Diffusion Limit for the Vlasov-Maxwell-Fokker-Planck System

Najoua El Ghani *

Abstract—**In this paper we study the diffusion limit of the Vlasov-Maxwell-Fokker-Planck System. Here, we generalize the one and one half dimensional case, obtained in [5] to the case of several space dimensions using global renormalized solutions. The limit equation consists in a Drift-Diffusion equation where the drift velocity is defined by means of the Maxwell relation.**

Keywords: Vlasov-Maxwell-Fokker-Planck system, Drift-Diffusion model, moment method, velocity averaging lemma, renormalized solution.

1 Introduction

We consider a plasma in which the dilute charged particles interact both through collisions and through the action of their self-consistent electro-magnetic field. The evolution of the plasma is governed by the following equations

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v.\nabla_x f^\varepsilon + (\frac{1}{\varepsilon}E^\varepsilon + \delta^2 v \wedge B^\varepsilon).\nabla_v f^\varepsilon = \frac{\theta}{\varepsilon^2}\mathcal{L}_{FP}(f^\varepsilon)$$

$$:= \frac{\theta}{\varepsilon^2}\nabla_v.(vf + \nabla_v f), \ \ (t,x,v) \in]0,T[\times \mathrm{R}^d \times \mathrm{R}^d, \qquad (1)$$

$$\partial_t E^\varepsilon - curl_x B^\varepsilon = -J^\varepsilon, \ \ (t,x) \in]0,T[\times \mathrm{R}^d, \qquad (2)$$

$$\delta^2 \partial_t B^\varepsilon + curl_x E^\varepsilon = 0, \ (t,x) \in]0,T[\times \mathrm{R}^d, \qquad (3)$$

$$div_x E^\varepsilon = \rho^\varepsilon, \ \ (t,x) \in]0,T[\times \mathrm{R}^d, \qquad (4)$$

$$div_x B^\varepsilon = 0, \ \ (t,x) \in]0,T[\times \mathrm{R}^d, \qquad (5)$$

where δ, θ and ε are dimensionless parameters. The system (1), (2), (3), (4), (5) is refered to as the rescaled Vlasov-Maxwell-Fokker-Planck (VMFP for short) system on $\mathrm{R}^d \times \mathrm{R}^d$ where $d \geq 1$. Here $f^\varepsilon(t,x,v) \geq 0$ is the distribution function of particles, E^ε, B^ε stand for the electric and magnetic fields respectively.

$$\rho^\varepsilon(t,x) = \int_{\mathrm{R}^d} f^\varepsilon dv, \quad J^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathrm{R}^d} v f^\varepsilon dv.$$

are respectively the charge and current densities of f^ε. The dimensionless parameter ε is proportional with the scaled thermal mean free path and also with the scaled macroscopic velocity. We are interested in the asymptotic regime $0 < \varepsilon << 1$. For simplicity, we take $\delta = \theta = 1$.
If we neglect the magnetic field we obtain the Vlasov-Poisson-Fokker-Planck system (VPFP for short) on $\mathrm{R}^d \times \mathrm{R}^d$ where $d \geq 2$,

$$\begin{cases} \partial_t f^\varepsilon + \frac{1}{\varepsilon} v.\nabla_x f^\varepsilon - \frac{1}{\varepsilon}\nabla_x \phi^\varepsilon.\nabla_v f^\varepsilon = \frac{\mathcal{L}_{FP}(f^\varepsilon)}{\varepsilon^2}, \\ \mathcal{L}_{FP}(f) = \nabla_v.(vf + \nabla_v f), \\ -\triangle \phi^\varepsilon = \rho^\varepsilon - p(x) \end{cases} \qquad (6)$$

where $p(x)$ is the density of a background of positive charges which is assumed to be fixed. The asymptotic behavior of this system when ε goes to 0 was studied in [11], [7] and [4]. It was shown that the limit $(\rho, \phi) = \lim_{\varepsilon \to 0}(\rho^\varepsilon, \phi^\varepsilon)$ solves the following Drift-Diffusion limit

$$\begin{cases} \partial_t \rho + \nabla_x.(-\nabla_x \rho - \rho \nabla_x \phi) = 0, \\ -\triangle_x \phi = \rho - p(x). \end{cases} \qquad (7)$$

By taking as a small parameter ε the square of the ratio of the thermal mean free path with respect to the Debye length and by assuming that the distance traveled by the light during the relaxation time is of the Debye length, Bostan-Goudon [5] has proved, in this case, the convergence of the following VMFP system in one and one half dimensional case, i.e $f^\varepsilon = f^\varepsilon(t,x,p_1,p_2)$, $E^\varepsilon = (E_1^\varepsilon(t,x), E_2^\varepsilon(t,x), 0)$ and $B^\varepsilon = (0,0,B^\varepsilon(t,x))$ for any $(t,x,p_1,p_2) \in [0,T] \times \mathrm{R}^3$.

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v_1(p).\partial_x f^\varepsilon + \Big(\frac{1}{\varepsilon}E_1^\varepsilon + \delta^2 v_2(p) B^\varepsilon\Big)\partial_{p_1} f^\varepsilon$$

$$+\Big(\frac{1}{\varepsilon}E_2^\varepsilon - \delta^2 v_1(p) B^\varepsilon\Big)\partial_{p_2} f^\varepsilon = \frac{\theta}{\varepsilon^2} L(f^\varepsilon)$$

$$:= \frac{\theta}{\varepsilon^2} div_p(\nabla_p f^\varepsilon + v(p) f^\varepsilon), \qquad (8)$$

$$\partial_t E_1^\varepsilon = -\frac{1}{\varepsilon}\int_{\mathrm{R}^d} v_1(p) f^\varepsilon(t,x,v) dv + J(t,x), \ (9)$$

*Département de Mathématiques, Faculté des Sciences de Tunis, TN-2092 El Manar Tunis, Tunisia. Tel: +216 22 871 798. Fax: +216 77 236 632. Email: Najoua.ElGhani@fst.rnu.tn

