On the Exit Laws for Semidynamical Systems and Bochner Subordination

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Abstract—**Let $\Phi : [0,\infty[\times E \mapsto E$ be a semidynamical system and $\beta = (\beta_t)_{t>0}$ be a Bochner subordinator. It is proved in this paper that, every β-Liapunov function l for Φ is of the form $l(x) = \int_0^\infty f(t,x)\,dt$ where $f :]0,\infty[\times E \mapsto [0,\infty[$ be a solution of the following functional equation**

$$\int_0^\infty f(t,\Phi(r,x))\,\beta_s(dr) = f(s+t,x), \quad s,t>0,\, x\in E.$$

We deduce an explicit formula for α-Liapunov functions defined by the fractional power subordinator of order $\alpha \in]0,1[$.

Keywords: semidynamical system, Bochner subordinator, exit law.

1 Introduction

Let $\Phi : [0,\infty[\times E \mapsto E$ be a measurable semidynamical system on a measurable space E and let $\mathcal{F}$ be the space of measurable finite functions defined on E. Let $\beta = (\beta_t)_{t>0}$ be a Bochner subordinator, i.e a convolution semigroup of probability measures on $[0,+\infty[$. We may define

$$Q_t u(x) := \int_0^\infty u(\Phi(s,x))\,\beta_t(ds), \quad u\in\mathcal{F},\, t\geq 0,\, x\in E.$$

A β-exit law associated to Φ is a family $f = (f_t)_{t>0}$ of positive measurable function satisfying the functional equation (using the notation $f_t := f(t,.)$)

$$Q_s f_t = f_{s+t}, \qquad s,\, t>0.$$

The integral representation in terms of exit law is originally given by Dynkin [4] and its studied by several authors [6, 7, 8, 9, 10] and [12, 13, 14, 15]. In this paper, we investigate first the representation by β-exit laws. In this case, if the function $\int_0^\infty f_t\,dt$ is finite then it belongs to the cone of **Q**-*Liapunov* functions defined by

$$L^\beta := \{u\in\mathcal{F} : u\geq 0,\ Q_t u\leq u,\ \lim_{t\to 0} Q_t u = u\}$$

Conversely, there are elementary examples for which elements from L^β do not admits an integral representation by a β-exit law (cf. [14], Example 2.7.1). In fact, as it is observed in many papers related to this problem (cf. [6, 7, 8, 9, 10, 12, 13, 14, 15]), some finiteness assumptions are needed, in order to represent elements of L^β in terms of β-exit laws. Along this paper, elements from L^β which is bounded on each trajectory of Φ will be called β-Liapunov functions.
For our context, it is proved in [14] that, for each η^α-Liapunov function l such that $\lim_{t\to\infty} Q_t^\alpha u = 0$, there exists a unique η^α-exit law $f^\alpha = (f_t^\alpha)_{t>0}$ such that

$$l(x) = \int_0^\infty f_t^\alpha(x)\,dt, \qquad x\in E \tag{1}$$

The aim of the present paper is to show that a similar, and in fact more general that (1). In what follows we shall denote by $\mathcal{K}$ the set of all Bochner subordinator β such that $t\to\beta_t$ is continuously differentiable from $]0,\infty[$ to the Banach algebra of complex borel measures on R such that $\|\beta_t'\|_S < \infty$ for each $t>0$. We prove the following integral representation result:
Let β be in $\mathcal{K}$. For each β-Liapunov function l there exists a unique (up to equivalence) β-exit law $f = (f_t)_{t>0}$ for Φ such that

$$l(x) = \int_0^\infty f_t(x)\,dt, \quad x\in E.$$

Moreover, $f = (f_t)_{t>0}$ is explicitly given by

$$f_t(x) = -\int_0^\infty l(\Phi(s,x))\,\frac{\partial}{\partial t}\beta_t(ds), \quad t>0,\, x\in E.$$

As application, we consider the fractional power subordinator $\eta^\alpha := (\eta_t^\alpha)$ of order $\alpha\in]0,1[$. It is defined by its Laplace transform $\mathcal{L}(\eta_t^\alpha)(r) = \exp(-tr^\alpha)$. In this case, under some regular assumption we prove that each η^α-Liapunov function l admits the integral representation.

$$l(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \varphi_t(x)\,t^{\alpha-1}\,dt, \qquad x\in E \tag{2}$$

where

$$\varphi_t(x) := \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \Big(l(\Phi(t,x)) - l(\Phi(s+t,x))\Big)\,\frac{ds}{s^{\alpha+1}} \tag{3}$$

Moreover, formulas like (2) and (3) will be also deduced for the Γ-subordinator and for the Poisson subordinator.

