

On Sequential Calibration for an Asset Price Model with Piecewise Lévy Processes

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Abstract— We propose simple sequential calibration for an asset price model driven by piecewise Lévy processes, for which simulation methods and Greeks formulas are available. The proposed methods are easy to implement and consist of fitting a sequence of Lévy processes to a return series such that they allow parameters to change at discrete points in time, so that the fitted process can be made consistent with option prices with a range of maturity dates. Given a sequence of implied characteristic functions obtained from quants calibration routine, three calibration criteria are discussed; calibration to implied probability densities, to implied option premiums, and directly to the market quotes. Numerical results on equity index volatilities indicates that without calibrating to market quotes through the Fourier inversion, our method achieves a sufficient calibration accuracy with significantly lighter computation load.

Keywords: characteristic function, implied volatility, option premium, Parseval theorem, piecewise Lévy processes.

1 Introduction

The two main features missing from the Black-Scholes model are non-Gaussianity and time-varying volatility of the log returns. To realize the non-Gaussianity, various asset price models have been proposed. Among such well-known models are the Heston model [10] and the SABR model [9], both of which are of stochastic volatility type. With those models, the plain vanilla premium or the implied volatility are given in nearly closed form. Those explicit formulas prove quite useful in calibration to market quotes. Other successful candidates are models involving jumps, in particular Lévy processes, such as the variance gamma model of Madan and Seneta [18], the NIG model of Barndorff-Nielsen [2], the Meixner model of Schoutens and Teugels [20], and the CGMY model of Carr et al. [5]. All those models have attracted much attention among market practitioners for the main purpose to interpolate or predict prices of contracts of the same type as the calibration instruments.

The time-varying volatility feature is however still missing from those models. Namely, their validity is in doubt as soon as several discrete points in time have to be considered for

calibration altogether, while this is obviously an essential situation when the purpose of calibration lies in applications, such as pricing exotics and structured products, hedging or risk management. To address this issue, further advanced stochastic volatility models have been proposed by, for example, Barndorff-Nielsen and Shephard [3], Benhamou et al. [4] and Carr et al. [6]. Those models indicate excellent fit to option premiums over a wide range of strikes and maturities, while they entail a large amount of computational effort for sample paths simulation. This is a serious drawback, as most recent structured products are so complex that they can no longer be evaluated without Monte Carlo simulation. In addition, computation of Greeks is in general difficult in such advanced models.

The aim of the present paper is to propose simple sequential calibration methods on a relatively simple asset price model driven by piecewise Lévy processes, with a view towards both pricing and hedging of exotics. In short, a piecewise Lévy process is a stochastic process with independent increments, which is a homogeneous Lévy process over each predetermined interval. The advantages of employing pure-jump processes in financial modeling are justified in [5], for example. As far as discrete observations are concerned, sample paths of piecewise Lévy processes can be simulated by using various existing simulation methods for genuine Lévy processes. Equally important is that Greeks formulas available for Lévy process models (for example, [13, 14, 15]) are directly applicable to our piecewise Lévy process model as well thanks to the independence of increments. Calibration with piecewise Lévy processes are considered in Eberlein and Kluge [8] in the context of interest rate term structure without describing calibration techniques. Our model setting and associated calibration methods are designed primarily to meet practical needs with significantly lighter computation load, rather than to develop sophisticated theoretical results.

The rest of this paper is organized as follows. Section 2 recalls some general notations and describes our problem setting. In Section 3, we describe sequential calibration and propose three criteria for calibration; calibration to implied probability densities, to implied option premiums, and as usual, to market quotes. The validity and applicability of those methods are investigated. Section 4 presents numerical results on equity volatility to support the effectiveness of the proposed methods. Finally, Section 5 gives some remarks and indicates

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the direction of future research.