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$$\partial_t E_2^\varepsilon + \partial_x B^\varepsilon = -\frac{1}{\varepsilon}\int_{\mathrm{R}^d} v_2(p) f^\varepsilon dv, \qquad (10)$$

$$\varepsilon^2\delta^2\partial_t B^\varepsilon + \partial_x E_2^\varepsilon = 0, \qquad (11)$$

$$\partial_x E_1^\varepsilon = \rho^\varepsilon(t,x) - D(t,x), \qquad (12)$$

where $D, J : [0,T]\times \mathrm{R} \longrightarrow \mathrm{R}$ are the charge and current densities of a background particle distribution of an opposite sign, satisfying $D \geq 0$ and the continuity equation

$$\partial_t D + \partial_x J = 0.$$

The following limit system was obtained

$$\begin{cases} \theta\partial_t E_1 + \rho E_1 - \partial_x^2 E_1 = \partial_x D + \theta J, \\ \partial_x E_1 = \rho - D \end{cases} \qquad (13)$$

In this paper, we generalize the study done in [5]. Indeed, if we want to work with global solutions we have only to deal with solutions to the VMFP system which are defined in the renormalized sense. Notice that in (1) the main difficulty relies on the non linear term $E^\varepsilon.\nabla_v f^\varepsilon$ which is the same order of magnitude in the diffusion Fokker-Planck term. The system is completed by prescribing the initial conditions for the distribution function f^ε and the electro-magnetic field $(E^\varepsilon, B^\varepsilon)$

$$f^\varepsilon(t=0,x,v) = f_0^\varepsilon(x,v), \qquad (14)$$

$$E^\varepsilon(t=0,x) = E_0^\varepsilon(x),\ B^\varepsilon(t=0,x) = B_0^\varepsilon(x), \qquad (15)$$

satisfying

$$div_x E_0^\varepsilon(x) = \int_{\mathrm{R}^d} f_0^\varepsilon(x,v)dv,\ div_x B_0^\varepsilon(x) = 0. \qquad (16)$$

Now, note that the Fokker-Planck operator can be written as the following

$$\mathcal{L}_{FP}(f) \quad = \quad div_v(e^{-\frac{|v|^2}{2}}\nabla_v(fe^{\frac{|v|^2}{2}})), \qquad (17)$$

and therefore (1) becomes

$$\partial_t f^\varepsilon + \frac{v}{\varepsilon}.\nabla_x f^\varepsilon + (v\wedge B^\varepsilon).\nabla_v f^\varepsilon$$

$$= \frac{1}{\varepsilon^2} div_v(e^{-\frac{|v-\varepsilon E^\varepsilon|^2}{2}}\nabla_v(fe^{\frac{|v-\varepsilon E^\varepsilon|^2}{2}})) \qquad (18)$$

We can expect from (18) that when $\varepsilon \to 0$, the distribution f^ε converge to

$$f^\varepsilon(t,x,v) \quad \simeq \quad \rho(t,x)M(v), \qquad (19)$$

where M is the normalized Maxwellian with zero mean velocity:

$$M(v) \quad = \quad (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}}. \qquad (20)$$

Let us define the Hilbert space $L_M^2(\mathrm{R}^d)$ as

$$L_M^2(\mathrm{R}^d) = \{f \in L^2(\mathrm{R}^d)/\int_{\mathrm{R}^d} f^2\frac{dv}{M} < +\infty\},$$

equipped with the inner product

$$< f, g >_{L_M^2(\mathrm{R}^d)} = \int_{\mathrm{R}^d} fg\frac{dv}{M}.$$

The operator $\mathcal{L}_{FP}$ acting on $L_M^2(\mathrm{R}^d)$ is unbounded, with domain

$$D(\mathcal{L}_{FP}) = \{f \in L_M^2(\mathrm{R}^d)/\nabla_v(M\nabla_v(\frac{f}{M})) \in L_M^2(\mathrm{R}^d)\},$$

and it satisfies the following proposition

Proposition 1.1 [4] *The operator $\mathcal{L}_{FP}$ is continuous on $L^1(dv)$ and satisfies*
1. $-\mathcal{L}_{FP}$ is self adjoint on $L_M^2(R^d)$.
2. $\mathcal{K}er(\mathcal{L}_{FP}) = RM$.
3. $\mathcal{R}(\mathcal{L}_{FP}) = \{g \in L_M^2(R^d)/\int_{R^d} g(v)dv = 0\}$,
4.For all $g \in \mathcal{R}(\mathcal{L}_{FP})$, there exists $f \in D(\mathcal{L}_{FP})$ such that $\mathcal{L}_{FP}(f) = g$. This solution is unique under the solvability condition $\int_{R^d} f(v)dv = 0$. We denote that $f = \mathcal{L}_{FP}^{-1}(g)$.
5. $\mathcal{L}_{FP}$ satisfies the following H-theorem for all $f \in L^1(dv)\cap D(\mathcal{L}_{FP})$,

$$\int_{R^d} \mathcal{L}_{FP}(f)\ln(\frac{f}{M})dv = -\int_{R^d} f|\nabla_v \ln(\frac{f}{M})|^2 dv.$$

The VMFP system is motivated to the plasma physics, as for instance in the theory of semiconductors, the evolution of laser-produced plasmas or the description of tokamaks. The diffusion limit has been analyzed in the kinetic theory of semiconductors in [10] and [9] and we can see also [4] for the VPFP system. Before giving the main result, let us assume that the sequence of initial data satisfies:
A1: $f_0^\varepsilon \geq 0$, $\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f_0^\varepsilon(1+|x|+\frac{|v|^2}{2}+|\ln(f_0^\varepsilon)|)\ dvdx < C$, for some constant $C > 0$ independent of ε.
A2: E_0^ε and B_0^ε are uniformly bounded in $L^2(\mathrm{R}^d)$.
Here, our main result to state as the following (see section 2. for the definition of renormalized solution)

