# Dual Equation of Indefinite Problem with Three Turning Points

R.Darzi, A. Dabbaghian, A. Neamaty and A. Jodayree Akbarfam *

*Abstract*—**We consider the differential equation**

$$-y'' + q(x)y = \rho^2\phi^2(x)y \quad for \quad x \in I := [0,1], \quad (i)$$

**where $I$ contain three turning points, that is here, zeros of $\phi$. Using of the asymptotic estimates provided in [5] for a special fundamental system of solutions of $(i)$ in $I$, we study the infinite product representation of solutions of $(i)$. Also, we use the infinite product representations of the solutions to derive dual differential equations of the second order.**

***Keywords:* Turning point, Asymptotic form, Hadamard factorization theorem, Infinite products**

## 1 Introduction

Differential equations with turning points have various applications in mathematics, elasticity, optics, geophysics and other branches of natural sciences(see[6,10,11]). The importance of asymptotic analysis in obtaining information on the solution of a Sturm-Liouville equation with multiple turning points was realized by Leung [11], Olver [15-16], Heading [6], and Eberhard, Freiling and schneider in [4]. The results of [10,2,3] bring important innovations to the asymptotic approximation of solutions of Sturm-Liouville equations with two turning points. Neamaty and Dabbaghian [13], authors obtained Asymptotic form of the solution of (i)with $m$ turning points of odd-even order. Marasi and Jodayree [12], authors considered that the weight function has $m$ turning points that one is of odd order and others are of even order. In [15], authors considered duality for an indefinite inverse Sturm-Liouville problem with one turning point. In this paper we obtain The canonical product of the solution of differential equation with turning points in a case where the weight function has three turning points that $x_1$ is of even order, $x_2$ is of odd order and $x_3$ is of even order. Such, we use the infinite product representations of the solutions to derive dual differential equations of the second order. In future paper we will apply the the dual equations to find the solution of an inverse problem.

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## 2 Notations

Let us consider the real second - order differential equation

$$-y'' + q(x)y = \lambda\phi^2(x)y \quad , \quad x \in I = [0,1], \qquad (1)$$

where $\lambda = \rho^2$ is a real parameter, $\phi^2$ and $q$ are functions. we suppose that

$$\phi^2(x) = \prod_{\nu=1}^{3}(x - x_\nu)^{l_\nu}\phi_0(x)$$

where $0 < x_1 < x_2 < x_3 < 1$, $l_\nu \in \mathcal{N}$, $\phi_0(x) > 0$ for $x \in I$, and $\phi_0$ is twice continuously differentiable on $I$. In the other words, $\phi^2$ has in $I$, three zeros $x_\nu$, of order $l_\nu$, $\nu = 1,2,3$, where $l_1$ is even, $l_2$ is odd and $l_3$ is even. In the terminology of [ 5 ], $x_1$ is of type I, $x_2$ is of type IV and $x_3$ is of type II. Zeros $x_\nu$ of $\phi^2$ are called turning points. We also assume that $q$ is bounded and integrable on $I$. Now let $C(x,\lambda)$ be the solution of (1) corresponding to the initial conditions $C(0,\lambda) = 1$, $C'(0,\lambda) = 0$. In order to represent the solution $C(x,\lambda)$ as on Asymptotic form we use a suitable fundamental system of solutions (FSS) for Eq.(1) as constructed in [5 ]. Introducing some terminology at this point we write:

$$[1] \equiv 1 + O(\frac{1}{\lambda}) \quad , \quad as \quad \lambda \longrightarrow \infty,$$

$[\alpha] \equiv \alpha + O(\frac{1}{\rho^{\sigma_0}})$ where $\alpha \in C$ and

$$\sigma_0 = \min\{\mu_1, \mu_2, \mu_3\},$$

$$\mu_\nu = \frac{1}{2 + l_\nu},$$

$D_{\nu,\epsilon} = [x_\nu + \epsilon, x_{\nu+1} - \epsilon]$ and $I_{\nu,\epsilon} = [x_{\nu-1} + \epsilon, x_\nu - \epsilon] \cup [x_\nu - \epsilon, x_\nu + \epsilon] \cup [x_\nu + \epsilon, x_{\nu+1} - \epsilon]$.

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We know from [5] that in the sector

$$S_{-1} = \{\rho | \arg \rho \in [-\frac{\pi}{4}, 0]\},$$

there exists an FSS of (1) $\{W_{1,1}(x,\rho), W_{2,1}(x,\rho)\}$ and such that

$$W_{1,1}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1], 0 \le x < x_1, \\ |\phi(x)|^{-\frac{1}{2}} \csc \pi\mu_1 \\ e^{\rho\int_{x_1}^{x}|\phi(t)|dt}[1], x_1 < x < x_2, \end{cases}$$

$$W_{2,1}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1], 0 \le x < x_1, \\ |\phi(x)|^{-\frac{1}{2}} \sin \pi\mu_1 \\ e^{-\rho\int_{x_1}^{x}|\phi(t)|dt}[1], x_1 < x < x_2. \end{cases} \quad (2)$$

Since $x_2$ is of type IV, we also have the following FSS

$$V_{1,2}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{\rho\int_{x_2}^{x}|\phi(t)|dt}[1], x_1 < x < x_2, \\ \frac{1}{2}|\phi(x)|^{-\frac{1}{2}} \csc \frac{\pi\mu_2}{2}\{e^{i\rho\int_{x_2}^{x}|\phi(t)|dt - i\frac{\pi}{4}}[1]+ \\ e^{-i\rho\int_{x_2}^{x}|\phi(t)|dt + i\frac{\pi}{4}}[1]\}, x_2 < x < x_3, \end{cases}$$

$$V_{2,2}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{-\rho\int_{x_2}^{x}|\phi(t)|dt}[1], x_1 < x < x_2, \\ 2|\phi(x)|^{-\frac{1}{2}} \sin \frac{\pi\mu_2}{2} \\ e^{-i\rho\int_{x_2}^{x}|\phi(t)|dt - i\frac{\pi}{4}}[1], x_2 < x < x_3, \end{cases} \quad (3)$$