2 Preliminaries

Let us begin with some general notations and definitions which will be used throughout the paper. We denote by \mathbb{R} the one-dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, and by $\mathcal{B}(\mathbb{R}_0)$ the Borel σ -field of \mathbb{R}_0 . A stochastic process $\{X_t : t \geq 0\}$ in \mathbb{R} is a Lévy process if it has stationary and independent increments, is stochastically continuous, and $\mathbb{P}(X_0 = 0) = 1$. By the Lévy-Khintchine representation theorem, its marginal distribution is uniquely characterized by the triplet (γ, σ, ν) in the following characteristic function,

$$\mathbb{E} [e^{iyX_t}] = \exp \left[t \left(iy\gamma - \frac{1}{2} \sigma^2 y^2 + \int_{\mathbb{R}_0} (e^{iyz} - 1 - iyz \mathbb{1}_{(0,1]}(|z|)) \nu(dz) \right) \right], \quad (1)$$

where $\gamma \in \mathbb{R}$, $\sigma > 0$, and ν is a Lévy measure on \mathbb{R}_0 , that is, a σ -finite measure satisfying $\int_{\mathbb{R}_0} (|z|^2 \wedge 1) \nu(dz) < +\infty$. Let φ denote the characteristic exponent of the characteristic function (1), that is, $\varphi(y) := t^{-1} \ln \mathbb{E}[e^{iyX_t}]$, with $\varphi(-i) := t^{-1} \ln \mathbb{E}[e^{X_t}]$ whenever the expectation is well defined. We say that a stochastic process $\{X_t : t \geq 0\}$ in \mathbb{R} is a *piecewise Lévy process* if, on some predetermined disjoint time partitions $[t_0, t_1), [t_1, t_2), [t_2, t_3), \dots$ with $t_0 := 0$, the stochastic process $\{X_t - X_{t_{k-1}} : t \in [t_{k-1}, t_k)\}$ is a Lévy process for each $k \in \mathbb{N}$. Finally, for convenience, we write ν_t, σ_t and φ_t , respectively, for the Lévy measure, the diffusion coefficient and the characteristic exponent of a piecewise Lévy process over the time interval I .

Consider a mean-correcting asset price dynamics $\{S_t : t \geq 0\}$ defined by

$$S_t := e^{rt} S_0 \frac{e^{X_t}}{\mathbb{E}[e^{X_t}]}, \quad (2)$$

where $\{X_t : t \geq 0\}$ is a Lévy process satisfying $\mathbb{E}[e^{X_1}] < +\infty$. The discounted dynamics $\{e^{-rt} S_t : t \geq 0\}$ is a martingale with respect to the natural filtration generated by $\{X_t : t \geq 0\}$, and

$$\mathbb{E} [e^{iy \ln S_t}] = e^{iy(rt + \ln S_0)} \frac{e^{t\varphi(y)}}{e^{iyt\varphi(-i)}}.$$

This suggests the use of Lévy processes for which the characteristic exponent φ is available in closed form.

Suppose we have obtained a sequence of implied Lévy measures $\{\nu_{[0,t_k]}\}_{k \in \mathbb{N}}$ and one of implied diffusion coefficients $\{\sigma_{[0,t_k]}\}_{k \in \mathbb{N}}$. (This assumption is not against reality; financial quants perform calibration of hedging instruments for every maturity as a frequent routine to keep track of the volatility smile, to check the robustness of model parameters, and to interpolate the volatility smile.) In order for those implied parameters to be consistent, they must satisfy the constraints; for each $k \in \mathbb{N}$,

$$t_k \nu_{[0,t_k]}(B) \leq t_{k+1} \nu_{[0,t_{k+1}]}(B), \quad B \in \mathcal{B}(\mathbb{R}_0). \quad (3)$$

When all those constraints are satisfied, the effective Lévy measure $\nu_{(t_k,t_{k+1}]}$ and the characteristic exponent $\varphi_{(t_k,t_{k+1}]}$ are given respectively by

$$\nu_{(t_k,t_{k+1}]}(B) = \frac{t_{k+1} \nu_{[0,t_{k+1}]}(B) - t_k \nu_{[0,t_k]}(B)}{t_{k+1} - t_k}, \quad B \in \mathcal{B}(\mathbb{R}_0),$$

$$\varphi_{(t_k,t_{k+1}]}(y) = \frac{t_{k+1} \varphi_{[0,t_{k+1}]}(y) - t_k \varphi_{[0,t_k]}(y)}{t_{k+1} - t_k}.$$

On the contrary, as soon as any single violation is found, for example, $t_k \nu_{[0,t_k]}(B) > t_{k+1} \nu_{[0,t_{k+1}]}(B)$ or $t_k \sigma_{[0,t_k]}^2 > t_{k+1} \sigma_{[0,t_{k+1}]}^2$ for some $k \in \mathbb{N}$ and $B \in \mathcal{B}(\mathbb{R}_0)$, then the calibration results for $[0, t_{k+1}]$ and thereafter simply do not make any sense. Based on our numerical experiments for various instruments and underlying Lévy processes, it seems very difficult to complete entire calibration procedure without such violations.

