An Optimal Replenishment Policy for Seasonal Items in Retailing

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Abstract—**The present study discusses the retailer's optimal replenishment policy for products with a seasonal demand pattern. The demand of seasonal merchandise such as clothes, sporting goods, children's toys and electrical home appearances tends to decrease with time. In this study, we focus on "Special Display Goods", which are heaped up in end displays or special areas at retail store. They are sold at a fast velocity when their quantity displayed is large, but are sold at a low velocity if the quantity becomes small. We develop the model with a finite time horizon (period of a season) to determine the optimal replenishment policy, which maximizes the retailer's total profit. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.**

Keywords: optimal replenishment policy, seasonal demand, special display goods

1 Introduction

The demand rate of seasonal merchandise such as clothes, sporting goods, children's toys and electrical home appearances tends to decrease with time. The seasonal items have a relatively short selling season (eight to 12 weeks), while they have a relatively long ordering leadtime (three to nine months)[1]. For this reason, the retailers have to commit themselves to a single order to purchase the seasonal items, prior to the start of the season. Recently, Quick Response (QR) system has widely used by manufacturing industries[2]. Quick Response is a vertical strategy where the manufacturer strives to provide products and services to its retail customers in exact quantities on a continuous basis with minimum lead time[3]. Appling the QR system to the manufacture and distribution allows the retailer to re-order the seasonal items during the selling season.

Inventory models with a finite planning horizon and time-varying demand patterns have extensively been studied in the inventory literature[4-10]. Resh et al.[4] and Donaldson[5] established an algorithm to determine the optimal number of replenishment cycles and the optimal replenishment time for a linearly increasing demand pattern. Barbosa and Friedman[6] and Henery[7] respectively extended the demand pattern to a power demand form and a log-concave function. Hariga and Goyal[8] and Teng[9] extended Donaldson's work by considering various types of shortages. For deteriorating items such as medicine, volatile liquids and blood banks, Dye[10] developed the inventory model under the circumstances where shortages are allowed and backlogging rate linearly depends on the total number of customers in the waiting line during the shortage period. However, there still remain many problems associated with replenishment policies for retailers that should theoretically be solved to provide them with effective indices. We focus on a case where *special display goods*[11, 12, 13] are dealt in. The special display goods are heaped up in the end displays or special areas at retail store. Retailers deal in such special display goods with a view to introducing and/or exposing new products or for the purpose of sales promotions in many cases. They are sold at a fast velocity when their quantity displayed is large, but are sold at a low velocity when their quantity becomes small. Baker[14] and Baker and Urban[15] dealt with a similar problem, but they expressed the demand rate simply as a function of a polynomial form without any practical meaning.

Our previous work has developed an inventory model for the special display goods with a seasonal demand rate over a finite time horizon to determine the optimal replenishment policy, which maximizes the retailer's total profit[16]. However, the salvage value which is the disposal value of unsold inventory at the end of the season was assumed to be equal to the purchase cost. In this study, we relax this restriction on the salvage value in order to derive a more general solution. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.

2 Notations and Assumptions

The main notations used in this paper are listed below:

H: planning horizon.
p: selling price per item.
c: acquisition cost per item.
h: inventory holding cost per item and unit of time.
K: ordering cost per lot.

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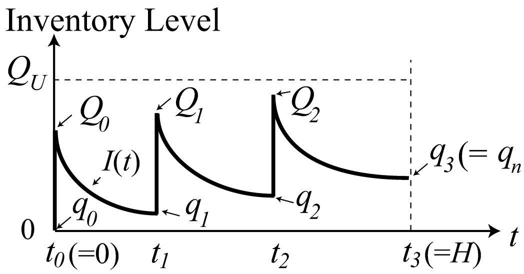


Figure 1: Transition of inventory level ($n = 3$)

θ: salvage value, per item, of unsold inventory at the end of the planning horizon.

$g(t)$: stock-dependent consumption rate parameter at time t ($\int g(t)dt = G(t) + C$).

$\mu(t)$: demand rate, at time t, which is independent of the quantity displayed.

Q_U: maximum inventory level.

Q_j, q_j: the order-up-to level and the re-order point, respectively, in the jth replenishment cycle($q_0 = 0$, $0 \le q_j < Q_j \le Q_U$, $j = 1, 2, \cdots, n$).

n: the number of replenishment cycles during the planning horizon.

t_j: the time of the jth replenishment ($t_{j-1} < t_j$, $t_0 = 0$, $t_n = H$).

The assumptions in this study are as follows:

(1) The finite planning horizon H is divided into n replenishment cycles.

