# Global Existence of Solution for Reaction Diffusion Systems

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*Abstract*—**The aim of this paper is to study the global existence in time of solutions for some class of reaction-diffusion systems. Our techniques of proof are based on Lyapunov functional methods and some $L^p$ estimates. Our goal is to show, under suitable assumptions, that the proposed model have a global solution for a large class of the functions $f$ and $g$.**

***Keywords: Global Existence, Reaction Diffusion Systems, Lyapunov Functional.***

## 1 Introduction

This paper is devoted to study the global existence of solutions of the following reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\triangle u = f(u,v) \qquad \text{in } ]0,+\infty[\times\Omega \tag{1}$$

$$\frac{\partial v}{\partial t} - b\triangle v = g(u,v) \qquad \text{in } ]0,+\infty[\times\Omega \tag{2}$$

with the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{in } ]0,+\infty[\times\partial\Omega. \tag{3}$$

Additionally we have initial conditions

$$u(0,.) = u_0,\ v(0,.) = v_0 \qquad \text{in } \Omega \tag{4}$$

where $u = u(t,x)$, $v = v(t,x)$, $x \in \Omega$, $\Delta$ denotes the Laplacian operator with respect to the $x$ variable, $\Omega$ is a regular and bounded domain of $\mathbb{R}^n$, $(n \geq 1)$, $a$ and $b$ are positive constants. The initial data are assumed to be nonnegative. Concerning the functions $f$ and $g$, we assume the following hypothesis: $f(r,s)$ and $g(r,s)$ are continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$, such that

$$f(0,s) \geq 0,\ \text{and } g(r,0) \geq 0 \qquad \forall r,s\ \geq 0 \tag{5}$$

Assume further that

$$\sup\left(\,|f(r,s)|,|g(r,s)|\right) \leq C(r+s+1)^m,\ \forall\, r,s \geq 0 \tag{6}$$

where $C$ is a positive constant and $m \geq 1$.

Also, we suppose that, one of the following conditions is satisfied:

• There exist $p \geq 2$, $c(p) > 0$ and positive numbers $(B_i(p))_{0\leq i\leq p}$ such that

$$B_i(p)f(r,s) + B_{i-1}(p)g(r,s) \leq c(p)(r+s+1) \tag{7}$$

where

$$B_i^2(p) \leq \frac{4ab}{(a+b)^2} B_{i-1}(p).B_{i+.1}(p). \tag{8}$$

• There exist $c(1) > 0$ and $B_i(1)$, $0 \leq i \leq 1$ such that

$$\begin{cases} B_1(1)f(r,s) + B_0(1)g(r,s) \leq c(1)(r+s+1) \\ B_0(1),\ B_1(1)\ > 0 \end{cases} \tag{9}$$

The study of the asymptotic behavior of the system (1)-(4) has been the object of intensive work. Many authors have discussed this problem in some particular situations, and several results concerning global existence and blow up have been established. We will mention some known results about the global existence of the system (1)-(4).

Note that, when $f(u,v) = -g(u,v) = -uv^\sigma$, Alikakos [1], studied the system (1)-(4) and established a global existence result under the assumption

$$1 < \sigma < \frac{n+2}{n}. \tag{10}$$

The method used in [1] based on some Sobolev embedding theorems. In [11] Masuda obtained a global existence result for a large class of the parameter $\sigma$. In fact, by using some $L^p$ estimates, he showed that the solution of problem (1)-(4) exists globally in time if $\sigma > 1$. We point out that it is early known by standard results for reaction diffusion systems that once we have the $L^\infty$ estimates then the global existence result will be a consequence of these estimates. While, the $L^1$-bound in

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time doesn't ensure the global existence of classical solutions except when $a = b$.

The same result in [11] was obtained by Hollis *et al* [8] by exploiting the duality arguments on $L^p$ techniques, allowing to derive the uniform boundeness of the solution.

Following Masuda's approach, Haraux and Youkana [6] established a global existence result of system (1)-(4) for a large class of the function $f$ and $g$. More precisely they showed that for

$$f(u,v) = -g(u,v) = -u\varphi(v) \tag{11}$$

the problem (1)-(4) admits a global solution provided that the following condition holds:

$$\lim_{v\to+\infty} \frac{[Log(1+\varphi(v))]}{v} = 0.$$

In the general case, that is to say for

$$f(u,v) = -g(u,v) \tag{12}$$

the positivity of the function $g(u,v)$ together with the maximum principle of the heat operator give the following uniform estimate of the solution in $L^\infty(\Omega)$

$$\|u(t)\|_\infty \le \|u_0(t)\|_\infty, \ \forall t \in [0, T_{\max}[$$

where $T_{\max}$ is the maximal time of existence. See Pazy [12] for more details.