3 Sequential Calibration with Piecewise Lévy Processes

To address the inconsistency issue, we employ a piecewise Lévy process by suppressing the stationarity of increments of the underlying homogeneous Lévy process in the base model (2) and propose sequential calibration with such piecewise Lévy processes. In what follows, we denote by $\{X_t : t \geq 0\}$ a piecewise Lévy process and restrict ourselves to the pure-jump settings, that is, the diffusion component is assumed to be degenerate. The asset price dynamics (2) then reads

$$\mathbb{E} [e^{iy \ln S_t}] = e^{iy(rt + \ln S_0)} \frac{\prod_{j=1}^k e^{(t_j \wedge t - t_{j-1}) \varphi_{(t_{j-1}, t_j]}(y)}}{\prod_{j=1}^k e^{iy(t_j \wedge t - t_{j-1}) \varphi_{(t_{j-1}, t_j]}(-i)}},$$

for $t \in (t_{k-1}, t_k]$. The discounted price dynamics $\{e^{-rt} S_t : t \geq 0\}$ is clearly a martingale with respect to the natural filtration generated by the underlying piecewise Lévy process $\{X_t : t \geq 0\}$. (The risk-free rate is left independent of time for simplicity, while it can easily be generalized.) Suppose again we have a sequence of implied characteristic functions for intervals $[0, t_k]$, $k \in \mathbb{N}$. We then wish to find a sequence $\{\varphi_{(t_k, t_{k+1}]}\}_{k \in \mathbb{N}}$ of characteristic exponents, in such a way that a set of approximations

$$\mathbb{E} [e^{iy \ln S_{t_{k+1}}}] \leftarrow \mathbb{E} [e^{iy \ln S_{t_k}}] e^{iyr(t_{k+1} - t_k)} \frac{e^{(t_{k+1} - t_k) \varphi_{(t_k, t_{k+1}]}(y)}}{e^{iy(t_{k+1} - t_k) \varphi_{(t_k, t_{k+1}](-i)}},$$

are adequately accurate.

3.1 Calibration to Implied Probability Densities

First, we consider choosing $\varphi_{(t_k, t_{k+1}]}$ through minimization of the following distance measure between two characteristic functions

$$\left[\int_{\mathbb{R}} |\widehat{\mu}_T(y) - \widehat{\mu}_P(y)|^2 dy \right]^{\frac{1}{2}}, \quad (4)$$

that is, the calibration of $\widehat{\mu}_P$ to the given target $\widehat{\mu}_T$, where in our case,

$$\begin{bmatrix} \widehat{\mu}_T \\ \widehat{\mu}_P \end{bmatrix} = \begin{bmatrix} \mathbb{E}[e^{iy \ln S_{t_{k+1}}}] \\ \mathbb{E}[e^{iy \ln S_{t_k}}] e^{iy r(t_{k+1}-t_k)} \frac{e^{(t_{k+1}-t_k)\varphi(t_k, t_{k+1})|^{(y)}}}{e^{iy(t_{k+1}-t_k)\varphi(t_k, t_{k+1})|^{(-i)}}} \end{bmatrix},$$

with given $\mathbb{E}[e^{iy \ln S_{t_{k+1}}}]$ and $\mathbb{E}[e^{iy \ln S_{t_k}}] e^{iy r(t_{k+1}-t_k)}$. (Hereafter, ‘‘T’’ and ‘‘P’’ indicate ‘‘Target’’ and ‘‘Proposal’’, respectively.) If the consistency constraint (3) is satisfied for $[0, t_k]$ and $[0, t_{k+1}]$, we may instead set

$$\begin{bmatrix} \widehat{\mu}_T \\ \widehat{\mu}_P \end{bmatrix} = \begin{bmatrix} \mathbb{E}[e^{iy \ln S_{t_{k+1}}}] / \mathbb{E}[e^{iy \ln S_{t_k}}] \\ e^{iy r(t_{k+1}-t_k)} \frac{e^{(t_{k+1}-t_k)\varphi(t_k, t_{k+1})|^{(y)}}}{e^{iy(t_{k+1}-t_k)\varphi(t_k, t_{k+1})|^{(-i)}}} \end{bmatrix}.$$

This would not make sense otherwise since then $\widehat{\mu}_T$ would no longer be a characteristic function of any infinitely divisible distribution (might even not be a characteristic function of any distribution).

