L-Fuzzy Ternary Subsemirings and L-Fuzzy Ideals in Ternary Semirings

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Abstract—**In this paper, we investigate some properties of L-fuzzy ternary subsemiring and L-fuzzy ideals in ternary semirings.**

Keywords: ***L-fuzzy subsets, L-fuzzy ternary subsemirings, L-fuzzy ideals, L-fuzzy characteristic ideals, normal L-fuzzy subsets***

1 Introduction

The notion of ternary algebraic system was introduced by Lehmer [13] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [14] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [1] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings ([1], [2], [3], [4], [5], [6], [7] and [10]).

The theory of fuzzy sets was first studied by Zadeh [17] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. In 1988, Zhang [18] studied prime L-fuzzy ideals and primary L-fuzzy ideals in rings where L is a completely distributive lattice. The concepts of L-fuzzy ideals in semirings were studied by Jun, Neggers and Kim in [8], [9], [15] and [16]. Recently Kavikumar, Khamis and Jun studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings in [11] and [12]. The main purpose of this paper is to study L-fuzzy ideals in ternary semirings.

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2 Preliminaries

In this section, we refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic systems that are necessary for this paper. For more details we refer to papers in the references.

Definition 2.1. A nonempty set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) is called a *semiring* if $(S,+)$ is a commutative semigroup, $(S,\cdot)$ is a semigroup and multiplicative distributes over addition both from the left and from the right, i.e., $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a,b,c \in S$.

Definition 2.2. A nonempty set S together with a binary operation and a ternary operation called addition $+$ and ternary multiplication, respectively, is said to be a *ternary semiring* if $(S,+)$ is a commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a+b)cd = acd+bcd$,

(iii) $a(b+c)d = abd+acd$ and

(iv) $ab(c+d) = abc+abd$ for all $a,b,c,d,e \in S$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider $\mathbb{Z}^-$, the set of all negative integers under usual addition and multiplication, we see that $\mathbb{Z}^-$ is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, $\mathbb{Z}^-$ is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 2.3. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0+x = x = x+0$ and $0xy = x0y = xy0 = 0$ for all $x,y \in S$, then 0 is called the *zero element* or simply the *zero* of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

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Definition 2.4. An additive subsemigroup T of S is called a *ternary subsemiring* of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.5. An additive subsemigroup I of S is called a *left [resp. right, lateral] ideal* of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an *ideal* of S.

Throughout this paper, let $L = (L, \leq, \wedge, \vee)$ be a completely distributive lattice which has the least and the greatest elements, say 0 and 1, respectively.

Definition 2.6. Let X be a nonempty set. An *L-fuzzy subset* of X is a map $\mu : X \to L$, and $F(X)$ denote the set of all L-fuzzy subsets of X.

For $\mu, \nu \in F(X)$, let $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$. It is easy to see that $F(X) = (F(X), \subseteq, \wedge, \vee)$ is a completely distributive lattice, which has the least and the greatest elements, say $\mathbf{0}$ and $\mathbf{1}$, respectively, where $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in X$.

Proposition 2.1([9]). Let f be a mapping from a set X to a set Y and $\mu \in F(X)$. Then for every $t \in L, t \neq 0$,

$$(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s}).$$

Definition 2.7. Let S and R be ternary semirings. A mapping $\varphi : S \to R$ is called a *homomorphism* if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xyz) = \varphi(x)\varphi(y)\varphi(z)$ for all $x, y, z \in S$.

Given any two sets X and Y, let $\mu \in F(X)$ and let $f : X \to Y$ be a function. Define $\nu \in F(Y)$ by for $y \in Y$,

$$\nu(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We call ν the *image* of μ under f which is denoted by $f(\mu)$. Conversely, for $\nu \in F(f(x))$, define $\mu \in F(X)$ by $\mu(x) = \nu(f(x))$ for all $x \in X$, and we call μ the *preimage* of ν under f which is denoted by $f^{-1}(\nu)$.

3 Main Results

Definition 3.1. An L-fuzzy subset μ of a ternary semiring S is called an *L-fuzzy ternary subsemiring* of S if $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in S$.

Definition 3.2. An L-fuzzy ternary subsemiring μ of a ternary semiring S is called an *L-fuzzy left [resp. right, lateral] ideal* of S if $\mu(xyz) \geq \mu(z)$ [resp. $\mu(xyz) \geq \mu(x), \mu(xyz) \geq \mu(y)$] for all $x, y, z \in S$. If μ is an L-fuzzy left, right and lateral ideal of S, then I is called an *L-fuzzy ideal* of S.