Since $x_3$, is of type II, we also have the following FSS

$$U_{1,3}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{i\rho\int_{x_3}^{x}|\phi(t)|dt}[1], x_2 < x < x_3, \\ |\phi(x)|^{-\frac{1}{2}} \csc \pi\mu_3\{e^{i\rho\int_{x_3}^{x}|\phi(t)|dt}[1] + i \\ \cos \pi\mu_3 e^{-i\rho\int_{x_3}^{x}|\phi(t)|dt}[1]\}, x_3 < x < 1, \end{cases}$$

$$U_{2,3}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}}\{e^{-i\rho\int_{x_3}^{x}|\phi(t)|dt}[1] + i \\ \cos \pi\mu_3 e^{i\rho\int_{x_3}^{x}|\phi(t)|dt}[1]\}, x_2 < x < x_3, \\ |\phi(x)|^{-\frac{1}{2}} \sin \pi\mu_3 \\ e^{i\rho\int_{x_3}^{x}|\phi(t)|dt}[1], x_3 < x < 1, \end{cases} \quad (4)$$

It follows that the Wronskian of FSS satisfies

$$W(\rho) \equiv W(W_{1,1}(x,\rho), W_{2,1}(x,\rho)) = -2\rho[1],$$
$$W(V_{1,2}(x,\rho), V_{2,2}(x,\rho)) = -2\rho[1],$$
$$W(U_{1,2}(x,\rho), U_{2,2}(x,\rho)) = -2i\rho[1]$$

as $\rho \to \infty$.
We also need
$\{W_{1,1}(x_1,\rho), W_{2,1}(x_1,\rho)\}$, $\{V_{1,2}(x_2,\rho), V_{2,2}(x_2,\rho)\}$ and $\{U_{1,3}(x_3,\rho), U_{2,3}(x_3,\rho)\}$. From [5] we have

$$W_{1,1}(x_1,\rho) = \frac{\sqrt{2\pi}}{2}(i\rho)^{\frac{1}{2}-\mu_1} csc\pi\mu_1 e^{i\pi(-\frac{1}{4}+\frac{\mu_1}{2})} \frac{2^{\mu_1}\psi(x_1)}{\Gamma(1-\mu_1)}[1],$$

$$W_{2,1}(x_1,\rho) = \frac{\sqrt{2\pi}}{2}(i\rho)^{\frac{1}{2}-\mu_1} e^{i\pi(-\frac{1}{4}+\frac{\mu_1}{2})} \frac{2^{\mu_1}\psi(x_1)}{\Gamma(1-\mu_1)}[1],$$

where

$$\psi(x_1) = \lim_{x\to x_1} \phi^{-\frac{1}{2}}(x)\{\int_{x_1}^{x}\phi(t)dt\}^{\frac{1}{2}-\mu_1}. \quad (3)$$

At the $x = x_2$, we have

$$V_{1,2}(x_2,\rho) = \frac{\sqrt{2\pi}}{2}(\rho)^{\frac{1}{2}-\mu_2} csc\pi\mu_2 \frac{2^{\mu_2}\psi(x_2)}{\Gamma(1-\mu_2)}[1],$$

$$V_{2,2}(x_2,\rho) = \frac{\sqrt{2\pi}}{2}(\rho)^{\frac{1}{2}-\mu_2} e^{-i\frac{\pi\mu_2}{2}} \sec(\frac{\pi\mu_2}{2}) \frac{2^{\mu_2}\psi(x_2)}{\Gamma(1-\mu_2)}[1].$$

At the $x = x_3$, we have

$$U_{1,3}(x_3,\rho) = \frac{\sqrt{2\pi}}{2}(\rho)^{\frac{1}{2}-\mu_3} csc\pi\mu_3 e^{i\pi(\frac{1}{4}-\frac{\mu_3}{2})} \frac{2^{\mu_3}\psi(x_3)}{\Gamma(1-\mu_3)}[1],$$

$$U_{2,3}(x_3,\rho) = \frac{\sqrt{2\pi}}{2}(\rho)^{\frac{1}{2}-\mu_3} e^{i\pi(\frac{1}{4}-\frac{\mu_3}{2})} \frac{2^{\mu_3}\psi(x_3)}{\Gamma(1-\mu_3)}[1].$$

## 3 Asymptotic form of the solution

We consider the differential equation (1) with the following conditions

$$C(0,\lambda) = 1 \quad , \quad C'(0,\lambda) = 0\cdot \quad (6)$$

Applying the FSS $\{W_{1,1}(x,\rho), W_{2,1}(x,\rho)\}$ for $x \in I_{1,\epsilon}$, we have

$$C(x,\rho) = c_1 W_{1,1}(x,\rho) + c_2 W_{2,1}(x,\rho), \quad (7)$$

that using of Cramer's rule leads to the equation

$$C(x,\rho) = \frac{1}{W(\rho)}(W'_{2,1}(0,\rho)W_{1,1}(x,\rho)$$

$$-W'_{1,1}(0,\rho)W_{2,1}(x,\rho)), \quad (8)$$

where $W(\rho) = -2\rho[1]$.
Taking (2) in to account we derive

$$C(x,\rho)=\begin{cases}\frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(e^{\rho\int_0^x|\phi(t)|dt}[1]\\ +e^{-\rho\int_0^x|\phi(t)|dt}[1]),0\le x<x_1,\\ \frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(\csc\pi\mu_1(\rho)\\ e^{\rho\int_{x_1}^x|\phi(t)|dt}[1]+\sin\pi\mu_1(\rho)\\ e^{-\rho\int_{x_1}^x|\phi(t)|dt}[1]),x_1<x<x_2.\end{cases}\tag{9}$$