The Parseval theorem adds a further meaning to the minimization of the distance (4). That is, by letting f_T and f_P be probability density functions, respectively, corresponding to the characteristic functions $\widehat{\mu}_T$ and $\widehat{\mu}_P$, it holds that

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\mu}_T(y) - \widehat{\mu}_P(y)|^2 dy = \int_{\mathbb{R}} |f_T(x) - f_P(x)|^2 dx. \quad (5)$$

This identity indicates that minimization of the distance (4) is equivalent to minimization of the $L^2(\mathbb{R})$ -distance between two probability density functions.

Note that only one integral computation (4) is required at each step of the minimization for a target maturity, no matter how many strike prices are under consideration. This helps reduce computation load, compared to the ordinary calibration which essentially requires computation of integrals as many times as the number of strike prices.

The numerical error for (4) can be decomposed into two types. One is the truncation error $\int_{|y|>U} |\widehat{\mu}_T(y) - \widehat{\mu}_P(y)|^2 dy$, while the other is the discretization error in evaluating the remaining integral

$$\int_{|y|\leq U} |\widehat{\mu}_T(y) - \widehat{\mu}_P(y)|^2 dy. \quad (6)$$

Recall that our base model (2) is build on the assumption of $\mathbb{E}[e^{X_t}] < +\infty$ for each t . This implies that the (polynomial) moment property of X_t depends only on the negative side of Lévy measure, that is, ν on $(-\infty, -1)$. Moreover, finite higher moments guarantee the differentiability of the characteristic function of the corresponding order. Hence, the integral (6) can be evaluated with a suitable Simpson-type rule. Coupled with the fact that the characteristic function is uniformly continuous and its modulus is uniformly bounded by 1, it can be expected that (6) can be evaluated without significant discretization error. The truncation error can also be kept negligibly small by setting the truncation level U sufficiently large.

Let us summarize our calibration procedure;

- (i) Find $\varphi_{(t_1, t_2]}$ through minimization of the distance (4) with

$$\begin{bmatrix} \widehat{\mu}_T \\ \widehat{\mu}_P \end{bmatrix} = \begin{bmatrix} \mathbb{E}[e^{iy \ln S_{t_2}}] \\ \mathbb{E}[e^{iy \ln S_{t_1}}] e^{iy r(t_2-t_1)} \frac{e^{(t_2-t_1)\varphi_{(t_1, t_2]}(y)}}{e^{iy(t_2-t_1)\varphi_{(t_1, t_2]}(-i)}} \end{bmatrix}.$$

- (ii) With $\varphi_{(t_1, t_2]}$ obtained in (i), find $\varphi_{(t_2, t_3]}$ through minimization of (4) with

$$\widehat{\mu}_T(y) = \mathbb{E}[e^{iy \ln S_{t_3}}],$$

and

$$\begin{aligned} \widehat{\mu}_P(y) = & \mathbb{E}[e^{iy \ln S_{t_1}}] e^{iy r(t_3-t_1)} \\ & \times \frac{e^{(t_2-t_1)\varphi_{(t_1, t_2]}(y)}}{e^{iy(t_2-t_1)\varphi_{(t_1, t_2]}(-i)}} \frac{e^{(t_3-t_2)\varphi_{(t_2, t_3]}(y)}}{e^{iy(t_3-t_2)\varphi_{(t_2, t_3]}(-i)}}. \end{aligned}$$

- (iii) Continue the procedure forward for $\varphi_{(t_3, t_4]}$, $\varphi_{(t_4, t_5]}$, and so on.