(2) The demand rate is deterministic and significantly depends on the quantity displayed: the items sell well if their quantity displayed is large, but do not when their quantity displayed becomes small. We express such a behavior of special display goods in terms of the following differential equation:

$$\frac{d}{dt}m_j(t) = g(t)\left[Q_{j-1} - m_j(t)\right] + \mu(t) \qquad (1)$$

where $m_j(t)$ denotes the cumulative quantity of the objective product sold during $[t_{j-1}, t]$ ($t < t_{j+1}$) and Q_{j-1} signifies the order-up-to level at the beginning of the jth replenishment cycle. A mathematically identical equation has been used to express the behavior of deteriorating items and their optimal ordering policy has been obtained by Abad[17]. Under the model proposed in this study, the demand depends on the quantity heaped and thus depends on time.

(3) The retailer deals in the seasonal merchandise. The demand rate rapidly reaches its maximum value, and then becomes a constant or decreases with time. The growth stage of this class of item is negligibly short, so that we focus on the maturity and saturation stages. We therefore assume that $\mu'(t) \le 0$ and $g'(t) \le 0$ ($0 \le t \le H$).

(4) The end of season is defined by the retailer as the point before which any units that are left in inventory are sold below cost, i.e., we assume that $\theta \ge c$.

(5) The rate of replenishment is infinite and the delivery is instantaneous.

(6) Backlogging and shortage are not allowed.

(7) The retailer orders $(Q_j - q_j)$ units when her/his inventory level reaches q_j. Figure 1 shows the transition of inventory level in the case of $n = 3$.

(8) We assume $v(t) = (p - c)g(t) - h > 0$. This assumption, $v(t) > 0$, is equivalent to $(p - c)(Q_j - q_j) > h\frac{Q_j - q_j}{g(t)}$. The left-hand-side of the inequality, $(p - c)(Q_j - q_j)$, denotes the cumulative gross profit during $[t_{j-1}, t_j)$, and the right-hand-side of the inequality, $h\frac{Q_j - q_j}{g(t)}$, approximately expresses the cumulative inventory holding cost during $[t_{j-1}, t_j)$. Therefore, $v(t) > 0$ signifies that the gross profit exceeds the inventory holding cost during one replenishment cycle.

3 Total Profit

By solving the differential equation in Eq. (1) with the boundary condition $m_j(t_{j-1}) = 0$, the cumulative quantity, $m_j(t)$, of demand for the product at time $t(\ge t_{j-1})$ is given by

$$\begin{aligned} m_j(t) &= Q_{j-1}\left\{1 - e^{-[G(t) - G(t_{j-1})]}\right\} \\ &\quad + e^{-G(t)} \int_{t_{j-1}}^{t} e^{G(u)}\mu(u)du. \qquad (2) \end{aligned}$$

Since we have $I(t_j) = q_j$, the inventory level of the product at time t becomes

$$\begin{aligned} I(t) &= Q_{j-1} - m_j(t) \\ &= e^{-G(t)}\left[q_j e^{G(t_j)} + \int_t^{t_j} e^{G(u)}\mu(u)du\right]. \qquad (3) \end{aligned}$$

Therefore, the initial inventory level in jth replenishment cycle is given by

$$\begin{aligned} Q_{j-1} &= I(t_{j-1}) \\ &= e^{-G(t_{j-1})} \\ &\quad \times \left[q_j e^{G(t_j)} + \int_{t_{j-1}}^{t_j} e^{G(u)}\mu(u)du\right]. \qquad (4) \end{aligned}$$

By letting $Q_{j-1} = I(t_{j-1})$ in Eq. (2), the cumulative quantity of demand during $[t_{j-1}, t_j)$ becomes

$$m(t_{j-1}, t_j) = q_j\left[e^{G(t_j) - G(t_{j-1})} - 1\right]$$

$$+e^{-G(t_{j-1})}\int_{t_{j-1}}^{t_j} e^{G(u)}\mu(u)du. \quad (5)$$

There obviously exists a time $t = t_j^U (> t_{j-1})$ at which the inventory level reaches zero, where t_j^U is unique finite positive solution to

$$e^{-G(t_{j-1})}\int_{t_{j-1}}^{t} e^{G(u)}\mu(u)du \quad = \quad Q_{j-1}. \quad (6)$$

The left-hand-side of Eq. (6) indicates that the cumulative demand of the product in jth replenishment cycle when the re-order point q_j is set to be zero. The maximum value of t_j can therefore be given by t_j^U.