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The case β be the Dirac subordinator is already investigated in [8, 9, 10, 12]. Moreover, similar results are obtained in other contexts in [6, 7, 13, 15] and in some related references.

2 Preliminary

Let $(E, \mathcal{E})$ be a measurable and separable space and let $\mathcal{B}$ be the space of measurable bounded functions defined on E. We denote by $\mathcal{F}$ the set of all finite functions defined on E and by $\mathcal{F}_+$ be the subset of positive elements of $\mathcal{F}$. Note that any linear operator defined on the space $\mathcal{B}$, may be extended to any positive measurable function in the usual way. The space $[0, \infty[\times E$ is always endowed with product σ-algebra $\mathcal{E} \otimes \mathcal{A}$. For every g defined on $]0, \infty[\times E$, we denote by g_t the function defined on E by putting $g_t(x) := g(t, x)$. Let $g, h :]0, \infty[\times E \to R$, we write $g_t = h_t$, λ-a.e. if, for each $x \in E$ the set $\{t \geq 0 : h_t(x) \neq g_t(x)\}$ is λ-negligible. In this section we summarize some known results (cf. [2, 11, 17]).

Definition 2.1 A *semidynamical system* (SDS) on E is a measurable mapping $\Phi : [0, \infty[\times E \mapsto E$ which satisfies

i) $\Phi(0, x) = x,\ x \in E$,

ii) $\Phi(s+t, x) = \Phi(s, \Phi(t, x)),\ s, t \geq 0,\ x \in E$. (Translation equation)

Let Φ be a SDS on E. For each $x \in E$, the set $T_x := \{\Phi(t, x) : t \geq 0\}$ is called *trajectory from x*. If there exists $a > 0$ such that $x = \Phi(a, x)$ then T_x is said to be *periodical*. By putting

$$H_t u(x) := u(\Phi(t, x)), \qquad u \in \mathcal{B},\ t \geq 0,\ x \in E,$$

we define a semigroup $\mathbf{H} := (H_t)_{t \geq 0}$ of linear operators on $\mathcal{B}$. $\mathbf{H}$ is the *deterministic* or *substitution* semigroup associated to the SDS Φ.

We consider R endowed with its Borel field, we denote by λ the Lebesgue measure on $[0, \infty[$ and by ε_t the Dirac measure at point t. Moreover, for each bounded measure μ on $[0, \infty[$, $\mathcal{L}$ denotes its Laplace transform, i.e. $\mathcal{L}(\mu)(r) := \int_0^\infty \exp(-rs)\, \mu(ds)$. If μ has a density δ with respect to λ, we denote by $\mathcal{L}(\delta) := \mathcal{L}(\delta.\lambda)$.
A *Bochner subordinator* is a convolution semigroup $\beta = (\beta_t)_{t>0}$ of probability measures on R such that

1. For each $t > 0$, the measure $\beta_t \neq \varepsilon_0$ and supported by $[0, \infty[$,
2. $\beta_s * \beta_t = \beta_{s+t}$ for all $s, t > 0$,
3. $\lim_{t \to 0} \beta_t = \varepsilon_0$, vaguely.

Let β be a Bochner subordinator. The associated *potential measure* is defined by $\kappa := \int_0^\infty \beta_s\, ds$. Following (cf. [2], Proposition 14.1) κ is a Borel measure. The associated *Bernstein function* k is defined by the Laplace transform $\mathcal{L}(\beta_t)(r) = \exp(-tk(r))$ for all $r, t > 0$. It is known that k admits the representation (cf. [2], Theorem 9.8)

$$k(r) = br + \int_0^\infty (1 - \exp(-rs))\, \nu(ds), \qquad r > 0 \quad (4)$$

where $b \geq 0$ and ν is a measure on $]0, \infty[$ verifying $\int_0^\infty \frac{s}{s+1}\, \nu(ds) < \infty$. Moreover, b and ν are uniquely determined, they are called *parameters* of the Bernstein function of β.