3.2 Calibration to Model Premiums of Implied Distribution

The next criterion is a variant of the previous one, that is, minimization of a modified distance measure between the characteristic functions inspired by the Carr-Madan formula [7]. Consider the distance measure

$$\begin{aligned} H_1(\widehat{\mu}_T, \widehat{\mu}_P) & := \left[\int_0^{+\infty} \frac{|\widehat{\mu}_T(y - (\alpha + 1)i) - \widehat{\mu}_P(y - (\alpha + 1)i)|^2}{(y^2 + \alpha^2)(y^2 + (\alpha + 1)^2)} dy \right]^{\frac{1}{2}}, \quad (7) \end{aligned}$$

and the root mean squared error (rmse) H_2 in premiums, defined by

$$H_2(\widehat{\mu}_T, \widehat{\mu}_P) := \left[\frac{1}{M} \sum_{m=1}^M [C_T(K_m) - C_P(K_m)]^2 \right]^{\frac{1}{2}},$$

where $\{K_m\}_{m=1, \dots, M}$ is a sequence of strike prices, and where $C_T(K)$ and $C_P(K)$ are call premiums at strike K , respectively, of the target characteristic function $\widehat{\mu}_T$ and of the proposal $\widehat{\mu}_P$, each of which models the marginal $\ln S_t$. The two quantities H_1 and H_2 can be related as follows.

Proposition 3.1. *It holds that for each pair $(\widehat{\mu}_T, \widehat{\mu}_P)$,*

$$H_2(\widehat{\mu}_T, \widehat{\mu}_P) \leq \left[\sum_{m=1}^M \frac{e^{-2r t - 2\alpha \ln K_m}}{M \pi^2} \right]^{\frac{1}{2}} H_1(\widehat{\mu}_T, \widehat{\mu}_P).$$

Proof. By the well known result of Carr and Madan [7], we have

$$C(K) = \frac{e^{-r t - \alpha \ln K}}{\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iy \ln K} \widehat{\mu}(y - (\alpha + 1)i)}{\alpha^2 + \alpha - y^2 + i(2\alpha + 1)y} \right] dy. \quad (8)$$

Hence, it holds that

$$C_T(K) - C_P(K) = \frac{e^{-rt - \alpha \ln K}}{\pi} \times \int_0^{+\infty} \operatorname{Re} \left[e^{-iy \ln K} \frac{\widehat{\mu}_T(y - (\alpha + 1)i) - \widehat{\mu}_P(y - (\alpha + 1)i)}{\alpha^2 + \alpha - y^2 + i(2\alpha + 1)y} \right] dy.$$

Therefore, we have

$$\begin{aligned} & H_2(\widehat{\mu}_T, \widehat{\mu}_P)^2 \\ & \leq \left[\sum_{m=1}^M \frac{e^{-2rt - 2\alpha \ln K_m}}{M\pi^2} \right] \\ & \quad \times \int_0^{+\infty} \left| e^{-iy \ln K_m} \frac{\widehat{\mu}_T(y - (\alpha + 1)i) - \widehat{\mu}_P(y - (\alpha + 1)i)}{\alpha^2 + \alpha - y^2 + i(2\alpha + 1)y} \right|^2 dy \\ & = \left[\sum_{m=1}^M \frac{e^{-2rt - 2\alpha \ln K_m}}{M\pi^2} \right] \\ & \quad \times \int_0^{+\infty} \frac{|\widehat{\mu}_T(y - (\alpha + 1)i) - \widehat{\mu}_P(y - (\alpha + 1)i)|^2}{(y^2 + \alpha^2)(y^2 + (\alpha + 1)^2)} dy, \end{aligned}$$

where the last equality holds by the properties of complex modulus; for $(z, w) \in \mathbb{C}^2$, $|zw| = |z||w|$ and $|z/w| = |z|/|w|$ when $|w| \neq 0$. \square

As in the method of Section 3.1, computation of integrals has to be done only once at each step of minimization since the integral (7) is independent of strike prices $\{K_m\}_{m \in \mathbb{N}}$. Minimizing H_1 is not identical to minimizing H_2 , while numerical illustrations later indicate the effectiveness of the inequality of Proposition 3.1.