Theorem 1.2 *Assume that the assumptions **A1** and **A2** are satisfied. Let $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ be a renormalized solution of the VMFP system (1)-(5). Then,*

$$\begin{aligned} &f^\varepsilon \to \rho M(v) && in\ L^1(0,T;\ L^1(R^d\times R^d)),\\ &E^\varepsilon \rightharpoonup E && in\ L^2(0,T;\ L^2(R^d)) - weakly.\\ &B^\varepsilon \rightharpoonup B && in\ L^2(0,T;\ L^2(R^d)) - weakly.\end{aligned}$$

In particular, ρ^ε converges weakly in $L^1(0,T;\ L^1(R^d))$ towards ρ and (ρ, E, B) is a weak solution of the Drift-

Diffusion-Maxwell system

$$\begin{cases} \partial_t \rho + \nabla_x . J = 0, \ \ (t,x) \in]0,T[\times R^d, \\ \partial_t E - curl_x B = -J, \ \ (t,x) \in]0,T[\times R^d, \\ \partial_t B + curl_x E = 0, \ \ (t,x) \in]0,T[\times R^d, \\ div_x E = \rho, \ \ (t,x) \in]0,T[\times R^d, \\ div_x B = 0, \ \ (t,x) \in]0,T[\times R^d, \\ \rho(t=0,x) = \rho_0(x) = \int_{R^d} f_0(x,v)dv, \end{cases} \tag{21}$$

where f_0 is the weak limit of f_0^ε and $J := -\nabla_x \rho + \rho E$.

The proof of this Theorem is as the following. In section 2, we recall the existence of renormalized solutions of the VMFP system. Then, in section 3, we establish some a priori uniform estimates. In section 4, we prove the compactness of ρ^ε using an averaging lemma. This result will be essential in order to pass to the limit in the equation which will be done in section 5. In the last section we prove the regularity estimates of (ρ, E) which will end the proof of Theorem 1.2.

2. Existence of renormalized solutions

Let us now present the definition of solutions we are going to deal with.

Definition 2.1 We say that $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ is a renormalized solution to the VMFP system (1)-(5) if it satisfies
1. $\forall\ \beta \in \mathcal{C}^1(\mathrm{R}^+)$, $|\beta(t)| \leq C\min(\sqrt{t}, 1)$, $|\sqrt{t}\beta'(t)| \leq C$ and $|t\beta''(t)| \leq C$, $\beta(f^\varepsilon)$ is a weak solution of

$$\begin{cases} \varepsilon \partial_t \beta(f^\varepsilon) + v.\nabla_x \beta(f^\varepsilon) + \nabla_v.((E^\varepsilon + \varepsilon(v\wedge B^\varepsilon))\beta(f^\varepsilon)) \\ \quad = \beta'(f^\varepsilon)\dfrac{\mathcal{L}_{FP}(f^\varepsilon)}{\varepsilon} \\ \beta(f^\varepsilon)(t=0) = \beta(f_0^\varepsilon). \end{cases} \tag{22}$$

2. $\forall\ \lambda > 0$, $\theta_{\varepsilon,\lambda} = \sqrt{f^\varepsilon + \lambda M}$ satisfies

$$\varepsilon\partial_t \theta_{\varepsilon,\lambda} + v.\nabla_x \theta_{\varepsilon,\lambda} + \nabla_v.((E^\varepsilon + \varepsilon(v\wedge B^\varepsilon))\theta_{\varepsilon,\lambda}) = \frac{\mathcal{L}_{FP}(f^\varepsilon)}{2\varepsilon\theta_{\varepsilon,\lambda}}$$

$$+ \quad \frac{\lambda M}{2\theta_{\varepsilon,\lambda}} v.(E^\varepsilon + \varepsilon(v\wedge B^\varepsilon)) \tag{23}$$

Remark 2.2 We point out here that the method is based on a double renormalization. First, we write the equation satisfied by $\beta_\delta(f^\varepsilon)$ where $\beta_\delta(s) = \frac{1}{\delta}\beta(\delta s)$ for all $s > 0$ and fixed parameter $\delta > 0$ and then weakly pass to the limit when ε goes to zero. then, we renormalize the resulting limit equation using the function $\sqrt{s + \lambda M}$.

Let us define the free energy functional

$$\mathcal{E}^\varepsilon(t) = \frac{1}{2}\int_{\mathrm{R}^d}(|E^\varepsilon|^2 + |B^\varepsilon|^2) + \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon(\frac{|v|^2}{2} + \ln(f^\varepsilon))$$

Proposition 2.3 *The VMFP system (1)-(5) has a renormalized solution in the sense of definition 2.1 which satisfies in addition*
1. The continuity equation

$$\partial_t \rho^\varepsilon + \nabla_x . J^\varepsilon \ \ = \ \ 0, \tag{24}$$

2. The entropy inequality

$$\mathcal{E}^\varepsilon(t) + \frac{2}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon \ln(\frac{f^\varepsilon}{\rho^\varepsilon M})dvdx \leq \mathcal{E}^\varepsilon(0). \tag{25}$$

Proof. After integration of (1) with respect to $v \in \mathrm{R}^d$ we deduce that the charge and current densities of the particles verify the conservation law:

$$\partial_t \rho^\varepsilon + div_x(J^\varepsilon) = 0.$$

Now, multiplying (1) by $\frac{|v|^2}{2}$ and integrating it with respect to (x,v), this implies that

$$\frac{d}{dt}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\frac{|v|^2}{2}f^\varepsilon dvdx - \int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\frac{1}{\varepsilon}E^\varepsilon.vf^\varepsilon dvdx$$

$$= -\frac{1}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(vf^\varepsilon + \nabla_v f^\varepsilon).v\ dvdx \tag{26}$$