Let S be the Banach algebra of complex borel measures on $[0, \infty[$, with convolution as multiplication, and normae by the total variation $\|.\|_S$. A Bochner subordintor $\beta = (\beta_t)_{t>0}$ is said to be in class $\mathcal{K}$ if :
$t \to \beta_t$ is continuously differentiable from $]0, \infty[$ to S such that $\|\beta_t'\|_S < \infty$ for each $t > 0$. This class of subordinators, is considered in [3].
The most important example of Bochner subordinator in the class $\mathcal{K}$ is the *one-sided or fractional power stable* subordinator of index $\alpha \in]0, 1[$.

Examples 2.2 Let β be a Bochner subordinator and let k be the associated Bernstein function given by (4). We shall give some sufficient condition for the Bernstein function in order to get a subordinator in $\mathcal{K}$. We exhibit such examples of subordinator be in $\mathcal{K}$ which contains a number of important functions, including fractional powers, the logarithm, the inverse hyperbolic cosine. We refer to [3] and [16].

1. If $\sup_{u \in S} |F^\beta(t, u)| = O(t^{-1}),\ t \downarrow 0$ where

   $$F^\beta(t, u) := \int_0^\infty \int_0^\infty u(r)\, \frac{\partial}{\partial r}(\beta_t(r-s) - \beta_t(r))\nu(ds)dr$$

   and S is the unit sphere of the complex space of exponential polynomials with respect to sup-norm on R_+. Then $\beta \in \mathcal{K}$ (cf. [16], Theorem 2). For examples:

   i) Let $\alpha \in [0, 1]$, $c \geq 0$ and $k(r) = (c + r)^\alpha - c^\alpha$. Then $\beta \in \mathcal{K}$.

   ii) Let $0 < \alpha < \gamma < 1$ and $k(r) = r^\alpha - (\exp(-(r)^\gamma) - 1)$. Then $\beta \in \mathcal{K}$.

2. Let $r \mapsto \beta_t([r-s, r))$ is monotone decreasing function on $[s, \infty)$ $(s \geq 0)$ for each sufficiently small $t > 0$. If

   $$\int_0^\infty \beta_t([0, s))\, \nu(ds) = O(t^{-1}) \text{ as } t \downarrow 0,$$

   then $\beta \in \mathcal{K}$ (cf, [16], Theorem 5). For examples:

i) Let $b > 0$ and $k(r) = \log(b+r) - \log b$, then $\beta \in \mathcal{K}$.

ii) Let $b, s \geq 0$ and $k(r) = \mathrm{acosh}(b+r) - \mathrm{acosh}\, b$, then $\beta \in \mathcal{K}$.

3. $\varepsilon * \beta$ is not in $\mathcal{K}$.

4. If β^1 and β^2 are in $\mathcal{K}$ then so is $\beta^1 * \beta^2$.

Let Φ be a SSD and β be a Bochner subordinator. Define $\mathbf{Q} = (Q_t)_{t>0}$ by

$$Q_t u(x) := \int_0^\infty u(\Phi(r,x))\, \beta_t(dr) \qquad (5)$$

for all $u \in \mathcal{B}$, $t \geq 0$ and $x \in E$. Then $\mathbf{Q}$ is a semigroup of linear operator on $\mathcal{B}$. This is clear by using the translation equation of Φ and semigroup property of β. The *potential kernel* associated to $\mathbf{Q}$ is defined by $V^\beta := \int_0^\infty Q_t\, dt$. By integration of (5), we get

$$V^\beta u(x) := \int_0^\infty Q_t u(x)\, dt = \int_0^\infty u(\Phi(t,x))\, \kappa(dt) \qquad (6)$$

for all $u \in \mathcal{B}$ and $x \in E$.

Definition 2.3 A positive measurable function $l \in \mathcal{F}$ is called $\mathbf{Q}$-*Liapunov function* for Φ if for any $x \in E$

(i) The function $t \to Q_t l(x)$ is decreasing,

(ii) $\lim_{t \to 0} Q_t l(x) = l(x)$,

We denote by L^β the cone of such functions.
Let $\mathrm{Im}(V^\beta) := \{V^\beta u : u \in \mathcal{F}, V^\beta u \in \mathcal{F}\}$. It is clear to see that $\mathrm{Im}(V^\beta) \subset L^\beta$. If we instead $\mathbf{Q}$ by the deterministic semigroup $\mathbf{H}$ associated to Φ then each function $l \in \mathcal{F}$ satisfying (i) and (ii) is called *classical Liapunov function* for Φ.