Let S be a ternary semiring and $\mu \in F(S)$. Let

$$\mu_t = \{x \in S \mid \mu(x) \geq t\},$$

which is called a t-level subset of μ. Note that

(i) for $s, t \in L, s < t$ implies $\mu_t \subseteq \mu_s$ and

(ii) for $t \in L$, $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s \in L$ such that $s > t$.

Theorem 3.1. Let S be a ternary semiring and $\mu \in F(S)$. The following statements are true.

(i) μ is an L-fuzzy ternary subsemiring of S if and only if for any $t \in L$ such that $\mu_t \neq \emptyset, \mu_t$ is a ternary subsemiring of S.

(ii) μ is an L-fuzzy left [resp. right, lateral] ideal of S if and only if for any $t \in L$ such that $\mu_t \neq \emptyset, \mu_t$ is a left [resp. right, lateral] ideal of S.

(iii) μ is an L-fuzzy ideal of S if and only if for any $t \in L$ such that $\mu_t \neq \emptyset, \mu_t$ is an ideal of S.

Proof. (i) Let μ be an L-fuzzy ternary subsemiring of S. Let $t \in L$ such that $\mu_t \neq \emptyset$. Let $x, y, z \in \mu_t$. Then $\mu(x), \mu(y), \mu(z) \geq t$. Then $\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \geq t$ and $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \geq t$. So $xyz \in \mu_t$. Hence μ_t is a ternary subsemiring of S. Conversely, let $x, y, z \in S$ and $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x), \mu(y) \geq t$. Thus $x, y \in \mu_t$. By assumption, $x + y \in \mu_t$. So $\mu(x + y) \geq t = \min\{\mu(x), \mu(y)\}$. Next, let $s = \min\{\mu(x), \mu(y), \mu(z)\}$. Then $\mu(x), \mu(y), \mu(z) \geq s$. Thus $x, y, z \in \mu_s$. By assumption, $xyz \in \mu_s$. So $\mu(xyz) \geq s = \min\{\mu(x), \mu(y), \mu(z)\}$. Therefore μ is an L-fuzzy ternary subsemiring of S.

(ii) Let μ be an L-fuzzy left ideal of S. Let $t \in L$ such that $\mu_t \neq \emptyset$. Let $x, y \in \mu_t$. Then $\mu(x), \mu(y) \geq t$. Then $\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \geq t$. Next, let $x, y \in S$ and $z \in \mu_t$. We have $\mu(xyz) \geq \mu(z) \geq t$. Thus $xyz \in \mu_t$. Therefore μ_t is a left ideal of S. Conversely, let $x, y, z \in S$ and $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x), \mu(y) \geq t$. Thus $x, y \in \mu_t$. By assumption, $x + y \in \mu_t$. So $\mu(x + y) \geq t = \min\{\mu(x), \mu(y)\}$. Next, let $s = \mu(z)$. Then $\mu(z) \geq s$. Thus $z \in \mu_s$. By assumption, $xyz \in \mu_s$. So $\mu(xyz) \geq s = \mu(z)$. Therefore μ is an L-fuzzy left ideal of S. The proofs of other cases are similar.

(iii) follows from (ii). □

Theorem 3.2. Let S be a ternary semiring. The following statements are true.

(i) If A is a ternary subsemiring of S, then there exists an L-fuzzy ternary subsemiring μ of S such that $\mu_t = A$ for some $t \in L$.

(ii) If A is a left [resp. right, lateral] ideal of S, then there exists an L-fuzzy left [resp. right, lateral] ideal μ of S such that $\mu_t = A$ for some $t \in L$.

(iii) If A is an ideal of S, then there exists an L-fuzzy ideal μ of S such that $\mu_t = A$ for some $t \in L$.

Proof. (i) Let $t \in L$ and define an L-fuzzy set of S by

$$\mu(x) = \begin{cases} t & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\mu_t = A$. For $s \in L$ we have

$$\mu_s = \begin{cases} S & \text{if } s = 0, \\ A & \text{if } 0 < s \leq t, \\ \emptyset & \text{otherwise .} \end{cases}$$

Since A and S are ternary subsemirings of S, it follows that every non-empty level subset μ_s of μ is a ternary subsemiring of S. By Theorem 3.1 (i), μ is an L-fuzzy ternary subsemiring of S.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Theorem 3.3. Let S be a ternary semiring with zero and $S_\mu = \{x \in S \mid \mu(x) \geq \mu(0)\}$. The following statements are true.

(i) If μ is an L-fuzzy ternary subsemiring of S, then S_μ is a ternary subsemiring of S.