Multiplying (2), (3) by E^ε, resp B^ε and integrating it with respect to x yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathrm{R}^d}(|E^\varepsilon|^2 + |B^\varepsilon|^2)dx = -\int_{\mathrm{R}^d} E^\varepsilon.J^\varepsilon dx \tag{27}$$

by combining (26) and (27), we obtain

$$\frac{d}{dt}(\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\frac{|v|^2}{2}f^\varepsilon dvdx + \frac{1}{2}\int_{\mathrm{R}^d}(|E^\varepsilon|^2 + |B^\varepsilon|^2)dx)$$

$$= -\frac{1}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(vf^\varepsilon + \nabla_v f^\varepsilon).v\ dvdx. \tag{28}$$

We multiply now (1) by $(1 + \ln(f^\varepsilon))$ and after integration with respect to (x,v), we get

$$\frac{d}{dt}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon \ln(f^\varepsilon)\ dvdx$$

$$= -\frac{1}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(vf^\varepsilon + \nabla_v f^\varepsilon).\frac{\nabla_v f^\varepsilon}{f^\varepsilon}\ dvdx. \tag{29}$$

Combining (28) and (29), we deduce that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}^\varepsilon &= -\frac{1}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(vf^\varepsilon + \nabla_v f^\varepsilon).(v + \frac{\nabla_v f^\varepsilon}{f^\varepsilon})\ dvdx \\ &= -\frac{1}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(vf^\varepsilon + \nabla_v f^\varepsilon)^2\frac{1}{f^\varepsilon}\ dvdx \\ &= -\frac{1}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(v\sqrt{f^\varepsilon} + 2\nabla_v\sqrt{f^\varepsilon})^2\ dvdx \\ &= -\frac{4}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|\nabla_v\sqrt{f^\varepsilon e^{\frac{|v|^2}{2}}}|^2 e^{-\frac{|v|^2}{2}}dvdx. \\ &\leq -\frac{2}{\varepsilon^2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon\ln(\frac{f^\varepsilon}{\rho^\varepsilon M})dvdx \end{aligned}$$

which is obtained by using the logarithmic Sobolev inequality.

3. Uniform estimates

The goal of this section is to derive uniform estimates needed for the proof of Theorem 1.2 one can establish for these renormalized solutions. We point out that we try here to generalize energy estimates obtained in [5]. The following Proposition states the usual bounds for the mass, energy and entropy.

Proposition 3.1 *Assume that assumptions A1 and A2 are satisfied. Then, there exists a renormalized solution* $(f^\varepsilon, E^\varepsilon, B^\varepsilon)$ *of the VMFP system (1)-(5)which satisfies the conclusions of Proposition 2.2.*
Besides, the following quantities are bounded for any $t \in [0,T]$, *with bounds which are independent of* ε *and* t:

$$\int_{R^d}\int_{R^d}(1+|x|+|v|^2+|\ln(f^\varepsilon)|)f^\varepsilon\, dvdx,$$

$$\int_{R^d}(|E^\varepsilon|^2+|B^\varepsilon|^2)\, dx$$

and

$$\frac{1}{\varepsilon^2}\int_0^t\int_{R^d}\int_{R^d}|\nabla_v\sqrt{f^\varepsilon e^{\frac{|v|^2}{2}}}|^2 e^{-\frac{|v|^2}{2}}\, dvdxds.$$

Moreover, f^ε *is weakly relatively compact in* $L^1((0,T)\times R^d\times R^d)$.

Proof. Integrating (1) with respect to (x,v) yields the mass conservation

$$\frac{d}{dt}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon(t,x,v)dvdx = 0,$$

which implies that

$$\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon(t,x,v)dvdx = \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f_0^\varepsilon(x,v)dvdx.$$

From Proposition 2.2 we deduce that

$$\mathcal{E}^\varepsilon(t)+\frac{4}{\varepsilon^2}\int_0^t\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|\nabla_v\sqrt{f^\varepsilon e^{\frac{|v|^2}{2}}}|^2 e^{-\frac{|v|^2}{2}} \le \mathcal{E}^\varepsilon(0). \quad (30)$$

In order to use the following inequality with $k=\frac{1}{4}$ which is based on classical arguments due to Carleman

$$\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon|\ln(f^\varepsilon)|dvdx \le \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f^\varepsilon\ln(f^\varepsilon)dvdx$$

$$+2k\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(|x|+\frac{|v|^2}{2})f^\varepsilon dvdx + C_k, \quad (31)$$

where the constant

$$C_k = 2C\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\exp(-\frac{k}{2}(|x|+\frac{|v|^2}{2}))\, dvdx$$

with $C=\sup_{0<y<1}\{-\sqrt{y}\ln(y)\}$. Let us multiply (1) by $|x|$ and integrate with respect to (x,v) . We deduce that

$$\frac{d}{dt}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|x|f^\varepsilon\, dvdx = \frac{1}{\varepsilon}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\frac{(v.x)}{|x|}f^\varepsilon\, dvdx$$

$$\le \frac{1}{2}\int_{\mathrm{R}^d}\rho^\varepsilon\, dx + \frac{1}{2}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}\Big|\frac{v\sqrt{f^\varepsilon}+2\nabla_v\sqrt{f^\varepsilon}}{\varepsilon}\Big|^2\, dvdx$$

implying that

$$\int_{\mathrm{R}^d}|x|f^\varepsilon\, dvdx \le \int_{\mathrm{R}^d}|x|f_0^\varepsilon\, dvdx + \frac{t}{2}\|f_0^\varepsilon\|_{L^1(\mathrm{R}^d\times\mathrm{R}^d)}$$

$$+\frac{2}{\varepsilon^2}\int_0^t\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|\nabla_v\sqrt{f^\varepsilon e^{\frac{|v|^2}{2}}}|^2 e^{-\frac{|v|^2}{2}}\, dvdxds. \quad (32)$$

Combining (30), (31) and (32) yields

$$\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}(|x|+\frac{|v|^2}{2}+|\ln(f^\varepsilon)|)f^\varepsilon\, dvdx$$

$$+\frac{1}{2}\int_{\mathrm{R}^d}(|E^\varepsilon|^2+|B^\varepsilon|^2)dx$$

$$+\frac{2}{\varepsilon^2}\int_0^t\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|\nabla_v\sqrt{f^\varepsilon e^{\frac{|v|^2}{2}}}|^2 e^{-\frac{|v|^2}{2}}\, dvdxds$$

$$\le C_{\frac{1}{4}} + \mathcal{E}^\varepsilon(0) + \|(\frac{t}{2}+|x|)f_0^\varepsilon\|_{L^1(\mathrm{R}^d\times\mathrm{R}^d)},$$

which leads to the desired results.