Let Φ be SDS and β be in $\mathcal{K}$. A β-exit law associated to Φ is a measurable function $f :]0,\infty[\times E \to [0,\infty[$ which satisfies:

$$\int_0^\infty f(t, \Phi(s,x))\, \beta_t(dr) = f(s+t, x) \qquad (7)$$

for all $s, t > 0$ and $x \in E$. The functional equation (7) is called β-exit equation. By (5) and the notation $f_t(x) := f(t,x)$, (7) is equivalent to

$$Q_s f_t(x) = f_{s+t}(x), \qquad s,t > 0,\ x \in E \qquad (8)$$

For example, for $u \in \mathcal{F}_+$, the function $(t,x) \to Q_t u(x)$ is a β-exit law for Φ whenever it is finite. This follows immediately from the semigroup property of $\mathbf{Q}$. Two β-exit laws f and ψ are said to be equivalent if $f_t = \psi_t$, λ-a.e.

Lemma 2.4 *Let $\beta \in \mathcal{K}$. Then*

$$\beta'_{s+t} = \beta'_s * \beta_t, \qquad s,t > 0 \qquad (9)$$

and

$$\beta_t = -\beta'_t * \kappa, \qquad t > 0 \qquad (10)$$

where $\beta'_t := \frac{\partial}{\partial t}\beta_t$ and $\kappa = \int_0^\infty \beta_t\, dt$.

Proof. Let $\beta \in \mathcal{K}$. Since $\mathcal{L}(\beta_t)(r) = \exp(-tf(r))$, then by differentiation with respect to t under the integral sign, we obtain

$$\mathcal{L}(\beta'_t) = \frac{\partial}{\partial t}\mathcal{L}(\beta_t)(r) = -f(r)\exp(-tf(r)); \qquad t, r > 0$$

Let $s, t, r > 0$, we get

$$\begin{aligned} \mathcal{L}(\beta'_s * \beta_t)(r) &= \mathcal{L}(\beta'_s)(r)\mathcal{L}(\beta_t)(r) \\ &= -f(r)\exp(-sf(r))\exp(-tf(r)) \\ &= -f(r)e^{-(s+t)f(r)} \\ &= \mathcal{L}(\beta'_{s+t})(r) \end{aligned}$$

Moreover, since $\mathcal{L}(\kappa)(r) = \frac{1}{f(r)}$ (cf. [2], Proposition 14.1) we have

$$\begin{aligned} \mathcal{L}(-\beta'_s * \kappa)(r) &= -\mathcal{L}(\beta'_s)(r)\mathcal{L}(\kappa)(r) \\ &= f(r)\exp(-sf(r))\frac{1}{f(r)} \\ &= \mathcal{L}(\beta_t)(r) \end{aligned}$$

We deduce (9) and (10) by the injectivity of Laplace transform.

3 Representation in terms of β-exit laws

Proposition 3.1 *Let Φ be a SDS and let $f = (f_t)_{t>0}$ be a β-exit law such that $l(x) := \int_0^\infty f_t(x)\, dt < \infty$. Then l is $\mathbf{Q}$-Liapunov function, moreover*

$$f_t(x) = -\frac{\partial}{\partial t} Q_t l(x), \qquad t > 0,\ x \in E \qquad (11)$$

Proof. By Fubini's Theorem and (8) we get for all $x \in E$

$$Q_t l(x) = \int_0^\infty Q_t f_s(x)\, ds = \int_t^\infty f_s(x)\, ds.$$

Therefore, $Q_t l$ is finite since $\int_0^\infty f_t\, dt < \infty$ and

$$Q_t l(x) = \int_t^\infty f_s(x)\, ds, \qquad t > 0,\ x \in E \qquad (12)$$

Now from (12), we easily deduce that l is $\mathbf{Q}$-Liapunov function. Moreover, by (12) again we have for $r, t > 0$

$$\frac{1}{r}(Q_{r+t} l - Q_t l) = -\frac{1}{r}\int_t^{r+t} f_s\, ds.$$

Hence we obtain (11).