On the other hand, the cumulative inventory, $A(t_{j-1}, t_j)$, held during $[t_{j-1}, t_j)$ $(t_j \le t_j^U)$ is expressed by

$$\begin{aligned} A(t_{j-1}, t_j) &= \int_{t_{j-1}}^{t_j} I(t)dt \\ &= \int_{t_{j-1}}^{t_j} \mu(u)e^{G(u)}\left(\int_{t_{j-1}}^{u} e^{-G(t)}dt\right)du \\ &\quad +q_j e^{G(t_j)}\int_{t_{j-1}}^{t_j} e^{-G(t)}dt. \quad (7)\end{aligned}$$

Hence, the total profit is given by

$$\begin{aligned} P_n &= \sum_{j=1}^{n}\Big[p\cdot m(t_{j-1}, t_j) - c\cdot(Q_{j-1}-q_{j-1}) \\ &\quad -h\cdot A(t_{j-1}, t_j)\Big] + \theta q_n - nK \\ &= (\theta - c)q_n - nK \\ &\quad +(p-c)\sum_{j=1}^{n}\Big\{q_j\left[e^{G(t_j)-G(t_{j-1})}-1\right] \\ &\quad +e^{-G(t_{j-1})}\int_{t_{j-1}}^{t_j} e^{G(u)}\mu(u)du\Big\} \\ &\quad -h\sum_{j=1}^{n}\Big\{q_j e^{G(t_j)}\int_{t_{j-1}}^{t_j} e^{-G(t)}dt \\ &\quad +\int_{t_{j-1}}^{t_j} e^{G(u)}\mu(u)\left(\int_{t_{j-1}}^{u} e^{-G(t)}dt\right)du\Big\}. \quad (8)\end{aligned}$$

4 Optimal Policy

This section analyzes the existence of the optimal policy $(Q_{j-1}, q_j, t_j) = (Q_{j-1}^*, q_j^*, t_j^*)$ for a given n $(j = 1, 2, \cdots, n)$, which maximizes P_n in Eq. (8).

4.1 Optimal Order-up-to Level and Reorder Point

In this subsection, we examine the existence of (Q_j^*, q_j^*), under a general form of $g(t)$, in case t_{j-1} and t_j are fixed to a suitable value.

Let $R(t_{j-1}, t_j)$ be defined by

$$\begin{aligned} R(t_{j-1}, t_j) &\equiv e^{-[G(t_j)-G(t_{j-1})]} \\ &\quad \times\left[Q_U - \int_{t_{j-1}}^{t_j} e^{G(u)-G(t_{j-1})}\mu(u)du\right] \\ &\quad (>0). \quad (9)\end{aligned}$$

The optimal order-up-to level and the optimal re-order point can be given by

$$(Q_j^*, q_j^*) \quad = \quad (Q_U, R(t_{j-1}, t_j)). \quad (10)$$

The proofs are given in Appendix A.

By letting $(Q_{j-1}, q_j) = (Q_U, R(t_{j-1}, t_j))$ in Eq. (8), the total profit on $(Q_{j-1}, q_j) = (Q_U, R(t_{j-1}, t_j))$ becomes

$$\begin{aligned} P_n &= (p-c)\sum_{j=1}^{n}\Big\{Q_U\left[1-e^{-\{G(t_j)-G(t_{j-1})\}}\right] \\ &\quad +e^{-G(t_j)}\int_{t_{j-1}}^{t_j} e^{G(u)}\mu(u)du\Big\} \\ &\quad -h\sum_{j=1}^{n}\Big\{\int_{t_{j-1}}^{t_j}\mu(u)e^{G(u)}\left(\int_{t_{j-1}}^{u} e^{-G(t)}dt\right)du \\ &\quad +\left[Q_U e^{G(t_{j-1})} - \int_{t_{j-1}}^{t_j} e^{G(u)}\mu(u)du\right] \\ &\quad \times\int_{t_{j-1}}^{t_j} e^{G(u)}du\Big\} + (\theta-c)R(t_{n-1}, t_n) \\ &\quad -nK. \quad (11)\end{aligned}$$

4.2 Optimal Replenishment Time

The analysis with respect to existence of $t_j = t_j^*$ becomes considerably complicated under a general form of $g(t)$. For this reason, we focus on the following two cases with $\lambda > 0$:

Case 1: $g(t) = \lambda$,

Case 2: $g(t) = \lambda\mu(t)$.

4.2.1 Case 1

This subsection makes an analysis of t_j^* that maximizes P_n, for a given (t_{j-1}, t_{j+1}), in the case of $g(t) = \lambda$. In this case, P_n in Eq. (11) can be rewritten as

$$\begin{aligned} P_n &= \tilde{v}\sum_{j=1}^{n}\left\{Q_U - e^{-\lambda(t_j - t_{j-1})}\left[Q_U - \tilde{m}(t_{j-1}, t_j)\right]\right\} \\ &\quad +h/\lambda\int_0^H \mu(u)du - nK \\ &\quad +(\theta - c)e^{-\lambda(t_n - t_{n-1})}\left[Q_U - \tilde{m}(t_{n-1}, t_n)\right], \quad (12)\end{aligned}$$

where

$$\tilde{v} = (p - c - h/\lambda), \tag{13}$$

$$\tilde{m}(t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} e^{\lambda(u - t_{j-1})}\mu(u)du. \tag{14}$$

Let us here $\varphi(t_j^U)$ be given by

$$\varphi(t_j^U) \equiv \frac{1}{\lambda}\ln\frac{\lambda Q_U + \mu(t_j^U)}{\mu(t_j^U)} + t_j^U. \tag{15}$$

We here summarize the result of analysis in relation to the optimal replenishment point t_j^*.