Lemma 3.2 *The current density* J^ε *is bounded in* $L^1((0,T)\times R^d)$.
Proof. The current density can be written as the following

$$J^\varepsilon = \frac{1}{\varepsilon}\int_{\mathrm{R}^d}(v\sqrt{f^\varepsilon}+2\nabla_v\sqrt{f^\varepsilon})\sqrt{f^\varepsilon}\, dv$$

So, by using Cauchy-Schwartz we get the estimate.

Corollary 3.3 *Assume that assumption A1 is satisfied. Then,* $|\nabla_v\sqrt{f^\varepsilon}|^2$ *is bounded in* $L^1((0,T)\times R^d\times R^d)$.
Proof. Notice that

$$|\nabla_v\sqrt{f^\varepsilon}|^2 = \frac{1}{4}|v\sqrt{f^\varepsilon}+2\nabla_v\sqrt{f^\varepsilon}|^2 - \frac{1}{4}|v|^2 f^\varepsilon$$

$$-v\sqrt{f^\varepsilon}.\nabla_v\sqrt{f^\varepsilon}.$$

Hence, by integrating with respect to (t,x,v) we conclude by using Proposition 3.1.

Corollary 3.4 ρ^ε *is weakly relatively compact in* $L^1((0,T)\times R^d)$.
Proof. Consider the convex function $\varphi : [0,+\infty[\to \mathrm{R}$,

$\varphi(s) = s\ln(s)$, $s > 0$, $\varphi(0) = 0$ and the probability measure on R^d $M(v)dv$. By applying Jensen inequality

$$\begin{aligned}\rho\ln(\rho) &= \varphi(\rho) = \varphi(\int_{\mathrm{R}^d} \frac{f}{M} M dv)\\ &\leq \int_{\mathrm{R}^d} \varphi(\frac{f}{M}) M dv\\ &= \int_{\mathrm{R}^d} f(\frac{|v|^2}{2} + \ln(f))dv + \frac{d}{2}\ln(2\pi)\int_{\mathrm{R}^d} f dv.\end{aligned}$$

Integrating with respect to x yields

$$\int_{\mathrm{R}^d} \rho\ln(\rho)dx$$

$$\leq \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f(\frac{|v|^2}{2}+\ln(f))dvdx+\frac{d}{2}\ln(2\pi)\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} f dvdx.$$

Thus, we have to use the inequality of Carlemann

$$\rho|\ln(\rho)| \leq \rho\ln(\rho) + |x|\rho + Ce^{-\frac{|x|}{4}},$$

with $C = \sup_{0<y<1}\{-\sqrt{y}\ln(y)\}$ and therefore, we deduce that

$$\int_{\mathrm{R}^d} \rho|\ln(\rho)|dx \leq \int_{\mathrm{R}^d} \rho(\ln(\rho) + |x|)dx + C\int_{\mathrm{R}^d} e^{-\frac{|x|}{4}}dx.$$

The proposition 3.1 lead to $\int_{\mathrm{R}^d}(1 + |\ln(\rho)|)\rho dx \leq C_T$, which implies the $L^1((0,T)\times \mathrm{R}^d)$ weak compactness of the sequence ρ^ε.

Let us define

$$r_\varepsilon = \frac{1}{\varepsilon\sqrt{M}}(\sqrt{f^\varepsilon} - \sqrt{\rho^\varepsilon M}).$$

Using the logarithmic Sobolev inequality and Young inequality, we deduce as in [4] that

Proposition 3.5 *r_ε is such that $|r_\varepsilon|^2 M$ is bounded in $L^1((0,T)\times R^d \times R^d)$, $\varepsilon|r_\varepsilon|^2|v|^2 M$ is bounded in $L^1((0,T)\times R^d\times R^d)$ and $\sqrt{\varepsilon}|r_\varepsilon|^2|v|M$ is bounded in $L^1((0,T)\times R^d \times R^d)$.*

4. Compactness of the density

Proposition 4.1 *The density ρ^ε is relatively compact in $L^1((0,T)\times R^d)$ and there exists $\rho \in L^1((0,T)\times R^d)$ such that, up to extraction of a subsequence if necessary,*

$$\rho^\varepsilon \to \rho \ \ in \ \ L^1 \ \ and \ a.e.$$

The proof of this proposition is done in two steps. We first prove the compactness of ρ^ε with respect to the x variable and then show the compactness in time.

In the sequel, we will use the following

$$\beta(s) = \frac{s}{1+s}, \qquad \beta_\delta(s) = \frac{1}{\delta}\beta(\delta s), \qquad \forall s > 0.$$

We recall that for all fixed parameter $\delta > 0$, we have

1. $0 \leq \beta_\delta(s) \leq \min(s, \frac{1}{\delta})$,
2. $|\beta_\delta(s)| \leq C_\delta \min(\sqrt{s}, 1)$,
3. $|\sqrt{s}\beta_\delta'(s)| \leq C_\delta$,
4. $|s\beta_\delta''(s)| \leq C_\delta$.

We remark that if we want to prove the Proposition 4.1, we only need to show for all (fixed) $\delta > 0$, the compactness $(\beta_\delta(f^\varepsilon))_\varepsilon$. This is a consequence of the following averaging lemma (see [10] for the proof).