Let $\mathcal{R}^\beta$ be the cone of functions $u := \int_0^\infty f_t\,dt$ such that f is an exit law for Φ and u is finite. From Proposition 3.1, it follows that

$$\mathrm{Im}(V^\beta) \subset \mathcal{R}^\beta \subset L^\beta.$$

But, the converse is not true in general, i.e. elements of L^β are not necessary on the form $u = \int_0^\infty f_s\,ds$ for some **Q**-exit laws f. As it is observed in many papers related to this problem (cf. [6, 7, 8, 9, 10, 12, 13, 14, 15]), we need some finiteness assumptions, in order to represent the **Q**-Liapunov functions in terms of the β-exit laws of Φ. In what follows, elements u of L^β for which there exists a $v \in \mathcal{F}_+$ such that $u(\Phi(t,x)) \le v(x)$ for each $t \ge 0$ and each $x \in E$ will be called β-Liapunov functions. This means that u is bounded on each trajectory of Φ.

Theorem 3.2 *Let Φ be a SDS, β in $\mathcal{K}$ and let l be an associated β-Liapunov function, then the function f defined by*

$$f_t(x) = -\int_0^\infty l(\Phi(s,x))\,\frac{\partial}{\partial t}\beta_t(ds), \quad t > 0,\ x \in E \quad (13)$$

is an exit law for Φ.

Proof. Let β be in $\mathcal{K}$ and let l be a β-Liapunov function. Since $l \circ \Phi_t \le v$ for each $t \ge 0$ and $\beta_t(]0,\infty[) = 1$, it follows that

$$Q_t l(x) = \int_0^\infty l(\Phi(r,x))\,\beta_t(dr) \le v(x).$$

Hence $Q_t l$ is a finite function. Now, since $l \circ \Phi_t \le v$ again and the total variation of β_t' is finite, the following function is well defined

$$f_t(x) := -\int_0^\infty l(\Phi(r,x))\,\beta_t'(dr), \qquad t > 0,\ x \in E,$$

and the differentiation with respect to t under the integral sign is justified in $Q_t l$. We may define

$$f_t(x) = -\frac{\partial}{\partial t} Q_t l(x), \qquad t > 0,\ x \in E \quad (14)$$

Now, since $t \to Q_t l(x)$ is decreasing, (14) allows us to conclude that $f_t \ge 0$ for all $t > 0$. Moreover, by Fubini Theorem's, (5) and (9), we have

$$\begin{aligned}
Q_t f_s(x) &= \int_0^\infty f_s(\Phi(m,x))\,\beta_t(dm)\\
&= -\int_0^\infty \int_0^\infty l(\Phi(r,\Phi(m,x)))\,\beta_s'(dr)\,\beta_t(dm)\\
&= -\int_0^\infty \int_0^\infty l(\Phi(r+m,x))\,\beta_s'(dr)\,\beta_t(dm)\\
&= -\int_0^\infty l(\Phi(r,x))\,(\beta_s' * \beta_t)(dr)\\
&= -\int_0^\infty l(\Phi(r,x))\,\beta_{s+t}'(dr)\\
&= f_{s+t}(x)
\end{aligned}$$

It follows that f is a **Q**-exit law.

Remarks 3.3 In [14] under the condition $\lim_{s\to\infty} Q_s l = 0$, we proved the representation given above by (17) of η^α-Liapunov function defined by the fractional power subordinator of order $\alpha \in]0,1[$ in terms of η^α-exit law.
Now we may obtain under the same condition the representation for all subordinator in $\mathcal{K}$. Indeed, from (14) it is easy to see that

$$Q_t l(x) - Q_s l(x) = \int_t^s f_r\,dr, \quad s,t > 0,\ x \in E \quad (15)$$

then, by letting $s \uparrow \infty$ in (15), we deduce that $r \mapsto f_r(x)$ is integrable at ∞ and

$$Q_t l(x) = \int_t^\infty f_r\,dr, \quad t > 0,\ x \in E \quad (16)$$

we conclude by letting $t \downarrow 0$ in (16).
In fact in Theorem 3.4 we prove that condition $\lim_{s\to\infty} Q_s l = 0$, is not necessary to get the representation of β-Liapunov functions in terms of β-exit law where β is a Bochner subordinator in the class $\mathcal{K}$.