The proofs are shown in Appendix B.

(1) $\{j < n-1\}$ or $\{j = n-1$ and $\theta = c\}$:

  (a) $t_{j+1} < \varphi(t_j^U)$:
  There exists a unique finite t_j^* $(t_{j-1} < t_j^* < \min(t_j^U, t_{j+1}))$ that maximizes P_n.

  (b) $t_{j+1} \geq \varphi(t_j^U)$:
  P_n is non-decreasing in t_j, and therefore $t_j^* = t_j^U$.

(2) $\{j = n-1$ and $\theta > c$ and $L_1'(t_n) < 0\}$:

In this subcase, we show the existence of the optimal replenishment time, t_{n-1}^*, at the beginning of the last cycle when $\theta > c$. A sufficient condition for $L_1'(t_{n-1}) < 0$ can be given by $\tilde{v} \geq (\theta - c)$ (> 0).

  (a) $\{L_1(t_n) < 0$ and $L(t_{n-1}^U) < 0\}$:
  There exists a unique finite t_{n-1}^* $(t_{j-1} < t_{n-1}^* < \min(t_{n-1}^U, t_n))$.

  (b) $\{L_1(t_n) \geq 0\}$ or $\{L_1(t_n) < 0$ and $L(t_{n-1}^U) \geq 0\}$:
  P_n is non-decreasing in t_{n-1}, and therefore $t_{n-1}^* = \min(t_{n-1}^U, t_n)$.

If there exists $t_j^* < t_j^U$ for all $j = 1, 2, \cdots, n-1$ and $\theta = c$, the total profit is given by

$$P_n = \tilde{v}\Bigg\{\frac{1}{\lambda}\sum_{j=1}^{n-1}\left[\lambda Q_U + \mu(t_j^*)\right]\left[1 - e^{-\lambda(t_{j+1}^* - t_j^*)}\right] \\ m(t_{n-1}^*, H)\Bigg\} + h/\lambda\int_0^H \mu(u)du - nK. \tag{16}$$

4.2.2 Case 2

In this subsection, we examine the existence of t_j^* in the case of $g(t) = \lambda\mu(t)$. In this case, P_n in Eq. (11) can be rewritten as

$$P_n = \frac{h}{\lambda}H - nK + n(p-c)\left(Q_U + \frac{1}{\lambda}\right) \\ - \left(Q_U + \frac{1}{\lambda}\right)\sum_{j=1}^{n}\Bigg\{(p-c)e^{-[G(t_j) - G(t_{j-1})]} \\ + he^{G(t_{j-1})}\int_{t_{j-1}}^{t_j} e^{-G(u)}du\Bigg\} + (\theta - c) \\ \times\left\{\left(Q_U + \frac{1}{\lambda}\right)e^{-[G(t_n) - G(t_{n-1})]} - \frac{1}{\lambda}\right\}. \tag{17}$$

In the following, the mathematical results are briefly summarized in the case of $L_2'(t_j) < 0$.

The proofs are presented in Appendix C.

(1) $\{j < n-1\}$ or $\{j = n-1$ and $\theta = c\}$:

  (a) $\{t_{j+1} \leq t_j^U\}$ or $\{L(t_j^U) < 0\}$.
  There exists a unique finite t_j^* $(t_{j-1} < t_j^* < \min(t_j^U, t_{j+1}))$.

  (b) $\{t_{j+1} > t_j^U\}$ and $\{L(t_j^U) \geq 0\}$.
  We have $\frac{\partial}{\partial t_j}P_n \geq 0$ and therefore $t_j^* = t_j^U$.

(2) $\{j < n-1\}$ or $\{j = n-1$ and $\theta > c\}$:

The classification necessary here is identical to that of Subcase(2) in 4.2.1.

5 Numerical Examples

This section presents numerical examples to illustrate the proposed model for the following two cases:

Case 1: $g(t) = \lambda$,

Case 2: $g(t) = \lambda\mu(t)$.