By virtue of (12), the following estimates are also valid:

$$C(x,\rho)=\begin{cases}\frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}e^{\rho\int_0^x|\phi(t)|dt}\\ E_k(x,\rho),0\le x<x_1\\ \\ \frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1 e^{\rho\int_0^x|\phi(t)|dt}\\ E_k(x,\rho),x_1<x<x_2,\end{cases}\tag{10}$$

where

$$E_k(x,\rho)=[1]+\sum_{n=1}^{\nu(x)}e^{\rho\alpha_k\beta_{kn}(x)}[b_{kn}(x)],$$

and $\alpha_{-2}=\alpha_1=-1$ , $\alpha_0=-\alpha_{-1}=i$ , $\beta_{k\nu}(x)\neq 0$ , $0<\delta\le\beta_{k1}(x)<\beta_{k2}(x)<\cdots\le\beta_{k\nu(x)}(x)\le 2\max\{R_+(1),R_-(1)\}$, where the integer-valued functions $\nu$ and $b_{kn}$ are constant in every interval $D_{j,\epsilon}$, $j=1,2,3$ and

$$R_+(x)=\int_0^x\sqrt{\max\{0,\phi^2(t)\}}dt\ \ ,$$

$$R_-(x)=\int_0^x\sqrt{\max\{0,-\phi^2(t)\}}dt\cdot\tag{11}$$

Similarly using of (5) and (11) for $x=x_1$ we find that

$$C(x_1,\rho)=\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}(i\rho)^{\frac{1}{2}-\mu_1}csc\pi\mu_1 e^{i\pi(-\frac{1}{4}+\frac{\mu_1}{2})}$$

$$\frac{2^{\mu_1}\psi(x_1)}{4\Gamma(1-\mu_1)}\times e^{\rho\int_0^{x_1}|\phi(t)|dt}E_k(x_1,\rho).\tag{12}$$

Hence we have estimated the solution of (1) defined by the initial conditions (9) in $I_{1,\epsilon}$. In order to find the solution in $I_{2\epsilon}$, we fix $x\in(x_1,x_2)$ and use (3) and Cramer's rule to determine the connection coefficients $N_1(\rho),N_2(\rho)$ with

$$\begin{cases}C(x,\rho)=N_1(\rho)V_{1,2}(x,\rho)+N_2(\rho)V_{2,2}(x,\rho),\\ C'(x,\rho)=N_1(\rho)V'_{1,2}(x,\rho)+N_2(\rho)V'_{2,2}(x,\rho).\end{cases}\tag{13}$$

Consequently

$$N_1(\rho)=\frac{1}{2}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1 e^{\rho\int_0^{x_2}|\phi(t)|dt}[1],$$

$$N_2(\rho)=\frac{1}{2}|\phi(0)|^{\frac{1}{2}}\sin\pi\mu_1 e^{-\rho\int_0^{x_2}|\phi(t)|dt}[1]\cdot\tag{14}$$

Substituting (12) and estimates of $V_{1,2}(x,\rho)$ and $V_{2,2}(x,\rho)$ from (3) in the case $x_2<x<x_3$ we derive the continuation of the solution to interval $(x_2,x_3)$ in the form:

$$C(x,\rho)=\frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(T_1(\rho)e^{i\rho\int_{x_2}^x|\phi(t)|dt}[1]+$$

$$T_2(\rho)e^{-i\rho\int_{x_2}^x|\phi(t)|dt}[1])\tag{15}$$

where

$$T_1(\rho)=\frac{1}{2}\csc\pi\mu_1\csc\frac{\pi\mu_2}{2}e^{\rho\int_0^{x_2}|\phi(t)|dt-i\frac{\pi}{4}},$$

$$T_2(\rho)=\frac{1}{2}\csc\pi\mu_1\csc\frac{\pi\mu_2}{2}e^{\rho\int_0^{x_2}|\phi(t)|dt+i\frac{\pi}{4}}$$

$$+2\sin\pi\mu_1\sin\frac{\pi\mu_2}{2}e^{-\rho\int_0^{x_2}|\phi(t)|dt-i\frac{\pi}{4}},$$

or

$$C(x,\rho)=\tfrac{1}{4}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1\csc\tfrac{\pi\mu_2}{2}$$

$$\times e^{\rho\int_0^{x_2}|\phi(t)|dt+i\rho\int_{x_2}^x|\phi(t)|dt-i\frac{\pi}{4}}E_k(x,\rho),x_2<x<x_3\cdot\tag{16}$$

In addition, the value of $C(x,\rho)$ at $x=x_2$ can be calculated by taking account of (7) and (16)

$$C(x_2,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}\rho^{\frac{1}{2}-\mu_2}2^{\mu_2}\psi(x_2)csc\pi\mu_1 csc\pi\mu_2}{4\Gamma(1-\mu_2)}$$

$$\times e^{\rho\int_0^{x_2}|\phi(t)|dt}E_k(x_2,\rho).\tag{17}$$

Now for fixed $x\in(x_2,x_3)$ and use (4) we determine the connection coefficients $B_1(\rho),B_2(\rho)$ with

$$C(x,\rho)=B_1(\rho)U_{1,3}(x,\rho)+B_2(\rho)U_{2,3}(x,\rho)\ \ ,\ \ x_2<x<x_3,$$

consequently

$$\begin{aligned}B_1(\rho)&=\tfrac{1}{2}|\phi(0)|^{\frac{1}{2}}T_1(\rho)e^{i\rho\int_{x_2}^{x_3}|\phi(t)|dt}[1]-\\ &\tfrac{i}{2}|\phi(0)|^{\frac{1}{2}}\cos\pi\mu_3 T_2(\rho)e^{-i\rho\int_{x_2}^{x_3}|\phi(t)|dt}[1],\\ B_2(\rho)&=\tfrac{i}{2}|\phi(0)|^{\frac{1}{2}}T_2(\rho)e^{-i\rho\int_{x_2}^{x_3}|\phi(t)|dt}[1]\cdot\end{aligned}\tag{18}$$