Theorem 3.4 *Let Φ be a SDS and let β in $\mathcal{K}$. For each β-Liapunov function l, there exists a unique (up to equivalence) β-exit law $f = (f_t)_{t>0}$ for Φ such that*

$$l(x) = \int_0^\infty f_t(x)\,dt, \qquad x \in E \quad (17)$$

Moreover, f is explicitly given by

$$f_t(x) = -\int_0^\infty l(\Phi(s,x))\,\frac{\partial}{\partial t}\beta_t(ds), \quad t > 0,\ x \in E \quad (18)$$

Proof. Let β be in $\mathcal{K}$ and let l be a β-Liapunov function. By Theorem 3.2 we may define

$$f_t(x) = -\frac{\partial}{\partial t} Q_t l(x), \qquad t > 0,\ x \in E.$$

By Fubini's Theorem, (5), (10) and (9) we have for fixed $s, t > 0$

$$\begin{aligned}
Q_{s+t} l &= \int_0^\infty H_r(Q_s l)\,\beta_t(dr)\\
&= -\int_0^\infty H_r(Q_s l)\,(\beta_t' * \kappa)(dr)\\
&= -\int_0^\infty \int_0^\infty H_{r+\ell}(Q_s l)\,\beta_t'(dr)\,\kappa(d\ell)\\
&= -\int_0^\infty \int_0^\infty \int_0^\infty H_{r+\ell}(Q_s l)\,\beta_t'(dr)\,\beta_q(d\ell)\,dq\\
&= -\int_0^\infty \left(\int_0^\infty H_r(Q_s l)(\beta_t' * \beta_q)(dr)\right) dq\\
&= -\int_0^\infty \int_0^\infty H_r\,(Q_s l)\ \beta_{t+q}'(dr)\,dq
\end{aligned}$$

$$\begin{aligned} &= -\int_0^\infty \frac{\partial}{\partial t} Q_{t+q} Q_s l\, dq \\ &= -\int_0^\infty \frac{\partial}{\partial t} Q_{t+q+s} l\, dq \\ &= \int_0^\infty f_{t+s+q}\, dq \\ &= \int_{t+s}^\infty f_q\, dq \end{aligned}$$

Therefore we obtain the representation

$$Q_t l = \int_t^\infty f_s\, ds, \qquad t > 0 \tag{19}$$

then by letting $t \downarrow 0$ in (19), we obtain (17). By Theorem 3.2, we get (18).

Corollary 3.5 *Let Φ be a SDS and let $\beta \in \mathcal{K}$. Let ℓ be a classical Liapunov function for Φ, then there exists a unique (up to equivalence) β-exit law f for Φ such that*

$$\ell(x) = \int_0^\infty f_t(x)\, dt.$$

Proof. Let ℓ be a classical Liapunov function for Φ. Since $t \to \ell(\Phi(t,x))$ is decreasing then $t \to Q_t\ell(x))$ is also decreasing. Moreover, $\lim_{t\to 0} Q_t\ell(x) = 0$ by the classical Lebesgue Theorem, the fact that $\lim_{t\to 0} \ell(\Phi(s,x)) = 0$ and $\ell(\Phi(s,x)) \le \ell(x)$. This means that ℓ is a β-Liapunov function and therefore Theorem 3.4 may be applied.

Remarks 3.6

1. Let (E,Φ) be a SDS. A cocycle for (E,Φ) is a measurable application $C : E \times [0,\infty[\to [0,\infty[$, satisfying the functional equation

$$C(s+t,x) = C(t,x).C(s,\Phi(t,x)), \qquad s,t > 0,\ x \in E.$$

In this paper, we may replace the deterministic semigroup $\mathbf{H}$ by a so called *lattice semigroup* $\mathbf{P} := (P_t)_{t\ge 0}$, i.e. $|P_t h| = P_t|h|$ for any $t \ge 0$ and $h \in \mathcal{B}$. Indeed, following [8], $\mathbf{P}$ admits the representation

$$P_t h(x) = C(t,x)h(\Phi(t,x)), \quad h \in \mathcal{B},\ t \ge 0,\ x \in E \tag{20}$$

where Φ is a SDS and C is a *cocycle* for Φ (cf. [10] for more details). Now in view of (20), it is straightforward that Theorem 3.4 may be generalized for $\mathbf{P}$ instead of $\mathbf{H}$.