Lemma 4.2 *[10] Assume that h^ε is bounded in $L^2((0,T)\times R^d \times R^d)$, that h_0^ε and h_1^ε are bounded in $L^1((0,T)\times R^d\times R^d)$, and that*

$$\varepsilon\partial_t h^\varepsilon + v.\nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_v.h_1^\varepsilon. \tag{33}$$

Then, for all $\psi \in \mathcal{C}_0^\infty(R^d)$,

$$\| \int_{R^d} (h^\varepsilon(t, x+y, v) - h^\varepsilon(t,x,v))\psi(v)\, dv\|_{L^1_{t,x}} \to 0 \tag{34}$$

when $y \to 0$ uniformly in ε where $h^\varepsilon(t,x,v)$ has been prolonged by 0 for $x \notin R^d$.

Remark 4.3 This lemma only gives the compactness in the x variable of the averages in v of $h^\varepsilon(t,x,v)$. This is due to the presence of an ε in front of the time derivative in 33.

Proof of Proposition 4.1. Let δ be a (fixed) nonnegative parameter. Let us verify that the rescaled VMFP system (in the renormalized sense) satisfies the assumptions of the previous lemma. Indeed, $\beta_\delta(f^\varepsilon)$ is a weak solution of

$$\varepsilon\partial_t\beta_\delta(f^\varepsilon) + v.\nabla_x\beta_\delta(f^\varepsilon) = h_0^\varepsilon + \nabla_v.h_1^\varepsilon. \tag{35}$$

where

$$h_0^\varepsilon = -\frac{(vf^\varepsilon + \nabla_v f^\varepsilon)}{\varepsilon}\nabla_v f^\varepsilon \beta_\delta''(f^\varepsilon)$$

and

$$h_1^\varepsilon = \frac{(vf^\varepsilon + \nabla_v f^\varepsilon)}{\varepsilon}\beta_\delta'(f^\varepsilon) - E^\varepsilon\beta_\delta(f^\varepsilon) - \varepsilon(v\wedge B)\beta_\delta(f^\varepsilon).$$

We remark that we can rewrite h_0^ε and h_1^ε as the following

$$h_0^\varepsilon = -2\frac{(vf^\varepsilon + \nabla_v f^\varepsilon)}{\varepsilon\sqrt{f^\varepsilon}}\nabla_v\sqrt{f^\varepsilon}\beta_\delta''(f^\varepsilon)f^\varepsilon$$

$$h_1^\varepsilon = \frac{(vf^\varepsilon + \nabla_v f^\varepsilon)}{\varepsilon\sqrt{f^\varepsilon}}\beta_\delta'(f^\varepsilon)\sqrt{f^\varepsilon} - E^\varepsilon\beta_\delta(f^\varepsilon)$$

$$-\varepsilon(v\sqrt{f^\varepsilon}\wedge B^\varepsilon)\frac{\beta_\delta(f^\varepsilon)}{\sqrt{f^\varepsilon}}.$$

The sequence $(\beta_\delta(f^\varepsilon))_\varepsilon$ is bounded in $L^\infty\cap L^1((0,T)\times \mathrm{R}^d\times\mathrm{R}^d)$ and hence in $L^2((0,T)\times\mathrm{R}^d\times\mathrm{R}^d)$. Moreover, by applying Hölder's inequality and using the uniform bound of $\beta_\delta(f^\varepsilon)$ in L^2 (for fixed δ) and by using the Proposition 3.1 and Corollary 3.3 and 3.4, we obtain

$$\|E^\varepsilon\beta_\delta(f^\varepsilon)\|_{L^1((0,T)\times\mathrm{R}^d,L^2(\mathrm{R}^d))}\le C\sup_{t\le T}\|E^\varepsilon\|_{L^2(\mathrm{R}^d)},$$

and

$$\|\varepsilon(v\wedge B^\varepsilon)\beta_\delta(f^\varepsilon)\|_{L^1((0,T)\times\mathrm{R}^d,L^2(\mathrm{R}^d))}$$
$$\le\frac{C}{2}(\sup_{t\le T}\|B^\varepsilon\|_{L^2(\mathrm{R}^d)}+\sup_{t\le T}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|v|^2f^\varepsilon dvdx),$$

The assumptions of the above lemma are satisfied and we get the compactness in x of $\int_{\mathrm{R}^d}\beta_\delta(f^\varepsilon)\psi(v)\ dv$ for all $\psi\in\mathcal{D}(\mathrm{R}^d)$, namely (34) holds with h^ε replaced by $\beta_\delta(f^\varepsilon)$.

Next, using that $(\beta_\delta(f^\varepsilon))_\varepsilon$ is bounded in $L^\infty(0,T;\ L^1((1+|v|^2)dxdv))$, we see that we can take $\psi(v)$ to be constant equal to 1 in (34) and hence we deduce, after also sending δ to 0 and using the equi-integrability of f^ε, that

$$\|\rho^\varepsilon(t,x+y)-\rho^\varepsilon(t,x)\|_{L^1_{t,x}}\to 0$$

when $y\to 0$ uniformly in ε. Finally, using that $\partial_t\rho^\varepsilon=-\nabla_x.J^\varepsilon$ is bounded in $L^1(0,T;\ W^{-1,1}(\mathrm{R}^d))$, we deduce that ρ^ε is relatively compact in $L^1((0,T)\times\mathrm{R}^d)$ which ends the proof of the proposition.

Remark 4.4 Notice that we are renormalizing the equation satisfied by f^ε to get the compactness of ρ^ε and we need the equation satisfied by $\theta_{\varepsilon,\lambda}$ to pass to the limit in J^ε.

5. Passage to the limit

We would like to pass to the limit in the continuity equation

$$\partial_t\rho^\varepsilon+\nabla_x.J^\varepsilon=0.$$

The question is to identify the limit of the current density.