By the continuation of the solution to the interval $(x_3,1)$ we have

$$C(x,\rho)=B_1(\rho)U_{1,3}(x,\rho)+B_2(\rho)U_{2,3}(x,\rho)\tag{19}$$

$, x_3 < x < 1,$

then by (4) we obtain

$$C(x,\rho) = |\phi(x)|^{-\frac{1}{2}}[D_1(\rho)e^{i\rho\int_{x_3}^{x}|\phi(t)|dt}[1]+$$

$$D_2(\rho)e^{-i\rho\int_{x_3}^{x}|\phi(t)|dt}[1]] \quad , \quad x_3 < x < 1, \qquad (20)$$

where

$$D_1(\rho) = B_1(\rho)\csc\pi\mu_3,$$

$$D_2(\rho) = iB_1(\rho)\cos\pi\mu_3 + B_2(\rho)\sin\pi\mu_3\cdot$$

By substituting (21) in (23)we obtain the leading term of $C(x,\rho)$ in $(x_3, 1)$ as follows:

$$C(x,\rho) = \tfrac{1}{4}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1\csc\tfrac{\pi\mu_2}{2}\csc\pi\mu_3$$
$$e^{\rho\int_0^{x_2}|\phi(t)|dt+i\rho\int_{x_2}^{x}|\phi(t)|dt-i\frac{\pi}{4}}E_k(x,\rho)\ ,\ x_3 < x < 1\cdot \qquad (21)$$

In addition from (8) and (22) we can get the value of $C(x,\rho)$ at $x = x_3$:

$$C(x_3,\rho) = \frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}\rho^{\frac{1}{2}-\mu_3}2^{\mu_3}\psi(x_3)e^{i\pi(\frac{1}{4}-\frac{\mu_3}{2})}}{8\Gamma(1-\mu_3)} \qquad (22)$$

$$csc\pi\mu_1 csc\frac{\pi\mu_2}{2}csc\pi\mu_3$$

$$\times e^{\rho\int_0^{x_2}|\phi(t)|dt+i\rho\int_{x_2}^{x_3}|\phi(t)|dt}E_k(x_3,\rho)$$

## 4 The Asymptotic representation of the canonical product

We consider the boundary value problem $L_1 = L_1(\phi^2(x), q(x), b)$ for Eq.(1) with boundary conditions

$$y(0,\lambda) = 1, \quad y'(0,\lambda) = 0, \quad y(b,\lambda) = 0.$$

The boundary value problem $L_1$ for $b \in (0, x_1)$ has a countable set of negative eigenvalues $\{\lambda_n^-(b)\}_{n\ge 0}$. The asymptotic distribution of each function $\lambda_n(b)$ is of the form

$$\sqrt{\lambda_n^-(b)} = i\frac{n\pi}{\int_0^b|\phi(t)|dt} + O(\frac{1}{n}) \qquad (23)$$

and for $x = x_1$ similarly from (15) we have

$$\sqrt{-\lambda_n^-(x_1)} = \frac{n\pi + (\frac{\pi\mu_1}{2} - \frac{\pi}{4})}{\int_0^{x_1}|\phi(t)|dt} + O(\frac{1}{n}). \qquad (24)$$

Such,for $b \in (x_1, x_2)$ has a countable set of negative eigenvalues $\{\lambda_n^-(b)\}_{n\ge 0}$:

$$\sqrt{\lambda_n^-(b)} = i\frac{n\pi}{\int_0^b|\phi(t)|dt} + O(\frac{1}{n}), \qquad (25)$$

and for $x = x_2$ similarly from (20) we have

$$\sqrt{-\lambda_n^-(x_2)} = \frac{n\pi + (\frac{\pi\mu_2}{2} - \frac{\pi}{4})}{\int_0^{x_2}|\phi(t)|dt} + O(\frac{1}{n}). \qquad (26)$$

For $x_\upsilon < x < x_{\upsilon+1}, \upsilon \ge 2$, the boundary value problem $L_1$, has an infinite number of positive and negative eigenvalues,

$$\sqrt{\lambda_n^+(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_{x_2}^b|\phi(t)|dt} + O(\frac{1}{n}), \qquad (27)$$

$$\sqrt{\lambda_n^-(b)} = i\frac{n\pi - \frac{\pi}{4}}{\int_0^{x_2}|\phi(t)|dt} + O(\frac{1}{n}). \qquad (28)$$

Similarly for $x = x_3$,

$$\sqrt{\lambda_n^+(x_3)} = \frac{n\pi + (\frac{\pi\mu_3}{2} - \frac{\pi}{2})}{\int_{x_2}^{x_3}|\phi(t)|dt} + O(\frac{1}{n}), \qquad (29)$$

$$\sqrt{\lambda_n^-(x_3)} = \frac{n\pi - \frac{\pi}{4}}{\int_{x_2}^{x_3}|\phi(t)|dt} + O(\frac{1}{n}). \qquad (30)$$

Since the solution $C(x,\rho)$ of the Sturm -Liouville equation defined by a fixed set of initial conditions is an entire function of $\rho$ for each fixed $x \in [0,1]$, thus it follows from the classical Hadamard's factorization theorem that such solution is expressible as an infinite product.
Therefore, by using Hadamard's theorem, $C(x,\lambda)$ can be represented in the form

$$C(x,\lambda) = c(b)\prod_{n=1}^{\infty}(1 - \frac{\lambda}{\lambda_n(b)}) \qquad (31)$$

where $c(b)$ is a function independent of $\lambda$ but may depend on $b$. The sequence of $\lambda_n$ is a zero set of $C(b,\lambda)$ for each $b$, so that $C(b,\lambda_n) = 0$, which corresponds to eigenvalues of the boundary value problem $L_1$

on the closed interval $[0, b]$, $0 < b < x_1$. We rewrite the infinite product as

$$C(b,\lambda) = c(b)\prod(1 - \frac{\lambda}{\lambda_n(b)}) = c_1(b)\prod\frac{\lambda - \lambda_n(b)}{z_n^2} \qquad (32)$$

with

$$c_1(b) := c(b)\prod\frac{-z_n^2}{\lambda_n(b)},$$

where $z_n = \frac{n\pi}{R_-(x)}$. Now (26) implies that $\frac{-z_n^2}{\lambda_n(b)} = 1 + O(\frac{1}{n^2})$. It follows from [7] that the infinite product $\prod \frac{-z_n^2}{\lambda_n(b)}$ is absolutely convergent on any compact subinterval of $(0, x_1)$. The function $\frac{-z_n^2}{\lambda_n(b)}$ is continuous and so the O-term is uniformly bounded in $b$.