Based on the Lyapunov functional method and for $f$ and $g$ satisfying (12), Kouachi [9] proved that the solution of problem (1)-(4) exists globally in time if

$$\lim_{v\to+\infty} \frac{[Log(1+f(u,v))]}{v} < \frac{8ab}{n(a-b)^2 \|u_0\|_\infty}$$

We also mention the result due to Bonaved and Schmitt [3] where the authors considered the problem (1)-(4) and proved a global existence result of the solution under the condition

$$\exists\, K>0,\ \sigma>0: \ |f+g| \le K(u+v+1)^\sigma$$

Recently, Pierre and Schmitt [13] have showed with a counter-example that if

$$f(u,v) + g(u,v) \le 0,$$

the solution of the above problem may blows up in finite time. For more general results on problems of reaction diffusion systems, the interested reader is referred to [4, 5, 10, 2, 13, 14, 3] and the references cited therein for more detailed account of the reaction diffusion systems.

In the present work we consider the problem (1)-(4), where the function $f$ and $g$ are assumed to satisfy the condition (6) and by adopting the Lyapunov method combined with some $L^p$ estimates we will establish a global existence result of the solution.

The content of this paper is as follows. In section 2, we introduce some notations and give a local existence result. Our main result is stated in section 3.

## 2 Local existence

In this section, we present some material that we shall use in this paper, and for the sake of completeness, we state a local existence result of the solution. By $(.,.)$ we denote the scalar product in $L^2(\Omega)$ i.e. $(u,v)(t) = \int_\Omega u(x,t)v(x,t)dx$, and we mean by $\|.\|_p$ the $L^p(\Omega)$ norm for $1 \le p \le \infty$, i.e. $\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p dx$ and $\|u\|_\infty = ess \sup_{x\in\Omega} |u(x)|$, also we denote by $\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|$, the usual norms in $C(\overline{\Omega})$.

Since the functions $f$ and $g$ are continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$ then, for any initial data in $C(\overline{\Omega})$ it is easy to check the Lipschitz continuity on bounded subsets of the domain associated to the operator

$$A := \begin{pmatrix} -a\Delta & 0 \\ 0 & -b\Delta \end{pmatrix}.$$

Then, from the basic existence theory ( see Pazy [12]) the problem (1)-(4) admits unique classical solution $(u,v)$ defined on $[0, T_{\max}[ \times \Omega$. More precisely, under the above assumptions, we have the following theorem.

**Theorem 2.1** *([9]. Proposition 2.1). System (1)-(4) admits a unique classical solution $(u,v)$ defined on $(0, T_{max}] \times \Omega$. Moreover, if $T_{max} < \infty$, then*

$$\lim_{t\to T_{max}} \{\|u(t,.)\|_\infty + \|v(t,.)\|_\infty\} = \infty.$$

*In this case $T_{max}(\|u_0\|_\infty, \|v_0\|_\infty)$ called the blowing up time.*

## 3 Global existence

In this section we state and prove our main result on the global existence of solution of problem (1)-(4). To do this, it is well known that, it suffices to derive an uniform estimate of the quantity

$$\sup\left(\|f(u,v)\|_q, \|g(u,v)\|_q\right)$$

for some $q > n/2$. (See [7] for instance).

The following lemma is the key element of the proof of our result.

**Lemma 3.1** *Let $(u(t,.), v(t,.))$ be a solution of (1)-(4). If one of the conditions (7) or (9) has been satisfied, there would exist an integer $p \geq 1$ and a continuous function $C_p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that*

$$\sup\left(\|u(t,.)\|_p, \|v(t,.)\|_p\right) \leq C_p(t), \; t < T_{\max}$$

**Proof.** Let us consider the function $L_p$ defined by

$$\begin{aligned} L_p(t) &= \int_\Omega \left(\sum_{i=0}^{p} C_p^i B_i(p) u^i v^{p-i}\right) dx \\ &= \int_\Omega \left(\sum_{i=0}^{p} \alpha_i(p) u^i v^{p-i}\right) dx \end{aligned} \tag{13}$$

where

$$\alpha_i(p) = C_p^i B_i(p), \qquad i = 0, ..., p. \tag{14}$$

Differentiating $L_p$ with respect to $t$ yields

$$L_p'(t) = \int_\Omega \left(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i}\right) \frac{\partial u}{\partial t} dx + \int_\Omega \left(\sum_{i=0}^{p-1} (p-i) \alpha_i(p) u^i v^{p-i-1}\right) \frac{\partial v}{\partial t} dx$$