Corollary 5.1

$$f^\varepsilon\to\rho M\ \ in\ \ L^1_{t,x,v}\ and\ a.e.$$

Proof. We write

$$f^\varepsilon-\rho M=(f^\varepsilon-\rho^\varepsilon M)+(\rho^\varepsilon-\rho)M.$$

By using the previous section, there exists $\rho\in L^1((0,T)\times\mathrm{R}^d)$ such that

$$\rho^\varepsilon\to\rho\ \ in\ \ L^1_{t,x}\ and\ a.e.$$

Hence, it remains to discuss $f^\varepsilon-\rho^\varepsilon M$. By the logarithmic Sobolev inequality, we obtain

$$\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}f^\varepsilon\ln(\frac{f^\varepsilon}{\rho^\varepsilon M})dvdx$$
$$\le\gamma\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|\nabla_v\sqrt{f^\varepsilon e^{\frac{|v|^2}{2}}}|^2e^{-\frac{|v|^2}{2}}dvdx$$

for some $\gamma>0$. By Proposition 3.1, after integration with respect to time, this quantity is dominated by $C_T\varepsilon^2$. Eventually, we conclude by using the Csiszar-Kullback-Pinsker inequality

$$(\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|f^\varepsilon-\rho^\varepsilon M|dvdx)^2\le\mu\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}f^\varepsilon\ln(\frac{f^\varepsilon}{\rho^\varepsilon M})dvdx$$

for some $\mu>0$ which implies that

$$\begin{aligned}
&\int_0^T\int_{\mathrm{R}^d}|\int_{\mathrm{R}^d}(f^\varepsilon-\rho^\varepsilon M)\ dx|\ dv\ dt\\
\le\ &\sqrt{T}(\int_0^T(\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}|f^\varepsilon-\rho^\varepsilon M|\ dv\ dx)^2\ dt)^{\frac12}\\
\le\ &\sqrt{\mu T}(\int_0^T\int_{\mathrm{R}^d}\int_{\mathrm{R}^d}f^\varepsilon\ln(\frac{f^\varepsilon}{\rho^\varepsilon M})\ dv\ dx\ dt)^{\frac12}\\
\le\ &\varepsilon\sqrt{\gamma\mu TC_T}\longrightarrow 0_{\varepsilon\to 0}
\end{aligned}$$

Let us denote by J the weak limit of J^ε when ε goes to zero and r is the weak limit of r_ε in $L^2((0,T)\times\mathrm{R}^d\times\mathrm{R}^d,\ M(v)dtdxdv)$.

Proposition 5.2

$$J^\varepsilon\rightharpoonup^{\mathcal{D}'}2\sqrt{\rho}\int_{R^d}rvM\ dv,\ \ in\ \ L^1_{t,x}.$$

Proof. Using r_ε, we can write

$$f^\varepsilon=\rho^\varepsilon M+2\varepsilon M\sqrt{\rho^\varepsilon}r_\varepsilon+\varepsilon^2r_\varepsilon^2M.$$

Then, we obtain

$$J^\varepsilon=2\sqrt{\rho^\varepsilon}\int_{\mathrm{R}^d}r_\varepsilon vM\ dv+\varepsilon\int_{\mathrm{R}^d}r_\varepsilon^2vM\ dv.$$

Moreover, we know that $\rho^\varepsilon\to\rho$ *in* $L^1_{t,x}$ *and a.e.* Since $(\sqrt{a}-\sqrt{b})^2\le|a-b|$ we have

$$\sqrt{\rho^\varepsilon}\to\sqrt{\rho}\ \ in\ \ L^2_{t,x}\ and\ a.e.$$

Thus, using Proposition 3.5 leads to

$$J^\varepsilon=2\sqrt{\rho^\varepsilon}\int_{\mathrm{R}^d}r_\varepsilon vM\ dv+\mathcal{O}(\sqrt{\varepsilon})_{L^1_{t,x,v}}$$
$$\rightharpoonup 2\sqrt{\rho}\int_{\mathrm{R}^d}rvM\ dv\ \ in\ \ L^1_{t,x}.$$

We denote by χ the unique solution in $[\mathcal{R}(\mathcal{L}_{FP})\cap D(\mathcal{L}_{FP})]^d$ of

$$\mathcal{L}_{FP}\chi=vM$$

Proposition 5.3

$$J^\varepsilon \rightharpoonup_{D'} J := -2\sqrt{\rho}\ \boldsymbol{D}.(\nabla_x\sqrt{\rho} - \frac{1}{2}E\ \sqrt{\rho})$$

where $\boldsymbol{D}$ *is the diffusion matrix defined by*

$$\boldsymbol{D} = -\int_{R^d} \chi(v) \otimes v\ dv.$$

Proof. We can pass to the limit in (23) for $\lambda > 0$, up to extraction of a subsequence, we get

$$v.\nabla_x\sqrt{(\rho+\lambda)M} + \nabla_v.(E\sqrt{(\rho+\lambda)M}) = \frac{\sqrt{\rho}\mathcal{L}_{FP}(rM)}{\sqrt{(\rho+\lambda)M}} + \frac{\lambda M v.E}{2\sqrt{(\rho+\lambda)M}} \tag{36}$$

where E is the $L^2_{t,x}-$weak limit of E^ε and we have used that f^ε converges strongly to ρM in $L^1_{t,x,v}$. Sending λ to 0, we infer that

$$(\nabla_x\sqrt{\rho} - \frac{1}{2}E\ \sqrt{\rho}).vM = \mathcal{L}_{FP}(rM). \tag{37}$$

Let us go back to the expression of the current density computed in Proposition 5.2. Using (37), we get

$$\begin{aligned} J &= 2\sqrt{\rho}\int_{\mathrm{R}^d} rvMdv \\ &= 2\sqrt{\rho}\int_{\mathrm{R}^d} rM\frac{\mathcal{L}_{FP}\chi}{M}\ dv \\ &= 2\sqrt{\rho}\int_{\mathrm{R}^d} \mathcal{L}_{FP}(rM)\chi\ \frac{dv}{M} \\ &= 2\sqrt{\rho}\int_{\mathrm{R}^d} [(\nabla_x\sqrt{\rho} - \frac{1}{2}E\ \sqrt{\rho}).vM]\chi\ \frac{dv}{M} \\ &= -2\sqrt{\rho}\ [-\int_{\mathrm{R}^d}\chi\otimes v\ dv].(\nabla_x\sqrt{\rho} - \frac{1}{2}E\ \sqrt{\rho}) \end{aligned}$$

6. Regularity of the density

Now, we would like to explain how we can rewrite the current J. Precisely, we shall prove that the limit $\rho \in L^2(0,T;\ L^2(\mathrm{R}^d))$ and that $\sqrt{\rho} \in L^2(0,T;\ H^1(\mathrm{R}^d))$.