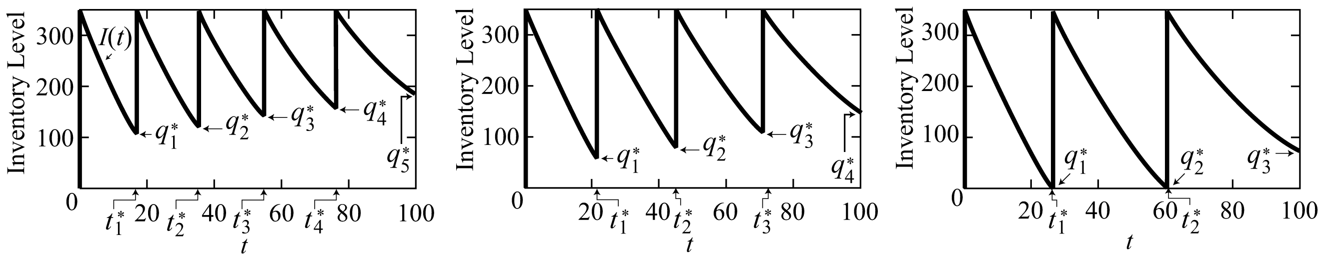
We here suppose the demand rate which is independent of the quantity displayed to be a linear function of time t, which is given by

$$\mu(t) = \beta - \alpha t \ (\alpha > 0,\ \beta > 0,\ \mu(t) > 0). \tag{18}$$

Figure 2 reveals the transition of inventory level along with behavior of (q_j^*, t_j^*) in the case of $g(t) = \lambda$ for $K = 5000, 7500, 1000$. In contrast, Figure 3 depicts the behavior of these values in the case of $g(t) = \lambda\mu(t)$ for $K = 7500, 10000, 12500$.

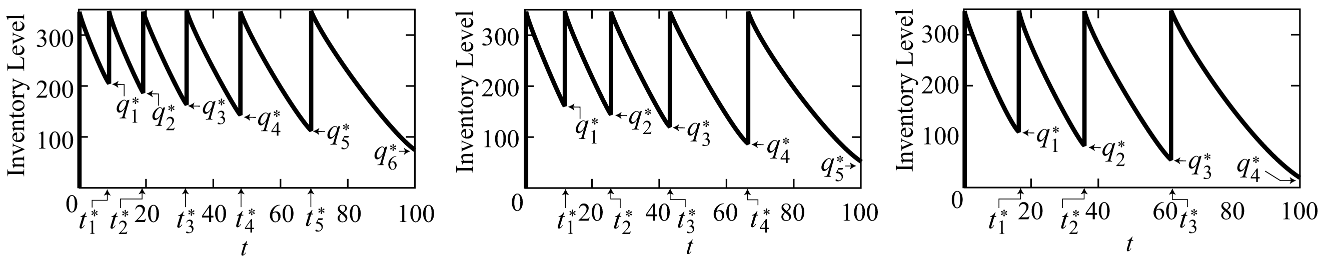
5.1 Case 1

Figure 2 illustrates the behavior of $I(t)$, q_j^* along with t_j^* in the case of $g(t) = \lambda$ with $(H, Q_U, \lambda, p, c, h, \theta, \alpha, \beta) = (100, 350, 0.01, 600, 300, 1, 300, 0.1, 13)$. It is observed in Fig. 2 that the number of replenishment cycles decreases



(a) In the case of K= 5000. (b) In the case of K= 7500. (c) In the case of K= 10000.

Figure 2: Sensitivity analysis (Case 1)



(a) In the case of K= 7500. (b) In the case of K= 10000. (c) In the case of K= 12500.

Figure 3: Sensitivity analysis (Case 2)

with increasing K. This is because when the ordering cost per lot becomes large, the total ordering cost should be slashed by means of increasing the time interval between replenishment cycles in order to decrease the number of its cycles.

It is also seen in Fig. 2 that q_j^* is non-decreasing in time t. This signifies that the cumulative quantity displayed in the jth replenishment cycle increases with increasing j. Heaping up the products to a large quantity reflects the situation where the demand velocity is large at the retail store. When the demand rate which is independent of the quantity displayed becomes small, the retailer can therefore maintain her/his profit as large as possible by increasing the quantity displayed.

5.2 Case 2

Figure 3 shows the behavior of $I(t)$, q_j^* as well as t_j^* in the case of $g(t) = \lambda\mu(t)$ with $(H, Q_U, \lambda, p, c, h, \theta, \alpha, \beta) = (100, 350, 0.0026, 600, 300, 1, 300, 0.05, 10)$.

It is observed in Fig. 3 that n^* decreases with increasing K, that is, the time intervals between replenishment cycles tend to increase with K. This tendency is quite similar to that in section 5.1.

We can also notice in Fig. 3 that q_j^* decreasing with increasing time t, which is significantly different from that in section 5.1. This is simply due to the effect of a large quantity on the demand of the product decreases with increasing time t, which can easily be confirmed by the form of $g(t)$.