**Theorem 1.** Let $C(x, \lambda)$ be the solution of (1) satisfying the initial conditions $C(0, \lambda) = 1$, $C'(0, \lambda) = 0$. Then for $0 < x < x_1$,

$$C(x, \lambda) = |\phi(x)|^{-\frac{1}{2}} |\phi(0)|^{\frac{1}{2}} R_-(x) \prod_{n\geq 1} \frac{\lambda - \lambda_n(x)}{z_n^2} \quad (33)$$

where $z_n = \frac{n\pi}{R_-(x)}$.

**proof.** Let $\{\lambda_n(x)\}$ be the eigenvalues of the boundary value problem $L_1$ on $[0, x]$, for fixed $x$, $0 < x < x_1$ then according to [9] we have

$$\prod (\frac{\lambda - \lambda_n(x)}{z_n^2}) = \frac{\sinh R_-(x)\sqrt{\lambda}}{R_-(x)\sqrt{\lambda}} (1 + O(\frac{\log n}{n})). \quad (34)$$

Thus from (13) and (35), we obtain

$$c_1(x) = |\phi(x)|^{-\frac{1}{2}} |\phi(0)|^{\frac{1}{2}} R_-(x).$$

Similarly for $b = x_1$ again by Hadamard's theorem we that

$$C(x_1, \lambda) = A \prod (1 - \frac{\lambda}{\lambda_n(x_1)}) \quad (35)$$

where $A$ is constant. Let $j_n$, $n = 1, 2, ...$ be the sequence of positive zeros of the Bessel function of order $\mu_1$, then

$$\frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)} = 1 + O(\frac{1}{n^2}),$$

so the infinite product

$$\prod \frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)}$$

are absolutely convergent.Consequently we may write as before,

$$C(x_1, \lambda) = A_1 \prod \frac{(\lambda - \lambda_n(x_1))R_-^2(x)}{j_n^2} \quad (36)$$

where

$$A_1 = A \prod \frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)}.$$

**Theorem 2.** For $b = x_1$,

$$C(x_1, \lambda) = \frac{|\phi(0)|^{\frac{1}{2}} \psi(x_1) R_-(x_1)^{\frac{1}{2}+\mu_1}}{2\mu_1}$$

$$\times \prod_{n\geq 1} \frac{(\lambda - \lambda_n(x_1))R_-^2(x_1)}{j_n^2} \quad (37)$$

where the sequence $\lambda_n(x_1)$ represents the sequence of negative eigenvalues of the boundary value problem $L_1$ on $[0, x_1]$.

**proof.** According to [14] the infinite product

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n(x_1))R_-^2(x_1)}{j_n^2} = 2^{\mu_1}\Gamma(1 + \mu_1)[i\sqrt{\lambda}R_-(x_1)]^{-\mu_1}$$

$$J_{\mu_1}(i\sqrt{\lambda}R_-(x_1))(1 + O(\frac{\log n}{n}))$$

uniformly on the Circles $|\lambda| = \frac{n^2\pi^2}{R_-^2(x_1)}$.

Thus it follows from (15), we obtain

$$A_1 = \frac{|\phi(0)|^{\frac{1}{2}} \psi(x_1) R_-(x_1)^{\frac{1}{2}+\mu_1}}{2\mu_1}.$$

**Theorem 3.** For $x_1 < x < x_2$,

$$C(x, \lambda) = |\phi(x)|^{-\frac{1}{2}} |\phi(0)|^{\frac{1}{2}} R_-(x) \csc \pi\mu_1$$

$$\prod_{m\geq 1} \frac{\lambda - \lambda_m(x)}{z_m^2} \quad (38)$$

**proof.** Similarly for $x_1 < x < x_2$, we use from (13) and (35).

**Theorem 4.** For $x = x_2$, we have

$$C(x_2, \lambda) = \frac{|\phi(0)|^{\frac{1}{2}} \psi(x_2) R_-(x_2)^{\frac{1}{2}+\mu_2} \csc \pi\mu_1}{2\mu_2}$$

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n(x_2))R_-^2(x_2)}{j_n^2}. \quad (39)$$

**proof.** The proof is similar in every respect to that of theorem 2 and so is omitted.

For $x_2 < x < x_3$, the boundary value problem $L_1$ on $[0, x]$ has an infinite number of positive and negative eigenvalues , say, respectively, $\{\lambda_n^+\}$, $\{\lambda_n^-\}$. By Hadamard's theoram, the solution on $[0, x]$, $x_2 < x < x_3$ is of the form

$$C(x, \lambda) = d(x) \prod (1 - \frac{\lambda}{\lambda_n^-(x)})(1 - \frac{\lambda}{\lambda_n^+(x)}). \quad (40)$$

Now let $\tilde{j}_n$, $n = 1, 2, 3, ...,$ be the positive zeros of $J_1'(z)$, derivative of the Bessel function of order one. The distribution of $\tilde{j}_n$ is of the form

$$\tilde{j}_n = m^2\pi^2 - \frac{m\pi^2}{2} + O(1), \quad (41)$$

(see [1]). Consequently, we have

$$\frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)} = 1 + O(\frac{1}{n^2}),$$

$$\frac{-\tilde{j}_n^2}{R_-^2(x)\lambda_n^-(x)} = 1 + O(\frac{1}{n^2}). \quad (42)$$

