and of the law of electromagnetic induction, for each axis, as $i, j, k \neq 0$ or $k = 0$, respectively.

We shall write an example for the first group:

$$\frac{\partial G^{1j}}{\partial x^j} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = J^1,$$
$$-\frac{\partial D_1}{\partial t} + \frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} = J^1. \qquad \text{(A.17)}$$

The sign minus before the first term of the last equation occurs because of the inversion of indices according to relation (A.12 f), and before the third term, also because of the inversion of the indices. Therefore, the equation of the magnetic circuital law, for the first axis, has been obtained.

We shall write an example for the second group when $i = 2, \quad j = 3, \quad k = 0$:

$$\frac{\partial F_{23}}{\partial x^0} + \frac{\partial F_{30}}{\partial x^2} + \frac{\partial F_{02}}{\partial x^3} = 0,$$
$$\frac{\partial B_1}{\partial t} + \frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} = 0, \qquad \text{(A.18)}$$

where the last equation has been obtained after having replaced the co-ordinate $x_0 = ct$.

Therefore, the local (differential) form of the local (differential) equation of the law of electromagnetic induction, for the first axis, has been obtained.

D. The Maxwell Equations for Polarized Media

In order to consider the equations for polarized media, it is necessary to introduce the polarization tensors of the substance.

For the first group of equations, we shall complete the equations of (A.15), according to relations (A.13 a, b). In order to bring the equations of the first group into a covariant form, with respect to any change of co-ordinates, we shall introduce, in the equation above, the symbols:

$$G^{0j} := G^{0j} + cP^j = c\varepsilon_0 \delta^{jj} E_j + cP^j, \quad \forall j \in [1, 3]; \qquad \text{(A.19 a)}$$

and

$$G^{i0} := -c\varepsilon_0 \delta^{ii} E_i - cP^i, \quad \forall i \in [1, 3]. \qquad \text{(A.19 b)}$$

In the case of an isotropic medium, we should have in view that the permittivity and permeability become $\chi_{\mathrm{e},kj} = \chi_{\mathrm{e}}$ and $\chi_{\mathrm{m},kj} = \chi_{\mathrm{m}}$. Then, from relations (A.13 a, d) and (A.19 a), it follows:

$$G^{0j} = \varepsilon_0 (1 + \chi_{\mathrm{e}}) E_j, \quad \forall j \in [1, 3]. \qquad \text{(A.20)}$$

Also, we have:

$$G^{ij} := G^{ij} - M^{ij}; \quad G^{ij} = \frac{1}{\mu_0} \cdot \frac{1}{c} F^{ij} - M^{ij}; \quad \forall i, j \in [1, 3] \qquad \text{(A.21 a, b)}$$

and

$$J^0 = c\rho_v; \quad P^0 = 0; \quad M^{0j} = 0, \quad \forall j \in [1, 3]. \qquad \text{(A.21 c)}$$

In the same case, as previously for (A.20), from relations (A.13 b, e) and (A.21 b), there follows:

$$\mu_0 (1 + \chi_{\mathrm{m}}) G^{ij} = \frac{1}{c} F^{ij}, \quad \forall i, j \in [1, 3]. \qquad \text{(A.22)}$$

In all definition formulae above, the terms of the form G^{ij} from the right-hand side are those given by relations (6 a, d) and are written like those of the left-hand side, with raised indices. In the same formulae, the terms of the form G^{ij} from the right-hand side are those given by relations (A.12 a, e). Consequently, equation (A.15) becomes:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i, \quad \forall i, j \in [0, 3], \quad i \neq j; \quad J^0 = c\rho_v; \quad P^0 = 0; \quad M^{0j} = 0. \qquad \text{(A.23 a-d)}$$

The equations of (A.23) correspond to equations of (A.15) above.

For the second group of equations, we shall obtain the same relations as for non-polarized media because the occurring quantities are not influenced by considering the polarization:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \quad \forall i, j \in [0, 3], \quad i \neq j \neq k. \qquad \text{(A.24)}$$

The equations (A.24) correspond to equations (A.16) above.

E. Establishing the Nature of the Four-Potential Vector

In many of the preceding equations, there occurred the four-potential vector A_i the nature of which should be known.

From the formulae of the general theory of the electromagnetic fields [16, pages 129, 134, 182], in the usual vector form, we get the formulae below, where the four-potential A_i also occurs. Firstly we shall examine the three-dimensional vector form:

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}; \quad \boldsymbol{B} = \mathrm{curl}\,\boldsymbol{A}. \qquad \text{(A.25 a, b)}$$

$$\boldsymbol{E} = -\mathrm{grad}V - \frac{\partial \boldsymbol{A}}{\partial t}. \qquad \text{(A.25 c)}$$

By the nature of a geometrical object considered a tensor, in particular a tensor of rank 1, i.e., a vector, is meant its character, hence if it has to be transformed when passing from one system of reference to another as a contravariant or as a covariant one. If the geometrical object, which will be defined as a tensor, is considered separately, with no relation with other tensors, each of the two variants can be chosen. If it is considered in relation with another tensor, the nature of which is known, the situation is different. For example, it is necessary to mention that if the product of two tensors yields a result of a certain nature, say a scalar (tensor of rank 0), the product should give the same result in any other system of reference.

If the character of one tensor is established and we have to choose the character of a geometrical object that will constitute the other tensor, one can use the known procedures: the tensor quotient law [5], or the theorem of tensor classification [12]. We shall extend the last one, using integral operators, in order to facilitate the operation.