(ii) If μ is an L-fuzzy left [resp. right, lateral] ideal of S, then S_μ is a left [resp. right, lateral] ideal of S.

(iii) If μ is an L-fuzzy ideal of S, then S_μ is an ideal of S.

Proof. (i) Let μ be an L-fuzzy ternary subsemiring of S and $x, y, z \in S_\mu$. So $\mu(x), \mu(y), \mu(z) \geq \mu(0)$. Thus $\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \geq \mu(0)$ and $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \geq \mu(0)$. So $x + y, xyz \in S_\mu$. Thus S_μ is a ternary subsemiring of S.

The proof of (ii) is similar to the proof of (i).

(iii) follows from (ii). □

Let S be a ternary semiring. If $\mu \in F(S)$ is an L-fuzzy ternary subsemiring of S, we call $\mu_t (\neq \emptyset)$ a level ternary subsemiring of μ. The level left ideals [resp. right ideals, lateral ideals, ideals] of μ are defined analogously.

Lemma 3.4. Let S be a ternary semiring and $\mu \in F(S)$. If μ is an L-fuzzy subset of S, then two level set μ_s and μ_t (with $s < t$ in L) of μ are equal if and only if there is no $x \in S$ such that $s \leq \mu(x) < t$.

Proof. Let $s, t \in L$ such that $s < t$ and $\mu_s = \mu_t$. If there exist $x \in S$ such that $s \leq \mu(x) < t$. Then $x \in \mu_s$ but $x \notin \mu_t$, a contradiction. Conversely, assume that there is no $x \in S$ such that $s \leq \mu(x) < t$. Let $y \in \mu_s$. Then $\mu(y) \geq s$. By assumption, $\mu(y) \geq t$. Hence $y \in \mu_t$. This implies that $\mu_s = \mu_t$. □

Let S be a ternary semiring. For any $\mu \in F(S)$, we denote by $Im(\mu)$ the image set of μ.

Theorem 3.5. Let S be a ternary semiring. The following statements are true.

(i) Let μ be an L-fuzzy subset of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of set μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level subset of μ.

(ii) Let μ be an L-fuzzy ternary subsemiring of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of ternary subsemirings μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level ternary subsemirings of μ.

(iii) Let μ be an L-fuzzy left [resp. right, lateral] ideal of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of left [resp. right, lateral] ideals μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level left [resp. right, lateral] ideals of μ.

(iv) Let μ be an L-fuzzy ideal of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of ideals μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level ideals of μ.

Proof. (i) If $t \in L$ with $t < t_1$, we have that $\mu_t = \mu_{t_1}$. If $t \in L$ with $t > t_n$, we have that $\mu_t = \emptyset$. If $t \in L$ with $t_i < t < t_{i+1}$ for some $i = 1, 2, \ldots, n-1$, By assumption, there is no $x \in R$ such that $t \leq \mu(x) < t_{i+1}$. By Lemma 3.4, $\mu_t = \mu_{t_{i+1}}$.

The proofs of (ii), (iii) and (iv) are similar to the proof of (i). □

Theorem 3.6. Let S and R be ternary semirings and $\varphi : S \to R$ be a onto homomorphism. The following statements are true.

(i) Let μ be an L-fuzzy ternary subsemiring of R. Then the preimage of μ under φ is an L-fuzzy ternary subsemiring of S.

(ii) Let μ be an L-fuzzy left [resp. right, lateral] ideal of R. Then the preimage of μ under φ is an L-fuzzy left [resp. right, lateral] ideal of S.

(iii) Let μ be an L-fuzzy ideal of R. Then the preimage of μ under φ is an L-fuzzy ideal of S.

Proof. (i) Let $\mu \in F(R)$ be an L-fuzzy ternary subsemiring and ν be the preimage of μ under φ. Then for any $x, y, z \in S$,

$$\begin{aligned}\nu(x+y) &= \mu(\varphi(x+y)) \\ &= \mu(\varphi(x)+\varphi(y)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y))\} \\ &= \min\{\nu(x), \nu(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu(xyz) &= \mu(\varphi(xyz)) \\ &= \mu(\varphi(x)\varphi(y)\varphi(z)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y)), \mu(\varphi(z))\} \\ &= \min\{\nu(x), \nu(y), \nu(z)\}\end{aligned}$$

This shows that ν is an L-fuzzy ternary subsemiring of S.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Theorem 3.7. Let S and R be ternary semirings and $\varphi : S \to R$ be an onto homomorphism. The following statements are true.

(i) Let μ be an L-fuzzy ternary subsemiring of S. Then the homomorphic image $\varphi(\mu)$ of μ under φ is an L-fuzzy ternary subsemiring of R.