2. Let $\varphi := (\varphi_t)_{t>0}$ be an $\mathbf{H}$-exit law and let $f := (f_t)_{t>0}$ the family defined by

$$f_t(x) := \int_0^\infty \varphi_s(x)\ \beta_t(ds), \qquad t > 0,\ x \in E \tag{21}$$

It can be easily verified that f is a β-exit law which is said to be *subordinated* to φ in the Bochner sense by means of β. Notice that if $\varphi_s = H_s h$ for some $h \in \mathcal{F}$ then (21) is just (5). Moreover, by the well definition of κ, we have

$$u(x) := \int_0^\infty f_t(x)\, dt = \int_0^\infty \varphi_t(x)\, \kappa(dt) \tag{22}$$

for all $x \in E$. Let $\mathcal{S}^\beta$ be the cone of finite functions on the form (22). From (5) and (22) again, we deduce that

$$\mathrm{Im}(V^\beta) \subset \mathcal{S}^\beta \subset \mathcal{R}^\beta.$$

3. We consider the function g_t be the density of $\eta^{\frac{1}{2}}$. It is easy to see that g_t is a $\mathbf{Q}$-exit law. Furthermore it is known that $\lim_{t\to 0} g_t(x) = 0$ for each $x \in R$. Hence $u := \int_0^\infty g_t\, dt \in \mathcal{R}^{\frac{1}{2}} \setminus \mathcal{S}^{\frac{1}{2}}$. (cf. [14] Example 2.7.2). Under some regular assumption we prove that $\mathcal{S}^\beta = \mathcal{R}^\beta$. Similar results of this problem are obtained in other contexts in [1].

4. Let Φ be a SDS and let β be in $\mathcal{K}$. A β-liapunov function l is said satisfies (C) if $s \to |l(\Phi(r,x)) - l(\Phi(r+s,x))|$ is ν integrable for all $x \in E$ and $r > 0$ where ν is the parameter of the associated Bernstein function given in (4).

5. Let Φ be a SDS and let β be in $\mathcal{K}$ with bounded associated Bernstein function. Then condition (C) is fulfilled for each β-liapunov function.

Theorem 3.7 *Let Φ be a SDS and let β be in $\mathcal{K}$. Then each β-Liapunov function l such that (C) holds, admits the integral representation*

$$l(x) = \int_0^\infty \varphi_t(x)\, \kappa(dt), \qquad x \in E \tag{23}$$

where

$$\varphi_t(x) := \int_0^\infty \Big(l(\Phi(t,x)) - l(\Phi(s+t,x)) \Big)\, \nu(ds).$$

Proof. Let β be in $\mathcal{K}$ and let l be a β-Liapunov function satisfying (C). Then by Theorem 3.4, there exist a unique β-exit law f such that $l(x) = \int_0^\infty f_t(x)\, dt$. By (18), we get

$$f_{t+s}(x) = -\int_0^\infty Q_t l(\Phi(r,x))\, \beta'_s(dr) = -\frac{\partial}{\partial s} Q_s Q_t l(x) \tag{24}$$

On the other hand, since $\beta_t([0,\infty[) = 1$ and the differentiation with respect to t under integral sing is justified in β_t, then $\int_0^\infty \beta'_t(dt) = 0$. Therefore, we have

$$\frac{\partial}{\partial t} Q_t u(x) = \int_0^\infty (u(\Phi(s,x)) - u(x))\beta'_t(ds),\ t > 0,\ x \in E.$$

Now since (C) holds, then for each $t > 0$ and $x \in E$, the following function is well defined

$$\varphi_t(x) := \int_0^\infty \Big(l(\Phi(t,x)) - l(\Phi(s+t,x)) \Big)\nu(ds).$$

By letting $s \downarrow 0$, (24) and ([17], p. 265), we get

$$f_t(x) = \int_0^\infty \Big(Q_t l(x) - Q_t l(\Phi(r,x) \Big)\, \nu(dr),\ t > 0,\ x \in E.$$

It follows from (5) that $f_t = \int_0^\infty \varphi_s\, \beta_t(ds)$ and we conclude by the well definition of κ to get (23).