In order to investigate discretization error in computation of the integral

$$\int_0^U \frac{|\widehat{\mu}_T(y - (\alpha + 1)i) - \widehat{\mu}_P(y - (\alpha + 1)i)|^2}{(y^2 + \alpha^2)(y^2 + (\alpha + 1)^2)} dy, \quad (9)$$

the differentiability of the integrand acts as a key. The denominator is strictly positive and in $C^\infty(\mathbb{R}_+; \mathbb{R}_+)$, while the differentiability of the numerator depends on the polynomial moments of underlying distribution. Similarly to the characteristic function, the function $\widehat{\mu}(y - (\alpha + 1)i)$ is also uniformly continuous and its modulus is bounded by $|\widehat{\mu}(-(\alpha + 1)i)|$. Depending on the order of differentiability, the integral (9) can be evaluated using a suitable Simpson-type rule. The truncation error here should be negligible as decay of the numerator at infinity is accelerated by the denominator of fourth polynomial order. Moreover, the integrand of (9) is defined only on the half real line. These also helps reduce the required computation time, compared to the computation for the $L^2(\mathbb{R})$ -distance (4).

Remark 3.2. In general, the integrals (4) and (7) are multimodal with respect to model parameters. Hence, minimization of those integrals with respect to model parameters are ill-posed inverse problems. Calibration results may be very unstable with respect to perturbation of implied characteristic

functions. Similarly to classical approaches to regularization of ill-posed problems, we may regularize results by making a suitable addition of a penalization term to (4) and (7). For example, letting $\widehat{\mu}_{T_\epsilon}$, $\widehat{\mu}_{P_\epsilon}$ and $\widehat{\mu}_P$ be characteristic functions respectively of an perturbed implied version, to be calibrated to $\widehat{\mu}_{T_\epsilon}$, and the one obtained from the calibration to $\widehat{\mu}_T$, we instead minimize

$$\left[\int_{\mathbb{R}} |\widehat{\mu}_{T_\epsilon}(y) - \widehat{\mu}_{P_\epsilon}(y)|^2 dy \right]^{\frac{1}{2}} + \beta \left[\int_{\mathbb{R}} |\widehat{\mu}_P(y) - \widehat{\mu}_{P_\epsilon}(y)|^2 dy \right]^{\frac{1}{2}}$$

or

$$H_1(\widehat{\mu}_{T_\epsilon}, \widehat{\mu}_{P_\epsilon}) + \beta H_1(\widehat{\mu}_P, \widehat{\mu}_{P_\epsilon}),$$

with an appropriate choice of the regularization parameter $\beta > 0$ and with the initial guess of $\widehat{\mu}_{P_\epsilon}$ being $\widehat{\mu}_P$. There is a large literature on methods for addressing ill-posed calibration issues, dating back to Avellaneda et al. [1]. \square

3.3 Calibration to Market Quotes

Let us finally describe direct calibration to market quotes. Just as in ordinary calibration, we try to find the best characteristic function by minimizing the root mean square error (rmse) between market premiums (market-pr) and model premiums (model-pr)

$$(\text{rmse}) := \left[\frac{1}{\#(\text{pr})} \sum \left[(\text{market-pr}) - (\text{model-pr}) \right]^2 \right]^{\frac{1}{2}}.$$

We should work with closed-form characteristic functions to apply the Carr-Madan equation (8). (See Lee [17] for its numerical error analysis.) We proceed with calibration as follows.

- (i) With the given implied characteristic function $\mathbb{E}[e^{iy \ln S_{t_1}}]$, find $\varphi_{(t_1, t_2)}$ through calibration of

$$\mathbb{E} \left[e^{iy \ln S_{t_1}} \right] e^{iy r(t_2 - t_1)} \frac{e^{(t_2 - t_1)\varphi_{(t_1, t_2)}(y)}}{e^{iy(t_2 - t_1)\varphi_{(t_1, t_2)}(-i)}}$$

to the market quotes for the maturity t_2 .

- (ii) With the implied characteristic function $\mathbb{E}[e^{iy \ln S_{t_1}}]$ and the implied characteristic exponent $\varphi_{(t_1, t_2)}$ obtained in (i), find $\varphi_{(t_2, t_3)}$ through calibration of

$$\mathbb{E} \left[e^{iy \ln S_{t_1}} \right] e^{iy r(t_3 - t_1)} \frac{e^{(t_2 - t_1)\varphi_{(t_1, t_2)}(y)}}{e^{iy(t_2 - t_1)\varphi_{(t_1, t_2)}(-i)}} \frac{e^{(t_3 - t_2)\varphi_{(t_2, t_3)}(y)}}{e^{iy(t_3 - t_2)\varphi_{(t_2, t_3)}(-i)}}$$

to the market quotes for the maturity t_3 .