Consequently,

$$L_p'(t) = \int_\Omega \left(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i}\right) \frac{\partial u}{\partial t} dx + \int_\Omega \left(\sum_{i=1}^{p} (p-i+1) \alpha_{i-1}(p) u^{i-1} v^{p-i}\right) \frac{\partial v}{\partial t} dx$$

Also, a simple computation leads

$$\begin{aligned} L_p'(t) &= \int_\Omega \left(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i}\right) \times (f(u,v) + a\Delta u) dx \\ &+ \int_\Omega \left(\sum_{i=1}^{p} (p-i+1) \alpha_{i-1}(p) u^{i-1} v^{p-i}\right) \times (g(u,v) + b\Delta v) dx \end{aligned}$$

which implies

$$\begin{aligned} L_p'(t) &= \left(\sum_{i=1}^{p} ai\alpha_i(p) u^{i-1} v^{p-i} \Delta u\right) dx \\ &+ \int_\Omega \left(\sum_{i=1}^{p} b(p-i+1) \alpha_{i-1}(p) u^{i-1} v^{p-i} \Delta v\right) dx \\ &+ \int_\Omega \left(\sum_{i=1}^{p} (p-i+1) \alpha_{i-1}(p) u^{i-1} v^{p-i} g(u,v)\right) dx \\ &+ \int_\Omega \left(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i} f(u,v)\right) dx \end{aligned}$$

From the above equality, it follows that

$$\begin{aligned} L_p'(t) &= \int_\Omega \sum_{i=1}^{p} i\alpha_i(p) f(u,v) u^{i-1} v^{p-i} dx \\ &+ \int_\Omega \sum_{i=1}^{p} (p-i+1) \alpha_{i-1}(p) g(u,v) u^{i-1} v^{p-i} dx \\ &+ \int_\Omega \sum_{i=1}^{p} \{ai\alpha_i(p) \Delta u\} u^{i-1} v^{p-i} dx \\ &+ \int_\Omega \sum_{i=1}^{p} (b(p-i+1) \alpha_{i-1}(p) \Delta v) u^{i-1} v^{p-i} dx \end{aligned} \tag{15}$$

At this point, we distinguish two cases:

**Case 1:** when $p = 1$, we obtain from (15)

$$L_1'(t) = \int_\Omega (a\alpha_1(1) \Delta u + b\alpha_0(1) \Delta v) dx + \int_\Omega (\alpha_1(1) f(u,v) + \alpha_0(1) g(u,v)) dx$$

By a simple use of Green's formula, we obtain

$$\begin{aligned} L_1'(t) &= \int_\Omega (\alpha_1(1) f(u,v) + \alpha_0(1) g(u,v)) dx \\ &= \int_\Omega (B_1(1) f(u,v) + B_0(1) g(u,v)) dx \end{aligned}$$

Using condition (9) we deduce,

$$\begin{aligned} L_1'(t) &\leq c(1) \int_\Omega (u+v+1) dx \\ &= c(1) \int_\Omega (u+v) dx + c(1)\, mes(\Omega) \end{aligned}$$

Then the functional $L_1$ satisfies

$$L_1'(t) \leq c_1(1) L_1(t) + c_2(1), \; \forall t < T_{\max}$$

where

$$c_1(1)=\frac{c(1)}{\min(\alpha_1(1),\alpha_0(1))},\; c_2(1)=c(1)\,mes(\Omega)$$

A simple integration of the above inequality gives, for all $t<T_{\max}$

$$L_1(t)\le\left[L_1(0)+\tfrac{c_2(1)}{c_1(1)}\right]\exp(c_1(1)t)-\tfrac{c_2(1)}{c_1(1)}.$$