(ii) Let μ be an L-fuzzy left [resp. right, lateral] ideal of S. Then the homomorphic image $\varphi(\mu)$ of μ under φ is an L-fuzzy left [resp. right, lateral] ideal of R.

(iii) Let μ be an L-fuzzy ideal of S. Then the homomorphic image $\varphi(\mu)$ of μ under φ is an L-fuzzy ideal of R.

Proof. (i) By Theorem 3.1 (i), it is sufficient to show that each nonempty level subset of $\varphi(\mu)$ is a ternary subsemiring of R. Let $t \in L$ such that $(\varphi(\mu))_t \neq \emptyset$. If $t = 0$, then $(\varphi(\mu))_t = R$. Assume that $t \neq 0$. By Proposition 2.1,

$$(\varphi(\mu))_t = \bigcap_{0<s<t} \varphi(\mu_{t-s}).$$

Then $\varphi(\mu_{t-s}) \neq \emptyset$ for all $0 < s < t$, and so $\mu_{t-s} \neq \emptyset$ for all $0 < s < t$. By Theorem 3.1 (i), μ_{t-s} is a ternary subsemiring of S for all $0 < s < t$. Since φ is an onto homomorphism, $\varphi(\mu_{t-s})$ is a ternary subsemiring of R for all $0 < s < t$. Then $(\varphi(\mu))_t = \bigcap_{0<s<t} \varphi(\mu_{t-s})$ is a ternary subsemiring of R.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Definition 3.3. A ternary subsemiring A of a ternary semiring S is said to be *characteristic* if $\varphi(A) = A$ for all $\varphi \in Aut(S)$ where $Aut(S)$ is the set of all automorphisms of S. The characteristic left ideals [resp. right ideals, lateral ideals, ideals] are defined analogously.

Definition 3.4. An L-fuzzy ternary subsemiring μ of S is said to be *L-fuzzy characteristic ternary subsemiring* if $\mu(\varphi(x)) = \mu(x)$ for all $x \in S$ and $\varphi \in Aut(S)$. The L-fuzzy characteristic left ideals [resp. right ideals, lateral ideals, ideals] are defined analogously.

Theorem 3.8. Let S be a ternary semiring, $\varphi : S \to S$ an onto homomorphism and $\mu \in F(S)$. Define $\mu^{\varphi} \in F(S)$ by $\mu^{\varphi}(x) = \mu(\varphi(x))$ for all $x \in S$. The following statements are true.

(i) If μ is an L-fuzzy ternary subsemiring of S, then μ^{φ} is an L-fuzzy ternary subsemiring of S.

(ii) If μ is an L-fuzzy left [resp. right, lateral] ideal of S, then μ^{φ} is an L-fuzzy left [resp. right, lateral] ideal of S.

(iii) If μ is an L-fuzzy ideal of S, then μ^{φ} is an L-fuzzy ideal of S.

Proof. (i) Let $x, y, z \in S$. We have

$$\begin{aligned}\mu^{\varphi}(x+y) &= \mu(\varphi(x+y)) \\ &= \mu(\varphi(x)+\varphi(y)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y))\} \\ &= \min\{\mu^{\varphi}(x), \mu^{\varphi}(y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^{\varphi}(xyz) &= \mu(\varphi(xyz)) \\ &= \mu(\varphi(x)\varphi(y)\varphi(z)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y)), \mu(\varphi(z))\} \\ &= \min\{\mu^{\varphi}(x), \mu^{\varphi}(y), \mu^{\varphi}(z)\}\end{aligned}$$

Therefore μ^{φ} is an L-fuzzy ternary subsemiring of S.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Theorem 3.9. Let S be a ternary semiring. The following statements are true.

(i) If μ is an L-fuzzy characteristic ternary subsemiring of S, then each level ternary subsemiring of μ is characteristic.

(ii) If μ is an L-fuzzy characteristic left [resp. right, lateral] ideal of S, then each level left [resp. right, lateral] ideal of μ is characteristic.

(iii) If μ is an L-fuzzy characteristic ideal of S, then each level ideal of μ is characteristic.

Proof. (i) Let μ be an L-fuzzy characteristic ternary subsemiring of $S, \varphi \in Aut(S)$ and $t \in L$. If $y \in \varphi(\mu_t)$. Then there exists $x \in S$ such that $\varphi(x) = y$ and so $\mu(y) = \mu(\varphi(x)) = \mu(x) \geq t$. Thus $y \in \mu_t$. Conversely, if $y \in \mu_t$, then $\mu(x) = \mu(\varphi(x)) = \mu(y) \geq t$ where $\varphi(x) = y$. It follows that $y \in \varphi(\mu_t)$. Thus $\varphi(\mu_t) = \mu_t$. Then μ_t is characteristic.