Lemma 6.1 *Let* ρ *be a positive function such that* $\rho \in L^\infty(0,T;\ L^1(R^d))$, *satisfying*

$$\left\{\begin{array}{l} \nabla_x\sqrt{\rho} - \frac{1}{2}E\ \sqrt{\rho} = G \in L^2(0,T;\ L^2(R^d)), \\ div_x E = \rho. \\ E \in L^\infty(0,T;\ L^2(R^d)), \end{array}\right. \tag{38}$$

then

$$\rho \in L^2(0,T;\ L^2(R^d)), \qquad \sqrt{\rho} \in L^2(0,T;\ H^1(R^d)),$$

and

$$E\sqrt{\rho} \in L^2(0,T;\ L^2(R^d)).$$

Proof. The first and the third equation of (38) imply that $\nabla_x\sqrt{\rho} \in L^1$.
Let β_δ be an approximation of identity, namely $\beta_\delta(s) = \frac{1}{\delta}\beta(\delta s)$ where β is a $\mathcal{C}^\infty$ function satisfying $\beta(s) = s$ for $-1 \le s \le 1$, $0 \le \beta'(s) \le 1$ for all s and $\beta(s) = 2$ for $|s| \ge 3$.
Hence, $\beta_\delta(s)$ goes to s and $\beta'_\delta(s)$ goes to 1, when δ goes to 0.
Now, we have

$$\nabla_x\beta_\delta(\sqrt{\rho}) = \nabla_x\sqrt{\rho}\beta'_\delta(\sqrt{\rho}).$$

Hence, we can renormalize the first equation appearing in (38), it gives

$$\nabla_x\beta_\delta(\sqrt{\rho}) - \frac{1}{2}E\ \beta'_\delta(\sqrt{\rho})\sqrt{\rho} = G\beta'_\delta(\sqrt{\rho}). \tag{39}$$

Then, using that for fixed $\delta > 0$,

$$|E\ \beta'_\delta(\sqrt{\rho})\sqrt{\rho}| \le \frac{3}{\delta}|E| \in L^2.$$

We deduce that $\nabla_x\beta_\delta(\sqrt{\rho}) \in\ L^2$ for fixed δ. Then, by taking the L^2 norm of (39), we get

$$\|\nabla_x\beta_\delta(\sqrt{\rho})\|^2_{L^2} + \frac{1}{4}\|E\ \beta'_\delta(\sqrt{\rho})\sqrt{\rho}\|^2_{L^2} - \int_0^T\int_{\mathrm{R}^d} E\nabla_x\beta_\delta(\sqrt{\rho})\beta'_\delta(\sqrt{\rho})\sqrt{\rho}dxdt \le \|G\|^2_{L^2}. \tag{40}$$

Let $\widetilde{\beta}$ be given by $\widetilde{\beta}(s) = \int_0^s \tau\beta'(\tau)^2 d\tau$ and $\widetilde{\beta}_\delta(s) = \frac{1}{\delta^2}\widetilde{\beta}(\delta s)$. Hence $\widetilde{\beta}_\delta(s)$ goes to s^2 when δ goes to 0.
Computing the third term in (40), we get

$$\begin{aligned} \int_{\mathrm{R}^d} E\nabla_x\beta_\delta(\sqrt{\rho})\beta'_\delta(\sqrt{\rho})\sqrt{\rho}dx &= \int_{\mathrm{R}^d} E\nabla_x\widetilde{\beta}_\delta(\sqrt{\rho})dx \\ &= -\int_{\mathrm{R}^d} \rho\widetilde{\beta}_\delta(\sqrt{\rho})dx. \end{aligned}$$

Hence, we deduce that for all $\delta > 0$,

$$\|\nabla_x\beta_\delta(\sqrt{\rho})\|^2_{L^2} + \frac{1}{4}\|E\ \beta'_\delta(\sqrt{\rho})\sqrt{\rho}\|^2_{L^2} + \int_0^T\int_{\mathrm{R}^d} \rho\widetilde{\beta}_\delta(\sqrt{\rho})dxdt \le \|G\|^2_{L^2}.$$

Letting δ go to 0, we get

$$\|\nabla_x\sqrt{\rho}\|^2_{L^2} + \frac{1}{4}\|E\sqrt{\rho}\|^2_{L^2} + \frac{1}{2}\int_0^T\int_{\mathrm{R}^d}|\rho|^2dxdt \le \|G\|^2_{L^2},$$

which end the proof of the Lemma.
Now, using the previous lemma, we can see easily that we can rewrite the current

$$\begin{aligned} J &= -2\sqrt{\rho}\,\mathbf{D}[\nabla_x\sqrt{\rho} - \tfrac{1}{2}\sqrt{\rho}\ E] \\ &= -\mathbf{D}[\nabla_x\rho - \rho\ E]. \end{aligned}$$

and we conclude by using that $\chi_i = v_iM$ and $\mathbf{D} = \mathrm{I}$.
Now, using that $E,\ B \in L^\infty(0,T;\ L^2(\mathrm{R}^d))$ we see that $\partial_t B = -curl_x E$ is bounded in $L^\infty(0,T;\ H^{-1}(\mathrm{R}^d))$ and $\partial_t E = curl_x B - J$ is bounded in $L^\infty(0,T;\ H^{-1}(\mathrm{R}^d)) + L^1((0,T)\times\mathrm{R}^d)$. This ends the proof of the main Theorem 1.2.

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