6 Conclusions

In this study, we have proposed an inventory model with a seasonal demand pattern over a finite time horizon (period of a season) to determine the optimal replenishment policy, which maximizes the retailer's total profit. We particularly focus on the case where the retailer is facing her/his customers' demand by dealing in a special display goods. Since the analysis in relation to an optimal replenishment policy is very complicated under the general form of $g(t)$, which expresses the stock-dependent consumption rate parameter, we focus on the following two cases for $\lambda > 0$: Case 1: $g(t) = \lambda$, Case 2: $g(t) = \lambda\mu(t)$. For each case in the above, we have clarified the existence of the optimal replenishment policy which maximizes the retailer's total profit. In the real circumstances, retailers frequently place a mirror at their display area, or they display products on a false bottom to increase their quantity displayed in appearance. Taking account of such factors is an interesting extension.

Appendix A

In this appendix, we discuss the existence of both the optimal order-up-to level and the re-order point $(Q_j, q_j) = (Q_j^*, q_j^*)$, under a general form of $g(t)$, in case n, t_{j-1} and t_j are fixed to a suitable value.

At retail stores, retailers have a maximum value for the inventory level arrowed for some reasons, which is denoted by Q_U. It can easily be shown from Eq. (4) that Q_{j-1} is a function of q_j $(0 \le q_j < Q_{j-1} \le Q_U)$, and furthermore, $Q_{j-1} \le Q_U$ agrees with

$$\begin{aligned} q_j &\le e^{-[G(t_j)-G(t_{j-1})]} \\ &\times \left[Q_U - \int_{t_{j-1}}^{t_j} e^{G(u)-G(t_{j-1})}\mu(u)du \right]. \quad \text{(A.1)} \end{aligned}$$

Let $R(t_{j-1}, t_j)$ express the right-hand-side of Inequality (A.1), which signifies the maximum value for the reorder point q_j.

By differentiating P_n in Eq. (8) with respect to q_j, we have

$$\begin{aligned} \frac{\partial}{\partial q_j}P_n &= (p-c)\left[e^{G(t_j)-G(t_{j-1})} - 1\right] \\ &\quad -he^{G(t_j)}\int_{t_{j-1}}^{t_j} e^{-G(u)}du \\ &> [(p-c)g(t_j) - h]e^{G(t_j)} \\ &\quad \times \int_{t_{j-1}}^{t_j} e^{-G(u)}du \ (>0). \end{aligned} \quad \text{(A.2)}$$

Since $v(t_{j-1}) = [(p-c)g(t_{j-1}) - h] > 0$ from assumption (8), we have $\frac{\partial}{\partial q_j}P_n > 0$, and consequently $(Q_{j-1}^*, q_j^*) = (Q_U, R(t_{j-1}, t_j))$.

Appendix B

In this appendix, we show the existence of t_j^* that maximizes P_n, for a given (t_{j-1}, t_{j+1}), in the case of $g(t) = \lambda$.

By differentiating P_n in Eq. (12) with respect to t_j, we have

$$\begin{aligned} \frac{\partial}{\partial t_j}P_n &= \tilde{v}\Big\{ \lambda e^{-\lambda(t_j - t_{j-1})}\left[Q_U - \tilde{m}(t_{j-1}, t_j)\right] \\ &\quad -\lambda Q_U e^{-\lambda(t_{j+1}-t_j)} \\ &\quad +\mu(t_j)\left[1 - e^{-\lambda(t_{j+1}-t_j)}\right]\Big\} \\ &\quad +\phi(j)(\theta - c)\zeta_1(t_{n-1}), \end{aligned} \quad \text{(B.1)}$$

where

$$\phi(j) \equiv \begin{cases} 0, & j < n-1, \\ 1, & j = n-1, \end{cases} \quad \text{(B.2)}$$

$$\zeta_1(t_{n-1}) = [\lambda Q_U + \mu(t_{n-1})]\, e^{-\lambda(t_n - t_{n-1})}. \quad \text{(B.3)}$$

Since $\tilde{v} = \lambda(p-c) - h > 0$ and $(\theta - c) > 0$, $\frac{\partial}{\partial t_j}P_n \ge 0$ agrees with

$$L_1(t_j) \ge 0, \quad \text{(B.4)}$$

where

$$\begin{aligned} L_1(t_j) &= \lambda e^{-\lambda(t_j - t_{j-1})}\left[Q_U - \tilde{m}(t_{j-1}, t_j)\right] \\ &\quad -\lambda Q_U e^{-\lambda(t_{j+1}-t_j)} \\ &\quad +\mu(t_j)\left[1 - e^{-\lambda(t_{j+1}-t_j)}\right] \\ &\quad +\phi(j)\frac{\theta - c}{\tilde{v}}\zeta_1(t_{n-1}). \end{aligned} \quad \text{(B.5)}$$