Consequently, the infinite products

$$\prod \frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)} \quad , \quad \prod \frac{-\tilde{j}_n^2}{R_-^2(x)\lambda_n^-(x)} \quad (43)$$

are absolutely convergent for each $x_2 < x < x_3$. Therefore we may write

$$C(x,\lambda) = c_2(x) \prod \frac{(\lambda - \lambda_n^-(x))R_-^2(x_2)}{\tilde{j}_n^2}$$

$$\prod \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2} \quad (44)$$

where

$$c_2(x) = d(x) \prod \frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)} \prod \frac{-\tilde{j}_n^2}{R_-^2(x)\lambda_n^-(x)}.$$

**Theorem 5.** For $x_2 < x < x_3$,

$$C(x,\lambda) = \frac{\pi}{8}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}$$

$$\times (R_-(x)R_+(x))^{\frac{1}{2}} \csc \pi\mu_1 \csc \pi\frac{\mu_2}{2}$$

$$\times \prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_2)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}. \quad (45)$$

**proof.** From [14] we have

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_2)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}$$

$$= \frac{4e^{R_-(x)\sqrt{\lambda}}}{\pi R_-^{\frac{1}{2}}(x)R_+^{\frac{1}{2}}(x)\sqrt{\lambda}}\{\cos(R_+(x)\sqrt{\lambda} - \frac{\pi}{4}) + O(\frac{1}{\sqrt{\lambda}})\}$$

as $\lambda \to \infty$. Thus we get

$$C_2(x) = \frac{\pi}{8}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(R_-(x)R_+(x))^{\frac{1}{2}}$$

$$\csc \pi\mu_1 \csc \pi\frac{\mu_2}{2}.$$

**Theorem 6.** For $x_3 < x < 1$,

$$C(x,\lambda) = \frac{\pi}{8}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(R_-(x)R_+(x))^{\frac{1}{2}}$$

$$\csc \pi\mu_1 \csc \pi\frac{\mu_2}{2} \csc \pi\mu_3 \quad (46)$$

$$\times \prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_2)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}.$$

**proof.** This follows from (24) and (47).
We can proceed similarly for $b = x_3$ to obtain

$$C(x_3,\lambda) = C_3(x) \prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x_3))R_-^2(x_2)}{\tilde{j}_n}$$

$$\prod_{n\geq 1} \frac{(\lambda_n^+(x_3) - \lambda)R_+^2(x_3)}{\tilde{r}_n}. \quad (47)$$

where $C_3(x)$ are constant and $\tilde{j}_n$ is the sequence of positive zeros $J_1'(z)$ and $\tilde{r}_n$ is the sequence of positive zeros of $J_{\mu_3+\frac{3}{2}}(z)$.
**Theorem 7.** For $x = x_3$, we have

$$C(x_3,\lambda) = \frac{\sqrt{\pi}|\phi(0)|^{\frac{1}{2}}\psi(x_3)R_-(x_2)^{\frac{1}{2}}R_+^{\mu_3}(x_3)}{4\Gamma(\mu_3 + \frac{1}{2})} \quad (48)$$

$$\times \csc \pi\mu_1 \csc \pi\frac{\mu_2}{2}$$

$$\times \prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x_3))R_-^2(x_2)}{\tilde{j}_n} \prod_{n\geq 1} \frac{(\lambda_n^+(x_3) - \lambda)R_+^2(x_3)}{\tilde{r}_n}.$$

**proof.** We have

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x_3))R_-^2(x_2)}{\tilde{j}_n} \prod_{n\geq 1} \frac{(\lambda_n^+(x_3) - \lambda)R_+^2(x_3)}{\tilde{r}_n}$$

$$= 2^{\mu_3+\frac{1}{2}}\Gamma(\mu_3 + \frac{1}{2})(R_+(x_3)\sqrt{\lambda})^{-(\mu_3-\frac{1}{2})}M, \quad (49)$$

where

$$M = \frac{1}{2\pi\rho\sqrt{R_+(x_3)R_-(x_2)}}e^{\rho R_-(x_2)+i\rho R_+(x_3)-\frac{\pi\mu_3}{2}i}[1],$$

thus, we get

$$C_3(x) = \frac{\sqrt{\pi}|\phi(0)|^{\frac{1}{2}}\psi(x_3)R_-(x_2)^{\frac{1}{2}}R_+^{\mu_3}(x_3)}{4\Gamma(\mu_3 + \frac{1}{2})}$$

$$\csc \pi\mu_1 \csc \pi\frac{\mu_2}{2}.$$

## 5 The dual equation

In this section, we drive the dual equations associated with (1) by use of infinite product representation.
**Theorem 8.** For $0 < x < x_1$, the sequences of functions $\{\lambda_n(x)\}$ satisfy

$$\lambda'' + \frac{2c'(x)\lambda_n'}{c(x)} + 2\lambda_n\lambda_n' \sum_{i\neq n, 1\leq i} \frac{\lambda'}{\lambda_i^2}(1 - \frac{\lambda_n(x)}{\lambda_i(x)})^{-1}$$

$$-2\frac{(\lambda_n')^2}{\lambda_n} = 0, \quad (50)$$

where $c(x)$ is defined in (35).

**proof.** For $0 < x < x_1$,the condition $C(x, \lambda_n(x)) = 0$ gives,

$$\frac{\partial C}{\partial x} + \frac{\partial C}{\partial \lambda}\lambda_n' = 0,$$

The sequence $\lambda_n$ represents the sequence of negative eigenvalues of $L_1$ on $(0, x_1)$. With differentiating again

$$\frac{\partial^2 C}{\partial x^2} + 2\frac{\partial^2 C}{\partial x \partial \lambda}\lambda_n' + \frac{\partial^2 C}{\partial \lambda^2}(\lambda_n')^2 + \frac{\partial C}{\partial \lambda}\lambda_n'' = 0. \quad (51)$$

The first term in (53) is zero at $(x,_n(x))$ by virtue of (1). Thus

$$2\frac{\partial^2 C}{\partial x \partial \lambda}\lambda_n' + \frac{\partial^2 C}{\partial \lambda^2}(\lambda_n')^2 + \frac{\partial C}{\partial \lambda}\lambda_n'' = 0. \quad (52)$$