It's not hard to see that from (13) we obtain

$$\begin{aligned}L_1(t)&\ge\min(\alpha_1(1),\alpha_0(1))\int_\Omega(u+v)\,dx\\&\ge\min(\alpha_1(1),\alpha_0(1))\sup\left(\|u(t,.)\|_1,\|v(t,.)\|_1\right)\end{aligned}$$

Then we get

$$\sup\|u(t,.)\|_1,\|v(t,.)\|_1\le c_1(t),\;\forall\;\; t\;<T_{\max}\qquad(16)$$

where

$$\begin{aligned}c_1(t)=&\tfrac{1}{\min(\alpha_1(1),\alpha_0(1))}\\&\times\left\{\left[L_1(0)+\tfrac{c_2(1)}{c_1(1)}\right]\exp(c_1(1)t)-\tfrac{c_2(1)}{c_1(1)}\right\}\end{aligned}$$

**Case 2:** when $p\ge2$, we set

$$\begin{aligned}T=&\int_\Omega\sum_{i=1}^{p}\{a_i\alpha_i(p)\Delta u\}u^{i-1}v^{p-i}dx\\&+\int_\Omega\sum_{i=1}^{p}\{b(p-i+1)\alpha_{i-1}(p)\Delta v\}u^{i-1}v^{p-i}dx\\&=\int_\Omega\sum_{i=1}^{p}\Delta\{ai\alpha_i(p)u\}u^{i-1}v^{p-i}dx\\&+\int_\Omega\sum_{i=1}^{p}\Delta(b(p-i+1)\alpha_{i-1}(p)v)u^{i-1}v^{p-i}dx\end{aligned}$$

which implies

$$\begin{aligned}T=&\sum_{i=1}^{p}\int_\Omega\Delta\{ai\alpha_i(p)u\}u^{i-1}v^{p-i}dx\\&+\sum_{i=1}^{p}\int_\Omega\Delta\{b(p-i+1)\alpha_{i-1}(p)v\}u^{i-1}v^{p-i}dx\end{aligned}$$

Then, Green's formula gives

$$\begin{aligned}T=&-\sum_{i=1}^{p}\int_\Omega\nabla\{ai\alpha_i(p)u\}\nabla\left(u^{i-1}v^{p-i}\right)dx\\&-\sum_{i=1}^{p}\int_\Omega\nabla\{b(p-i+1)\alpha_{i-1}(p)v\}\nabla\left(u^{i-1}v^{p-i}\right)dx\end{aligned}$$

which implies

$$\begin{aligned}T=&-\int_\Omega\sum_{i=2}^{p}a(i-1)i\alpha_i(p)u^{i-2}v^{p-i}\nabla^2udx\\&+\int_\Omega\sum_{i=1}^{p-1}ai(p-i)u^{i-1}v^{p-i-1}\nabla u\nabla vdx\\&+\int_\Omega\sum_{i=2}^{p}b(i-1)(p-i+1)\alpha_{i-1}(p)u^{i-2}v^{p-i}\nabla u\nabla vdx\\&+\int_\Omega\sum_{i=1}^{p-1}b(p-i+1)(p-i)\alpha_{i-1}(p)u^{i-1}v^{p-i-1}\nabla^2vdx\end{aligned}$$

and therefore,

$$\begin{aligned}T=-\Big\{&\int_\Omega\sum_{i=1}^{p-1}[ai(i+1)\alpha_{i+1}(p)\nabla^2u\\&+d(a+b)i(p-i)\alpha_i(p)\nabla u\nabla v\\&+b(p-i)(p-i+1)\alpha_{i-1}(p)\nabla^2v]u^{i-1}v^{p-i-1}dx\Big\}\end{aligned}$$

Hence, (15) becomes

$$\begin{aligned}L_p'(t)=&\int_\Omega\sum_{i=1}^{p}\{i\alpha_i(p)f\}u^{i-1}v^{p-i}dx\\&+\int_\Omega\sum_{i=1}^{p}\{(p-i+1)\alpha_{i-1}(p)g\}u^{i-1}v^{p-i}dx\\&-\int_\Omega\Big\{\sum_{i=1}^{p-1}ai(i+1)\alpha_{i+1}(p)\nabla^2u\\&+(a+b)i(p-i)\alpha_i(p)\nabla u\nabla v\\&+b(p-i)(p-i+1)\alpha_{i-1}(p)\nabla^2v]u^{i-1}v^{p-i-1}\Big\}dx\end{aligned}$$