- (iii) Continue the procedure forward for $\varphi_{(t_3, t_4)}$, $\varphi_{(t_4, t_5)}$, and so on.

4 Numerical Illustration: Equity Volatilities

In this section, we present some numerical results for the Nikkei 225 index options. To avoid overloading the paper with results of somewhat repetitional nature, we only report

results on a single day of June 30, 2006. We consider the premiums available at the strikes (60P, 80P, 90P, 95P, 100C, 105C, 110C, 120C, 150C), each of which indicates moneyness in percentage points with ATM 100, for each maturity $(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)=(1, 3, 6, 12, 24, 36, 48, 60)$, given in month. Here, P and C in the strikes respectively stand for “Put” and “Call”. We use the put-call parity to compute the put premiums. Both the interest rates and the dividend yields are assumed to be deterministic. We do not use the quotes at the strikes 60P and 150C for the two short maturities t_1 and t_2 , for the reason of unreliability of data due to extreme illiquidity.

We use an extension of the CGMY process of [5] for the piecewise Lévy process $\{X_t : t \geq 0\}$ in the asset price model (2). It is a Lévy process without Gaussian component and whose Lévy measure is given by

$$v(dz) = \left[C_n \frac{e^{-G|z|}}{|z|^{1+Y_n}} \mathbb{1}(z < 0) + C_p \frac{e^{-M|z|}}{|z|^{1+Y_p}} \mathbb{1}(z > 0) \right] dz,$$

defined on \mathbb{R}_0 , where $C_n, C_p, G, M > 0$ and $Y_n, Y_p \in (-\infty, 2)$. Its characteristic exponent is as simple as

$$\begin{aligned} \varphi(y) = & C_n \Gamma(-Y_n) ((G + iy)^{Y_n} - G^{Y_n}) \\ & + C_p \Gamma(-Y_p) ((M - iy)^{Y_p} - M^{Y_p}), \end{aligned}$$

provided that $Y_n \neq 1$ and $Y_p \neq 1$. The CGMY process can serve as an appropriate underlying process, not only because it proves capable of reproducing the implied volatility structure, but also because the simulation methods [16] and the Greeks formulas [15] are available. In order that our model be well defined, the expectation $\mathbb{E}[e^{X_1}]$ is required to be finite. By Theorem 25.17 of Sato [19], it is equivalent to $\int_{|z|>1} e^z v(dz) < +\infty$, which yields $M \geq 1$ if $Y_p > 0$ and $M > 1$ otherwise. We henceforth impose this condition with $Y_n \neq 1$ and $Y_p \neq 1$, throughout the calibration procedure.

First of all, let us give in Table 1 the estimated parameters through the preliminary calibration to the market quotes separately for $[0, t_k], k = 1, \dots, 8$. We use the Carr-Madan equation (8) to compute the premiums, with the integrand evaluated at 2^{14} points with equidistant spacing of 0.25, with the adjustment constant $\alpha = 1$, and with the Simpson rule. To apply (8), the additional condition $\mathbb{E}[S_t^{\alpha+1}] < +\infty$ must be satisfied. By Theorem 25.17 of Sato [19], this condition is equivalent to $M \geq \alpha + 1$ if $Y_p > 0$ and $M > \alpha + 1$ otherwise. We perform the Nelder-Mead direct search method to minimize the difference between market premiums and model premiums in rmse. Regardless of the successful results, the condition (3) is however violated, for example, we can find that at $z = 2$,

$$t_2 C_p(2) \frac{e^{-G(2)|z|}}{|z|^{Y_p(2)+1}} > t_3 C_p(3) \frac{e^{-G(3)|z|}}{|z|^{Y_p(3)+1}}.$$

The calibration results thus indicates inconsistency as dynamics. Let us remark here that the primal aim of the preliminary calibration is to obtain a sequence of implied characteristic functions with very high calibration accuracy. Hence, at this

stage, the underlying process does not have to be in a class of Lévy processes, but could be even a certain stochastic volatility model, so long as a characteristic function is available.