The proofs of (ii) and (iii) are similar to the proof of (i). □

The following theorem is the converse of Theorem 3.9.

Theorem 3.10. Let S be a ternary semiring and $\mu \in F(S)$. The following statements are true.

(i) If each level ternary subsemiring of μ is characteristic, then μ is an L-fuzzy characteristic ternary subsemiring of S.

(ii) If each level left [resp. right, lateral] ideal is characteristic, then μ is an L-fuzzy characteristic left [resp. right, lateral] ideal of S.

(iii) If each level ideal is characteristic of μ, then μ is an L-fuzzy characteristic ideal of S.

Proof. (i) Let $x \in S, \varphi \in Aut(S)$ and $t = \mu(x)$. Then $x \in \mu_t$ and $x \notin \mu_s$ for all $s \in L$ with $s > t$. Since each level ternary subsemiring of μ is characteristic, $\varphi(x) \in \varphi(\mu_t) = \mu_t$. Thus $\mu(\varphi(x)) \geq t$. Suppose $\mu(\varphi(x)) = r > t$. Then $\varphi(x) \in \mu_r = \varphi(\mu_r)$. This implies that $x \in \mu_r$, a contradiction. Hence $\mu(\varphi(x)) = \mu(x)$. Therefore μ is an L-fuzzy characteristic ternary subsemiring of S.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Definition 3.5. An L-fuzzy subset μ of a ternary semiring S is said to be *normal* if $\mu(0) = 1$.

Let S be a ternary semiring and $\mu \in F(S)$. Define an L-fuzzy subset μ^+ of S by

$$\mu^+(x) = \mu(x) + 1 - \mu(0) \text{ for all } x \in S.$$

Proposition 3.11. Let S be a ternary semiring and $\mu \in F(S)$. The following statements are true.

(i) μ^+ is a normal L-fuzzy subset of S containing μ.

(ii) $(\mu^+)^+ = \mu^+$.

(iii) μ is normal if and only if $\mu = \mu^+$.

Proof. (i) We can see that $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1$ and for $x \in S, \mu(x) \leq \mu^+(x)$, completing the proof.

(ii) By (i), we have $(\mu^+)^+(x) = \mu^+(x) + 1 - \mu^+(0) = \mu^+(x) + 1 - 1 = \mu^+(x)$, completing the proof.

(iii) Assume that μ is normal. Then $\mu^+(x) = \mu(x) + 1 - \mu(0) = \mu(x) + 1 - 1 = \mu(x)$. The converse is obvious by (i). □

Corollary 3.12. Let S be a ternary semiring, $\mu \in F(S)$ and $x \in S$. If $\mu^+(x) = 0$, then $\mu(x) = 0$.

Proof. By Proposition 3.11 (i), we have $\mu(x) \leq \mu^+(x)$, this implies that $\mu(x) = 0$. □

Theorem 3.13. Let S be a ternary semiring and $\mu \in F(S)$. The following statements are true.

(i) If μ is an L-fuzzy ternary subsemiring of S, then μ^+ is a normal L-fuzzy ternary subsemiring of S containing μ.

(ii) If μ is an L-fuzzy left [resp. right, lateral] ideal of S, then μ^+ is a normal L-fuzzy left [resp. right, lateral] ideal of S containing μ.

(iii) If μ is an L-fuzzy ideal of S, then μ^+ is a normal L-fuzzy ideal of S containing μ.

Proof. (i) Let $x, y, z \in S$. Then

$$\begin{aligned}\mu^+(x+y) &= \mu(x+y) + 1 - \mu(0)\\ &\geq \min\{\mu(x), \mu(y)\} + 1 - \mu(0)\\ &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\}\\ &= \min\{\mu^+(x), \mu^+(y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^+(xyz) &= \mu(xyz) + 1 - \mu(0)\\ &\geq \min\{\mu(x), \mu(y), \mu(z)\} + 1 - \mu(0)\\ &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0),\\ &\quad \mu(z) + 1 - \mu(0)\}\\ &= \min\{\mu^+(x), \mu^+(y), \mu^+(z)\}\end{aligned}$$

Hence μ^+ is an L-fuzzy ternary subsemiring of S. By Proposition 3.11 (i), μ^+ is a normal L-fuzzy ternary subsemiring of S containing μ.

The proofs of (ii) and (iii) are similar to the proof of (i) □

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