Since $\alpha_i(p)=C_p^iB_i(p),\; i=0,...,p$ then,

$$\begin{aligned}L_p'(t)&=\int_\Omega\sum_{i=1}^{p}\left\{iC_p^iB_i(p)f\right\}u^{i-1}v^{p-i}dx\\&+\int_\Omega\Big\{\sum_{i=1}^{p}\left\{(p-i+1)C_p^{i-1}B_{i-1}(p)g\right\}\\&\times\left(u^{i-1}v^{p-i}\right)\Big\}dx\\&-\int_\Omega\{\sum_{i=1}^{p-1}[ai(i+1)C_p^{i+1}B_{i+1}(p)\nabla^2u\\&+(a+b)i(p-i)C_p^iB_i(p)\nabla u\nabla v\\&+b(p-i)(p-i+1)\\&\times C_p^{i-1}B_{i-1}(p)\nabla^2v]u^{i-1}v^{p-i-1}\}dx\end{aligned}$$

Using the fact that

$$iC_p^i = (p-i+1)\,C_p^{i-1} = pC_{p-1}^{i-1}.$$

and also

$$\begin{aligned} i\,(i+1)\,C_p^{i+1} &= i\,(p-i)\,C_p^i \\ &= (p-i)\,(p-i+1)\,C_p^{i-1} \\ &= p\,(p-1)\,C_{p-2}^{i-1}. \end{aligned}$$

we conclude

$$\begin{aligned} L_p'(t) &= \int_\Omega \Big\{\sum_{i=1}^{p} pC_{p-1}^{i-1}\left[B_i(p)\,f + B_{i-1}(p)\,g\right] \\ &\times \left(u^{i-1}v^{p-i}\right)\Big\}dx \\ &- p\,(p-1)\int_\Omega \{\sum_{i=1}^{p-1} C_{p-2}^{i-1}[aB_{i+1}(p)\,\nabla^2 u \\ &+ (a+b)\,B_i(p)\,\nabla u\nabla v \\ &+ bB_{i-1}(p)\,\nabla^2 v]u^{i-1}v^{p-i-1}\}dx. \end{aligned}$$

The quadratic forms

$$aB_{i+1}(p)\,\nabla^2 u + (a+b)\,B_i(p)\,\nabla u\nabla v + bB_{i-1}(p)\,\nabla^2 v$$

are positive since from (8) we have

$$B_i^2(p) \le \frac{4ab}{(a+b)^2} B_{i-1}(p)\,B_{i+1}(p)$$

Consequently,

$$\begin{aligned} L_p'(t) \le\ & p\int_\Omega \sum_{i=1}^{p} C_{p-1}^{i-1}\{B_i(p)\,f(u,v) \\ & +B_{i-1}(p)\,g(u,v)\}u^{i-1}v^{p-i}dx \end{aligned}$$

Using condition (7) we deduce that

$$\begin{aligned} L_p'(t) &\le c'(p)\int_\Omega \left(\sum_{i=1}^{p} C_{p-1}^{i-1}(u+v+1)\,u^{i-1}v^{p-i}\right)dx \\ &\le c'(p)\int_\Omega \sum_{i=1}^{p} C_{p-1}^{i-1}u^i v^{p-i}dx \\ &+ c'(p)\int_\Omega \sum_{i=1}^{p} C_{p-1}^{i-1}u^{i-1}v^{p-i+1}dx \\ &+ c'(p)\int_\Omega \sum_{i=1}^{p} C_{p-1}^{i-1}u^{i-1}v^{p-i}dx \\ &\le c'(p)\int_\Omega \sum_{i=1}^{p} C_{p-1}^{i-1}u^i v^{p-i}dx \\ &+ c'(p)\int_\Omega \sum_{i=0}^{p-1} C_{p-1}^{i}u^i v^{p-i}dx \\ &+ c'(p)\int_\Omega \sum_{i=0}^{p-1} C_{p-1}^{i}u^i v^{p-i-1}dx \\ &\le c'(p)\int_\Omega \left(\sum_{i=0}^{p} C_p^i u^i v^{p-i}\right)dx \\ &+ c'(p)\int_\Omega \left(\sum_{i=0}^{p-1} C_{p-1}^{i}u^i v^{p-i-1}\right)dx \end{aligned}$$