In addition, from Eq. (B.5), we have

$$\begin{aligned} L_1'(t_j) &= -\lambda\Big\{\lambda e^{-\lambda(t_j - t_{j-1})}\left[Q_U - \tilde{m}(t_{j-1}, t_j)\right] \\ &\quad +\mu(t_j) + e^{-\lambda(t_{j+1}-t_j)}\left[\lambda Q_U + \mu(t_j)\right]\Big\} \\ &\quad +\mu'(t_j)\left[1 - e^{-\lambda(t_{j+1}-t_j)}\right] \\ &\quad +\phi(j)\frac{\theta - c}{\tilde{v}}\Big\{\lambda[\lambda Q_U + \mu(t_{n-1})] \\ &\qquad +\mu'(t_{n-1})\Big\}e^{-\lambda(t_n - t_{n-1})}, \end{aligned} \quad \text{(B.6)}$$

$$\begin{aligned} L_1(t_{j-1}) &= [\lambda Q_U + \mu(t_{j-1})] \\ &\quad \times\left[1 - e^{-\lambda(t_{j+1}-t_{j-1})}\right] \\ &\quad +\phi(j)\frac{\theta - c}{\tilde{v}}\zeta_1(t_{n-2}) \ (>0), \end{aligned} \quad \text{(B.7)}$$

$$\begin{aligned} L_1(t_{j+1}) &= -\lambda\Big\{e^{-\lambda(t_{j+1}-t_{j-1})}\tilde{m}(t_{j-1}, t_{j+1}) \\ &\quad +Q_U\left[1 - e^{-\lambda(t_{j+1}-t_{j-1})}\right]\Big\} \\ &\quad +\phi(j)\frac{\theta - c}{\tilde{v}}\zeta_1(t_n), \end{aligned} \quad \text{(B.8)}$$

$$\begin{aligned} L_1(t_j^U) &= \mu(t_j^U)\left[1 - e^{-\lambda(t_{j+1}-t_j^U)}\right] \\ &\quad -\lambda Q_U e^{-\lambda(t_{j+1}-t_j^U)}. \\ &\quad +\phi(j)\frac{\theta - c}{\tilde{v}}\zeta_1(t_n^U). \end{aligned} \quad \text{(B.9)}$$

In the case of $\{j < n-2\}$ or $\{j = n-1 \text{ and } \theta = c\}$, Eqs. (B.6) and (B.8) respectively yield

$$\begin{aligned} L_1'(t_j) &= -\lambda\Big\{\lambda e^{-\lambda(t_j - t_{j-1})}\left[Q_U - \tilde{m}(t_{j-1}, t_j)\right] \\ &\quad +\mu(t_j) + e^{-\lambda(t_{j+1}-t_j)}\left[\lambda Q_U + \mu(t_j)\right]\Big\} \\ &\quad +\mu'(t_j)\left[1 - e^{-\lambda(t_{j+1}-t_j)}\right] \ (<0), \end{aligned} \quad \text{(B.10)}$$

$$\begin{aligned} L_1(t_{j+1}) &= -\lambda\Big\{e^{-\lambda(t_{j+1}-t_{j-1})}\tilde{m}(t_{j-1}, t_{j+1}) \\ &\quad +Q_U\left[1 - e^{-\lambda(t_{j+1}-t_{j-1})}\right]\Big\} \\ &\quad (<0), \end{aligned} \quad \text{(B.11)}$$

and furthermore, $L_1(t_j^U) < 0$ is equivalent to

$$t_{j+1} \quad < \quad \frac{1}{\lambda} \ln \frac{\lambda Q_U + \mu(t_j^U)}{\mu(t_j^U)} + t_j^U. \qquad \text{(B.12)}$$

Let us denote, by $\varphi(t_j^U)$, the right-hand-side of Inequality (B.12).

On the basis of the above results, for a given (t_{j-1}, t_{j+1}), we show below that an optimal replenishment time t_j^* exists:

(1) $\{j < n-1\}$ or $\{j = n-1$ and $\theta = c\}$:
In this subcase, we can confirm from Eqs. (B.7), (B.10) and (B.11) that $L_1'(t_1) < 0$ and $L_1(t_{j-1}) > 0 > L_1(t_{j+1})$.

- (a) $t_{j+1} < \varphi(t_j^U)$:
The sign of $\frac{\partial}{\partial t_j} P_n$ changes from positive to negative only once, and thus there exists a unique finite t_j^* $(t_{j-1} < t_j^* < \min(t_j^U, t_{j+1}))$ that maximizes P_n.
- (b) $t_{j+1} \geq \varphi(t_j^U)$:
P_n is non-decreasing in t_j, and consequently $t_j^* = t_j^U$.

(2) $\{j = n-1$ and $\theta > c$ and $L_1'(t_n) < 0\}$:
In this subcase, we show the existence of the optimal replenishment time, t_{n-1}^*, at the beginning of the last cycle when $\theta > c$. It is easily shown from Eq. (B.6) that a sufficient condition for $L_1'(t_{n-1}) < 0$ can be given by $\tilde{v} \geq (\theta - c)$ (> 0).