In order to fix the ideas, we shall establish the nature of the set of four quantities A_i that may be functions of co-ordinates. Several explanations have been given in literature on this subject, some of them referring to formula (A.4), but without a precise conclusion [14].

We shall make a first verification in order to examine the previous choice. We shall consider relation (A.1 b), and calculate the flux of vector ***B*** through any simply connected surface bounded by a closed curve Γ in a three-dimensional continuum. As mentioned, we shall use an integral form. According to Stokes theorem, we shall replace the calculation of the flux, of the right-hand side of the theorem relationship, by the calculation of the circulation of ***A*** along that curve, which will be given by the integral of the left-hand side of the relation.

We shall assume that the flux is a scalar. Then, the covariant vector component A_i, multiplied with the circulation curve element, namely the contravariant component $\mathrm{d}l^i$, should yield a scalar, namely $A_i\,\mathrm{d}l^i$, hence in accordance with the physical meaning of the considered case. Therefore, the vector of components A_i will be a covariant one.

It is useful to add that in the calculation of the circulation, hence of the magnetic flux, the component A_0 does not occur.

It is to be noted that the preceding explanation, concerning the nature of the set A_i, although mentioned in literature, refers to the three-dimensional continuum and does not satisfy the case of four-dimensional continuum.

For this reason, we consider that this analysis may be carried out as follows We shall refer to the Section (3.10) of [16], where there is examined in Galilean systems of reference the transition from one system of reference K to another system of reference K' in motion with a constant velocity v relatively to the former.

There, the known relations between the components used for expressing the electromagnetic field state quantities of these systems are given. However, using the tensors of the present paper, the same relations may be obtained, but much easier. By a direct way, like in [16], one can obtain the following known relations:

$$A'_x = \alpha\left(A_x - \frac{v}{c^2}V\right); \quad A'_y = A_y; \quad A'_z = A_z; \qquad V' = \alpha\left(V - v A_x\right). \tag{A.26}$$

The known transformation relations between the co-ordinates of the both systems, using symbols like those of the present paper, will be:

$$x'^1 = \alpha\left(x^1 - \frac{v}{c}x^0\right); \quad x'^2 = x^2; \quad x'^3 = x^3; \qquad x'^0 = \alpha\left(x^0 - \frac{v}{c}x^1\right). \tag{A.27}$$

If we proceed to a scaling of the quantities A_i, by replacing A_i by $c\,A_i$, for $i \neq 0$, and replace these quantities in relation (A.26), we shall obtain:

$$A'_1 = \alpha\left(A_1 + \frac{v}{c}A_0\right), \quad A'_2 = A_2; \quad A'_3 = A_3; \qquad A'_0 = \alpha\left(A_0 + \frac{v}{c}A_1\right). \tag{A.28}$$

By comparing the two systems of relations, (A.27) and (A.28), it follows that the initial sets A_0, A_1, A_2, A_3 and x^0, x^1, x^2, x^3 change in opposite manners. Therefore the set x^i being a contravariant four-vector, the set A_i will be a covariant four-vector. Then, the product of the tensor A_i and an element of a space curve will give a scalar, result which remains unchanged in any other system of reference. In respect to Linear Algebra, if the set V, A_x, A_y, A_z is a covariant tensor of rank 1, each set obtained from the previous one, by multiplying each element (component) by any constant factor, will also be a covariant four-vector, hence a tensor of rank 1. Therefore the set $A_i, \forall i \in [0, 3]$, in any form, will be a covariant four-vector.

List of Symbols

A_i	–	component of the four-vector potential;
A	–	electrodynamic vector potential;
B_{ij}	–	twice covariant tensor component of magnetic induction, yielding B_k;
B_k	–	component of the magnetic induction along axis k, considered as a usual three-dimensional vector;
B	–	magnetic induction vector;
c	–	velocity of light in empty space, supposed to be constant;
D_i	–	component of the electric displacement, considered as a usual three-dimensional vector;
D^i	–	contravariant component of the electric displacement yielding D_k or D_i considered as a usual three-dimensional vector;
D	–	electric displacement (electric induction) vector;
E_i	–	covariant component of the electric field strength, as well as component of the electric field strength along axis i, as a usual three-dimensional vector;

E^i – contravariant component of the electric field strength;
E – electric field strength vector;
e_{ii} – axis coefficient, for the axis ***i*** of the Galilean reference frame;
F_{ij} – component of the covariant tensor of rank 2, yielding B_{ij} for i and j non-zero;
F_{i0} – component of the previous covariant tensor, and yielding the component E_i of the electric field strength, considered as a usual three-dimensional vector;
f_k – four-vector component of the volume density of the electromagnetic force;
G^{ij} – contravariant tensor of rank 2, yielding H^{ij};
G_{i0}, G^{i0} – components of the covariant and contravariant tensors, and yielding the component D^i;
H_k – component of the magnetic field strength along axis k, considered as a usual three-dimensional vector;
H – magnetic field strength vector;
J^i – component of a contravariant four-vector, for i non-zero, density of the conduction electric current;
J – conduction electric current density vector;
V – electric potential;
x^i – co-ordinate along axis i;
δ_{ij} – symbol equal to unity for equal indices, and equal to zero for different ones (Kronecker symbol);
ε – electric permittivity, *in vacuo* it is ε_0;
μ – magnetic permeability, *in vacuo* it is μ_0;
ρ_v – volume density of the free electric charge.

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