Using the fact that

$$\sum_{i=0}^{p-1} C_{p-1}^{i}u^i v^{p-i-1} = (u+v)^{p-1}$$

Therefore, the last inequality can be written as

$$L_p'(t) \le c_1(p)\,L_p(t) + c'(p)\int_\Omega (u+v)^{p-1}\,dx$$

Applying Hôlder's inequality to the second term in the right hand side of the above inequality, we obtain

$$\begin{aligned} L_p'(t) &\le c_1(p)\,L_p(t) + c'(p) \\ &\times (mes(\Omega))^{\frac{1}{p}}\left(\int_\Omega (u+v)^p\,dx\right)^{\frac{p-1}{p}} \end{aligned}$$

Since the following inequality holds,

$$(u+v)^p = \sum_{i=0}^{p} C_p^i u^i v^{p-i} \le \frac{\sup\limits_{0\le i\le p} C_p^i}{\min\limits_{0\le i\le p}\alpha_i(p)} \sum_{i=0}^{p}\alpha_i(p)\,u^i v^{p-i}$$

Then, we have

$$L_p'(t) \leq c_1(p) L_p(t) + c'(p)(mes\Omega)^{\frac{1}{p}} \times \left(\frac{\sup\limits_{0\leq i\leq p} C_p^i}{\min\limits_{0\leq i\leq p} \alpha_i(p)}\right)^{\frac{p-1}{p}} (L_p(t))^{\frac{p-1}{p}}$$

Hence, the functional $L_p$ satisfies the following differential inequality

$$L_p'(t) \leq c_1(p) L_p(t) + c_2(p)(L_p(t))^{\frac{p-1}{p}}, \forall t < T_{\max} \tag{17}$$

$$c_2(p) = c'(p)(mes\Omega)^{\frac{1}{p}} \left(\frac{\sup\limits_{0\leq i\leq p} C_p^i}{\min\limits_{0\leq i\leq p} \alpha_i(p)}\right)^{\frac{p-1}{p}}$$

which gives us, by a simple integration

$$(L_p(t))^{\frac{1}{p}} \leq \left[(L_p(0))^{\frac{1}{p}} + \frac{c_2'(p)}{c_1'(p)}\right] \exp(c_1'(p)t) - \frac{c_2'(p)}{c_1'(p)}$$

where

$$c_1'(p) = \frac{c_1(p)}{p} \text{ and } c_2'(p) = \frac{c_2(p)}{p}$$

By using the inequality

$$\begin{aligned} L_p(t) &= \int_\Omega \left(\sum_{i=0}^{p} \alpha_i(p) u^i v^{p-i}\right) dx \\ &\geq \int_\Omega [\alpha_p(p) u^p + \alpha_0(p) v^p] dx \end{aligned} \tag{18}$$

It follows that

$$L_p(t) \geq \min(\alpha_0(p),\ \alpha_p(p)) \sup\left(\int_\Omega u^p dx, \int_\Omega v^p dx\right)$$

Hence,

$$\begin{aligned} (L_p(t))^{\frac{1}{p}} &\geq [\min(\alpha_0(p),\ \alpha_p(p))]^{\frac{1}{p}} \\ &\times \sup\left(\left(\int_\Omega u^p dx\right)^{\frac{1}{p}}, \left(\int_\Omega v^p dx\right)^{\frac{1}{p}}\right) \end{aligned}$$

And therefore, for all $t < T_{\max}$

$$\sup\left(\|u(t,.)\|_p, \|v(t,.)\|_p\right) \leq \frac{(L_p(t))^{\frac{1}{p}}}{[\min(\alpha_0(p),\ \alpha_p(p))]^{\frac{1}{p}}}, \tag{19}$$

With (18) and (19) we obtain

$$\sup\left(\|u(t,.)\|_p, \|v(t,.)\|_p\right) \leq c_p(t),\ \ \forall\ t < T_{\max} \tag{20}$$

where

$$\begin{aligned} c_p(t) &= \frac{1}{[\min(\alpha_0(p), \alpha_p(p))]^{\frac{1}{p}}} \\ &\times \left\{\left((L_p(0))^{\frac{1}{p}} + \frac{c_2'(p)}{c_1'(p)}\right) e^{\left(c_1'(p)t\right)} - \frac{c_2'(p)}{c_1'(p)}\right\}. \end{aligned} \tag{21}$$

The proof of Lemma 3.1 is complete.