Now, we first calculate the various derivatives of $C(x, \lambda)$. In the case, from (35), it can be written

$$C(x, \lambda) = c(x)\prod_{k=1}^{\infty}(1 - \frac{\lambda}{\lambda_k(x)}). \quad (53)$$

We calculate $\frac{\partial C}{\partial \lambda}$, $\frac{\partial^2 C}{\partial \lambda^2}$ and $\frac{\partial^2 C}{\partial x \partial \lambda}$ at the points $(x, \lambda_n(x))$ by using (56). In forming $\frac{\partial^2 C}{\partial \lambda \partial x}$ from (56), the interchange of summation and differentiation in

$$\frac{d}{dx}\sum_{k \geq 1}(1 - \frac{\lambda}{\lambda_k(x)})$$

will be valid if the differentiated series

$$\sum_{k \neq n}\frac{-\lambda_n(x)\lambda_k'(x)}{(\lambda_k(x) - \lambda_n(x))\lambda_k(x)}$$

is uniformly convergent which is the case from [16]. We define $T_n$ by

$$T_n = T_n(x, \lambda_n(x)) = \prod_{k \neq n, 1 \leq k}(1 - \frac{\lambda_n(x)}{\lambda_k(x)}). \quad (54)$$

We have

$$\frac{\partial C}{\partial \lambda}(x, \lambda_n) = \frac{-cT_n}{\lambda_n(x)},$$

$$\frac{\partial^2 C}{\partial \lambda^2}(x, \lambda_n) = \frac{2cT_n}{\lambda_n(x)}\sum_{i \neq n, 1 \leq i}\frac{1}{\lambda_i}(1 - \frac{\lambda_n(x)}{\lambda_i(x)})^{-1},$$

$$\frac{\partial^2 C}{\partial \lambda \partial x}(x, \lambda_n) = \frac{-c'T_n}{\lambda_n(x)} + \frac{c\lambda_n' T_n}{\lambda_n^2(x)}$$

$$-c(x)T_n\sum_{i \neq n, 1 \leq i}\frac{\lambda_i'}{\lambda_i^2}(1 - \frac{\lambda_n(x)}{\lambda_i(x)})^{-1} -$$

$$\frac{c\lambda_n' T_n}{\lambda_n}\sum_{i \neq n, 1 \leq i}\frac{1}{\lambda_i}(1 - \frac{\lambda_n(x)}{\lambda_i(x)})^{-1}.$$

Placing these terms into (55), we obtain

$$\lambda'' + \frac{2c'(x)\lambda_n'}{c(x)} +$$

$$2\lambda_n\lambda_n'\sum_{i \neq n, 1 \leq i}\frac{\lambda'}{\lambda_i^2}(1 - \frac{\lambda_n(x)}{\lambda_i(x)})^{-1} - 2\frac{(\lambda_n')^2}{\lambda_n} = 0. \quad (55)$$

where $c(x)$ is defined in (35).

We note that dividing Eq. (53) by $\lambda_n'$ and integrating from a fixed number $\alpha \neq -1$ up to $x$, we obtain

$$\lambda_n'(x) = \frac{\lambda_n^2(x)\lambda_n'(\alpha)c^2(\alpha)}{\lambda_n^2(\alpha)c^2(x)}e^{-2s_n(x,\lambda_n)}, \quad (56)$$

where

$$S_n(x, \lambda_n) = \sum_{i \neq n}\int_{\alpha}^{x}\frac{\lambda_i'\lambda_n}{\lambda_i}(\lambda_i - \lambda_n)^{-1}.$$

Similarly, for $x_1 < x < x_2$, using $\lambda_n$, the sequence of negative eigenvalues on $(x_1, x_2)$ and

$$C(x, \lambda) = u(x)\prod_{k=1}^{\infty}(1 - \frac{\lambda}{\lambda_k(x)}), \quad (57)$$

where $u(x) = |\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}R_-(x)\csc\pi\mu_1$, we obtain

$$\lambda_n'(x) = \frac{\lambda_n^2(x)\lambda_n'(\alpha)u^2(\alpha)}{\lambda_n^2(\alpha)u^2(x)}e^{-2s_n(x,\lambda_n)}, \quad (58)$$

$$S_n(x, \lambda_n) = \sum_{i \neq n}\int_{\beta}^{x}\frac{\lambda_i'\lambda_n}{\lambda_i}(\lambda_i - \lambda_n)^{-1}.$$

where $x_1 < \beta < x < x_2$.

$$2\frac{\partial^2 C}{\partial x \partial \lambda}\lambda_n' + \frac{\partial^2 C}{\partial \lambda^2}(\lambda_n')^2 + \frac{\partial C}{\partial \lambda}\lambda_n'' = 0. \quad (59)$$

**Theorem 9.** For $x_2 < x < x_3$, the sequences of functions $\{\lambda_n^+\}$ and $\{\lambda_n^-\}$ satisfy

$$\lambda_n^{+''} + \frac{2d'(x)\lambda_n^{+'}}{d(x)}$$

$$+2\lambda_n^+\lambda_n^{+'}\{\sum_{i \neq n, 1 \leq i}\frac{\lambda_i^{+'}(\lambda_i^+(x) - \lambda_n^+(x))^{-1}}{\lambda_i^+}$$

$$+\sum_{i\neq n}\frac{\lambda_i^{-'}}{\lambda_i^-}(\lambda_i^+(x)-\lambda_n^+(x))^{-1}\}-2\frac{(\lambda_n^{+'})^2}{\lambda_n^+}=0, \quad (60)$$

$$\lambda_n^{-''}+\frac{2d'(x)\lambda_n^{-'}}{d(x)}$$

$$+2\lambda_n^-\lambda_n^{+'}\{\sum_{i\neq n,1\leq i}\frac{\lambda_i^{-'}}{\lambda_i^-}(\lambda_i^-(x)-\lambda_n^-(x))^{-1}$$

$$+\sum_{i\neq n}\frac{\lambda_i^{+'}(\lambda_i^-(x)-\lambda_n^-(x))^{-1}}{\lambda_i^+}\}-2\frac{(\lambda_n^{-'})^2}{\lambda_n^-}=0, \quad (61)$$