- (a) $\{L_1(t_n) < 0$ and $L(t_{n-1}^U) < 0\}$:
The sign of $\frac{\partial}{\partial t_{n-1}} P_n$ varies from positive to negative only once, and consequently there exists a unique finite t_{n-1}^* $(t_{n-2} < t_{n-1}^* < \min(t_{n-1}^U, t_n))$.
- (b) $\{L_1(t_n) \geq 0\}$ or $\{L_1(t_n) < 0$ and $L(t_{n-1}^U) \geq 0\}$:
We have $\frac{\partial}{\partial t_{n-1}} P_n \leq 0$ and therefore $t_{n-1}^* = \min(t_{n-1}^U, t_n)$.

Appendix C

In this appendix, we examine the existence of t_j^* in the case of $g(t) = \lambda\mu(t)$.

Let us here $L_2(t_j)$ be given by

$$\begin{aligned} L_2(t_j) \quad \equiv \quad & (p-c)g(t_j) \\ & \times \left\{ e^{-[G(t_j)-G(t_{j-1})]} - e^{-[G(t_{j+1})-G(t_j)]} \right\} \\ & + h \left\{ 1 - e^{-[G(t_j)-G(t_{j-1})]} \right. \\ & \left. - g(t_j) \int_{t_j}^{t_{j+1}} e^{-[G(u)-G(t_j)]} du \right\} \\ & + (\theta - c)\phi(j)\zeta_2(t_{n-1}), \qquad \text{(C.1)} \end{aligned}$$

where

$$\zeta_2(t_{n-1}) \quad \equiv \quad g(t_{n-1}) e^{-[G(t_n)-G(t_{n-1})]}. \qquad \text{(C.2)}$$

By differentiating P_n in Eq. (17) with respect to t_j, we have

$$\frac{\partial}{\partial t_j} P_n \quad = \quad \left(Q_U + \frac{1}{\lambda} \right) L_2(t_j). \qquad \text{(C.3)}$$

Since we have $\left(Q_U + \frac{1}{\lambda}\right) > 0$, $\frac{\partial}{\partial t_j} P_n \geq 0$ agrees with $L_2(t_j) \geq 0$. Furthermore, we have

$$\begin{aligned} L_2(t_{j-1}) \quad = \quad & (p-c)g(t_{j-1}) \left\{ 1 - e^{-[G(t_{j+1})-G(t_{j-1})]} \right\} \\ & - h g(t_{j-1}) e^{G(t_{j-1})} \int_{t_{j-1}}^{t_{j+1}} e^{-G(u)} du \\ & + (\theta - c)\phi(j)\zeta_2(t_{n-2}) \\ > \quad & g(t_{j-1}) e^{G(t_{j-1})} v(t_{j+1}) \int_{t_{j-1}}^{t_{j+1}} e^{-G(u)} du \\ & + (\theta - c)\phi(j)\zeta_2(t_{n-2}) \ (> 0), \qquad \text{(C.4)} \\ L_2(t_{j+1}) \quad = \quad & -v(t_{j+1}) \left\{ 1 - e^{-[G(t_{j+1})-G(t_{j-1})]} \right\} \\ & + (\theta - c)\phi(j)\zeta_2(t_n). \qquad \text{(C.5)} \end{aligned}$$

For $\{j < n-2\}$ or $\{j = n-2$ and $\theta = c\}$, Eq. (C.5) yields

$$\begin{aligned} L_2(t_{j+1}) \quad = \quad & -v(t_{j+1}) \left\{ 1 - e^{-[G(t_{j+1})-G(t_{j-1})]} \right\} \\ & (< 0). \qquad \text{(C.6)} \end{aligned}$$

Based on above results, we can show the conditions where an optimal replenishment time t_j^* exists in the case of $L_2'(t_j) < 0$:

(1) $\{j < n-1\}$ or $\{j = n-1$ and $\theta = c\}$:

- (a) $\{t_{j+1} \leq t_j^U\}$ or $\{L(t_j^U) < 0\}$.
The sign of $\frac{\partial}{\partial t_j} P_n$ varies from positive to negative only once, and hence there exists a unique finite t_j^* $(t_{j-1} < t_j^* < \min(t_j^U, t_{j+1}))$.
- (b) $\{t_{j+1} > t_j^U\}$ and $\{L(t_j^U) \geq 0\}$.
We have $\frac{\partial}{\partial t_j} P_n \geq 0$ and therefore $t_j^* = t_j^U$.

(2) $\{j < n-1\}$ or $\{j = n-1$ and $\theta > c\}$:
The classification necessary here is identical to that of Subcase(2) in Appendix B.

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