Our main result of this paper reads as follows:

**Theorem 3.1** *Let $(u(t,.), v(t,.))$ be a solution of problem (1)-(4). We assume that the condition (6) holds and one of the conditions (7) or (9) are satisfied . In addition if $p > \frac{mn}{2}$, then the solution $(u(t,.), v(t,.))$ exists globally in time.*

**Proof.** From (6) we have

$$\sup(\ |f\ (u,v)|, |g(u,v)|) \leq C(u+v+1)^m.$$

Then, it follows that

$$\sup\left(\int_\Omega |f\ (u,v)|^{\frac{p}{m}} dx, \int_\Omega |g(u,v)|^{\frac{p}{m}} dx\right) \leq C^{\frac{p}{m}} \int_\Omega (u+v+1)^p dx$$

which implies

$$\sup\left(\|f(u,v)\|_{\frac{p}{m}}^{\frac{p}{m}}, \|g(u,v)\|_{\frac{p}{m}}^{\frac{p}{m}}\right) \leq C^{\frac{p}{m}} \int_\Omega (u+v+1)^p dx. \tag{22}$$

On the other hand, we have

$$\begin{aligned} \int_\Omega (u+v+1)^p dx &= \int_\Omega \left[\sum_{k=0}^{p} C_p^k (u+v)^k\right] dx \\ &= \int_\Omega [1+(u+v)^p] dx \\ &+ \sum_{k=1}^{p-1} C_p^k \int_\Omega (u+v)^k dx \end{aligned}$$

An application of Hôlder's inequality leads

$$\sum_{k=1}^{p-1} \int_\Omega (u+v)^k dx$$

$$\leq \sum_{k=1}^{p-1} C_p^k \left[\left(\int_\Omega 1^{\frac{p}{p-k}} dx\right)^{\frac{p-k}{p}} \left(\int_\Omega (u+v)^p dx\right)^{\frac{k}{p}}\right].$$

Hence

$$\begin{aligned}&\int_\Omega (u+v+1)^p\,dx\\ &\le mes\,(\Omega)+\int_\Omega (u+v)^p\,dx\\ &+\sum_{k=1}^{p-1} C_p^k\,(mes\,(\Omega))^{\frac{p-k}{p}}\left(\int_\Omega (u+v)^p\,dx\right)^{\frac{k}{p}}\end{aligned} \tag{23}$$

using (21) we get

$$\begin{aligned}\left(\int_\Omega (u+v)^p\,dx\right)^{\frac{1}{p}} &= \|u(t,.)+v(t,.)\|_p\\ &\le \|u(t,.)\|_p+\|v(t,.)\|_p\\ &\le 2c_p(t)\end{aligned}$$

and the inequality (23) can be written as follows

$$\begin{aligned}&\int_\Omega (u+v+1)^p\,dx\\ &\le mes\,(\Omega)+2^p\,(c_p(t))^p+\sum_{k=1}^{p-1}2^kC_p^k\,(mes\,(\Omega))\,(c_p(t))^k\\ &\le \sum_{k=0}^{p}2^kC_p^k\,(mes\,(\Omega))\,(c_p(t))^k\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup\left(\|f(u,v)\|_{\frac{p}{m}}^{\frac{p}{m}},\|g(u,v)\|_{\frac{p}{m}}^{\frac{p}{m}}\right)\\ &\le C^{\frac{p}{m}}\left[\sum_{k=0}^{p}2^kC_p^k\,(mes\,(\Omega))\,(c_p(t))^k\right]\end{aligned}$$

which gives that

$$\sup\|f(u,v)\|_{\frac{p}{m}},\|g(u,v)\|_{\frac{p}{m}}\le c_{p,m}(t)\,,\ \forall t<T_{\max} \tag{24}$$

where

$$c_{p,m}(t)=c\left[\sum_{k=0}^{p}2^kC_p^k\,(mes\,(\Omega))\,(c_p(t))^k\right]^{\frac{m}{p}}\ \text{and}\ \frac{p}{m}>\frac{n}{2} \tag{25}$$

**Remark 3.1** *It's clear that conditions (5) implies the positivity of the solution on its interval of existence. See [7], for more details.*

**Remark 3.2** *From both Lemma 3.1 and Theorem 3.1, we have obtained an uniform estimate of* $\sup\left(\|f(u,v)\|_q,\|g(u,v)\|_q\right)$ *with* $q=\frac{p}{m}>\frac{n}{2}$. *By the preliminary remarks, we conclude that the solution of the given problem exists globally in time.*

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