**proof.** The conditions $\varphi(x,\lambda_n^+(x))=0$ and $\varphi(x,\lambda_n^-(x))=0$ give the equations

$$2\frac{\partial^2\varphi}{\partial x\partial\lambda}\lambda_n^{+'}+\frac{\partial^2\varphi}{\partial\lambda^2}(\lambda_n^{+'})^2+\frac{\partial\varphi}{\partial\lambda}\lambda_n^{+''}=0,$$

$$2\frac{\partial^2\varphi}{\partial x\partial\lambda}\lambda_n^{-'}+\frac{\partial^2\varphi}{\partial\lambda^2}(\lambda_n^{-'})^2+\frac{\partial\varphi}{\partial\lambda}\lambda_n^{-''}=0. \quad (62)$$

From (43), we have

$$C(x,\lambda)=d(x)\prod(1-\frac{\lambda}{\lambda_n^-(x)})(1-\frac{\lambda}{\lambda_n^+(x)}). \quad (63)$$

As before, we calculate the various derivatives of $C(x,\lambda)$ and evaluate these at the fixed points $(x,\lambda_n^+(x))$, $(x,\lambda_n^-(x))$. Suppose

$$G_n=G_n(x,\lambda_n^+(x))=\prod_{k\neq n,1\leq k}(1-\frac{\lambda_n^+(x)}{\lambda_k^+(x)}), \quad (64)$$

$$H_n=H_n(x,\lambda_n^+(x))=\prod_{1\leq k}(1-\frac{\lambda_n^+(x)}{\lambda_k^-(x)}). \quad (65)$$

We have

$$\frac{\partial C}{\partial\lambda}(x,\lambda_n^+)=\frac{-dH_nG_n}{\lambda_n^+(x)},$$

$$\frac{\partial^2 C}{\partial\lambda^2}(x,\lambda_n^+)=\frac{2dH_nG_n}{\lambda_n^+(x)}\sum_{1\leq i}\frac{1}{\lambda_i^-(x)-\lambda_n^+(x)}+\frac{2dH_nG_n}{\lambda_n^+(x)}$$

$$\sum_{1\leq i,i\neq n}\frac{1}{\lambda_i^+(x)-\lambda_n^+(x)},$$

$$\frac{\partial^2 C}{\partial\lambda\partial x}(x,\lambda_n^+)=\frac{-d'(x)H_nG_n}{\lambda_n^+(x)}+\frac{d(x)\lambda_n^{+'}H_nG_n}{\lambda_n^{+2}(x)}$$

$$-\frac{d(x)\lambda_n^{+'}H_nG_n}{\lambda_n^+(x)}\sum_{1\leq i}\frac{1}{\lambda_i^-(x)-\lambda_n^+(x)}$$

$$-d(x)H_nG_n\sum_{1\leq i}\frac{\lambda_i^{-'}}{\lambda_i^-}(\lambda_i^-(x)-\lambda_n^+(x))^{-1}$$

$$-d(x)H_nG_n\sum_{1\leq i,i\neq n}\frac{\lambda_i^{+'}}{\lambda_i^+}(\lambda_i^+(x)-\lambda_n^+(x))^{-1}$$

$$-\frac{d(x)\lambda_n^{+'}H_nG_n}{\lambda_n^+(x)}\sum_{1\leq i,i\neq n}\frac{1}{\lambda_i^+(x)-\lambda_n^+(x)}$$

Placing these terms into (65), we obtain (63).
Similarly for negative Eigenvalue $\lambda_n^-(x)$ we get (64). In this case, dividing Eq. (63) by $\lambda_n^{+'}$, Eq. (64) by $\lambda_n^{-'}$ and integrating from $b$ up to $x$, we obtain

$$\lambda_n^{+'}(x)=\frac{\lambda_n^{+2}(x)\lambda_n^{+'}(b)d^2(b)}{\lambda_n^{+2}(b)d^2(x)}e^{2Z_n(x,\lambda_n^+,\lambda_n^-)}, \quad (66)$$

$$\lambda_n^{-'}(x)=\frac{\lambda_n^{-2}(x)\lambda_n^{-'}(b)d^2(b)}{\lambda_n^{-2}(b)d^2(x)}e^{2Z_n(x,\lambda_n^-,\lambda_n^+)}, \quad (67)$$

where

$$Z_n(x,\lambda_n^-,\lambda_n^+)=\sum_{i\neq n}\int_b^x\frac{\lambda_i^{+'}\lambda_n^+}{\lambda_i^+}(\lambda_i^+-\lambda_n^+)^{-1}d\nu$$

$$+\sum_i\int_b^x\frac{\lambda_i^{-'}\lambda_n^+}{\lambda_i^-}(\lambda_i^--\lambda_n^+)^{-1}d\nu \quad (68)$$

Similarly, for $x_3<x<1$,

$$\lambda_n^{+'}(x)=\frac{\lambda_n^{+2}(x)\lambda_n^{+'}(1)f^2(1)}{\lambda_n^{+2}(1)f^2(x)}e^{2Z_n(x,\lambda_n^+,\lambda_n^-)}, \quad (69)$$

$$\lambda_n^{-'}(x)=\frac{\lambda_n^{-2}(x)\lambda_n^{-'}(1)f^2(1)}{\lambda_n^{-2}(1)f^2(x)}e^{2Z_n(x,\lambda_n^-,\lambda_n^+)}, \quad (70)$$

where

$$Z_n(x,\lambda_n^-,\lambda_n^+)=\sum_{i\neq n}\int_x^1\frac{\lambda_i^{+'}\lambda_n^+}{\lambda_i^+}(\lambda_i^+-\lambda_n^+)^{-1}d\nu$$

$$+\sum_i\int_x^1\frac{\lambda_i^{-'}\lambda_n^+}{\lambda_i^-}(\lambda_i^--\lambda_n^+)^{-1}d\nu. \quad (71)$$

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