4 Applications

1. One-sided stable subordinator: Let η^α be the *one-sided stable* subordinator of order $\alpha \in]0,1[$, i.e the unique convolution semigroup $\eta^\alpha = (\eta_t^\alpha)_{t>0}$ on $[0,\infty[$ such that for each $t > 0$, the Laplace Transform $\mathcal{L}(\eta_t^\alpha)(r) = \exp(-tr^\alpha)$ for $r > 0$. Moreover, following ([17], p.263), the measure η_t^α has a density, denoted by g_t^α, with respect to λ. If we consider $\alpha = \frac{1}{2}$, then the subordinator $\eta^{\frac{1}{2}}$ is called the *Inverse Gaussian subordinator* (cf. [3], p. 869). In this case (cf. [18], p. 268)

$$g_t^{\frac{1}{2}}(s) := 1_{]0,\infty[}(s)\frac{1}{\sqrt{4\pi}}\, ts^{\frac{-3}{2}}\exp(\frac{-t^2}{4s}), \qquad t > 0.$$

Following (cf. [3], p. 869), for each $\alpha \in]0,1[$, $\eta^\alpha \in \mathcal{K}$.
Let Φ be a SDS and let l be a η^α-Liapunov function. Following Theorem 3.4, in the special case if $\alpha = \frac{1}{2}$, l is on the form

$$l(x) = \frac{1}{\sqrt{4\pi}}\int_0^\infty\int_0^\infty l(\Phi(s,x))\, s^{\frac{-3}{2}}\Big(\frac{2t^2}{4s} - 1\Big)e^{\frac{-t^2}{4s}}\, ds\, dt$$

for all $x \in E$. Moreover, if (C) holds then by Theorem 3.7 each η^α-Liapunov function l admits the integral representation

$$l(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \varphi_t(x)\, t^{\alpha-1}\, dt, \qquad x \in E,$$

where

$$\varphi_t(x) := \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \Big(l(\Phi(t,x)) - l(\Phi(s+t,x))\Big)\frac{ds}{s^{\alpha+1}}.$$

2. Gamma subordinator: The Γ-subordinator $\gamma := (\gamma_t)_{t>0}$ is given by $\gamma_t := h_t \cdot \lambda$ where

$$h_t(s) := 1_{]0,\infty[}(s)\frac{1}{\Gamma(t)}\, s^{t-1}\exp(-s), \qquad t > 0.$$

In this case $\kappa := \int_0^\infty \gamma_t dt = d \cdot \lambda$ where

$$d(t) := \exp(-t)\int_0^\infty \frac{1}{\Gamma(s)}\, t^{s-1}\, ds.$$

Moreover $\gamma \in \mathcal{K}$ (cf. [3], p. 874). Let Φ be a SDS, by application of Theorem 3.4, each Γ-Liapunov function admits the integral representation

$$l(x) = \int_0^\infty\int_0^\infty l(\Phi(s,x))\,\frac{s^{t-1}}{\Gamma(t)}\Big(\frac{\Gamma'(t)}{\Gamma(t)} - \log s\Big)\, e^{-s}\, ds\, dt,$$

for all $t > 0$ and $x \in E$. Moreover, if (C) holds then by Theorem 3.7 each Γ-Liapunov function l admits the integral representation

$$l(x) = \int_0^\infty \varphi_t(x)\, k(t)\, dt, \qquad x \in E,$$

where

$$\varphi_t(x) = \int_0^\infty \Big(l(\Phi(s+t,x)) - l(\Phi(t,x))\Big)s^{-1}\exp(-s)\, ds.$$

3. Compound Poisson subordinator: Let q be an arbitrary probability measure on $[0,\infty[$. With $q_j := \{q\}^{*j}$ such that $q_0 \equiv \varepsilon_0$ and fixed $c > 0$, the following semigroup (cf. [3], p. 870)

$$\tau_t := e^{-ct}\sum_{j=0}^\infty \frac{(ct)^j}{j!}q_j, \qquad t > 0,$$

is called *Compound Poisson subordinator*. Moreover, the Bernstein function associated to $\tau := (\tau_t)_{t>0}$ which is bounded is given by $f(r) = c\mathcal{L}(\varepsilon_0 - q)(r),\ r > 0$. Note that $\tau \in \mathcal{K}$. For $q = \varepsilon_1$, we obtain the Poisson subordinator with jump c. In particular, if we consider the Poisson subordinator with jump 1 by Theorem 3.7 and Remark 3.6.4 each τ-Liapunov function l is on the form

$$l(x) = \sum_{n=0}^{n=\infty} f_n(x),$$

where

$$f_t(x) = l(\Phi(t,x)) - l(\Phi(t+1,x)), \quad t > 0.$$

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