Table 2 indicates how significant the mis-pricing would be if one used the implied parameters fitted to the longest interval $[0, t_8]$ to evaluate the option premiums for shorter maturities. One can see that this homogeneous Lévy process model tends to yield lower premiums around ATM, while higher far from ATM.

Next, Table 3 presents results of calibration to the implied characteristic function, or equivalently, to the implied probability density function by (5). The characteristic functions are evaluated at 2^{13} points with equidistant spacing of 0.25 and with the Simpson rule. Minimization of (4) is performed again with the Nelder-Mead direct search method. The $L^2(\mathbb{R})$ -distance of two probability density functions remains quite small, and the rmse does not seem to accumulate as the maturity gets longer.

Results of calibration to the model premiums of the implied distribution are given in Table 4, with H_1 indicating minimized values of (7). In the Nelder-Mead minimization procedure, the integrand of (7) is evaluated at 2^{12} points with equidistant spacing of 0.02 and with $\alpha = 1$, and with Simpson rule. Compared to the rmses in Table 3, they are here much smaller. Moreover, the required computation time for the calibration procedure (with the same initial guess of CGMY parameters) is approximately 90% shorter. For clear illustration, we give in Figure 1 model option premiums (“—”) with market quotes (“o”). The market premiums at the money and out of the money ($K \geq 100$) are of call options, while those in the money ($K < 100$) are of put options. The discontinuity at $K = 100$ is due to the dividend yield.

Finally, Table 5 presents results of calibration directly to market quotes. We use the Carr-Madan equation (8) to transform the closed-form characteristic function directly into option premiums, where the characteristic functions are evaluated at 2^{14} points with equidistant spacing of 0.25, with the adjustment constant $\alpha = 1$. We again impose the conditions induced by $\mathbb{E}[S_t^{\alpha+1}] < +\infty$ in the Nelder-Mead direct search method. As opposed to the accurate calibration, the computational overhead required here is much more than calibration to the implied distributions, as the integration (8) has to be repeated nine times (for nine strikes). Regardless of similar calibration accuracy through three criteria, the estimated parameters are significantly different for different criteria.

5 Concluding Remarks

We have proposed sequential calibration on an asset price model driven by piecewise Lévy processes. Our model settings are kept reasonably simple with a view towards practical use in the sense that the existing Monte Carlo simulation methods and the Greeks formulas for genuine Lévy processes are directly applicable. Three calibration criteria are investi-

gated; calibration to implied probability densities, to implied model premiums, and directly to market quotes. It is remarkable, from a practical point of view, that calibration to implied model premiums achieves an adequate degree of accuracy with significantly lighter computation load, relative to direct calibration to market quotes.

We did not present in this paper an exhaustive study of its range of applicability relative to different index options or underlying Lévy processes, which is significantly large. With successful calibration, the valuation of various structured products, such as TARN, snowball, or even more complicated ones, are certainly within reach through straightforward Monte Carlo simulation. Finally, combined with various other methods and models, such as a numerical inversion method [11] and a multivariate model with linear correlation [12], our methods may pave the way to evaluate and hedge even more intricate exotics, consisting of a basket of underlying assets over several discrete points in time, without employing intricate stochastic volatility formulations.

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Table 1: Estimated parameters through calibration separately for $[0, t_k], k = 1, \dots, 8$.

	C_n	C_p	G	M	Y_n	Y_p	rmse
$[0, t_1]$.01555	.009194	3.642	3.893	1.793	1.090	.0085
$[0, t_2]$.01616	.019597	2.526	3.801	1.761	.5650	.0082
$[0, t_3]$.01625	.011869	2.174	3.866	1.711	1.158	.0126
$[0, t_4]$.02332	.007939	2.143	4.027	1.520	1.595	.0140
$[0, t_5]$.04123	.015499	2.318	4.025	1.295	1.368	.0106
$[0, t_6]$.07213	.025245	2.491	4.029	1.020	1.206	.0063
$[0, t_7]$.11600	.035387	2.606	4.042	.7248	1.104	.0042
$[0, t_8]$.16832	.049414	2.685	4.029	.4708	.9595	.0049

Table 2: Price gap (model premium)-(market quote) when the asset price dynamics is modeled with a homogeneous Lévy process of the parameters for $[0, t_8]$.

	60P	80P	90P	95P	100C	105C	110C	120C	150C	rmse
$[0, t_1]$	n/a	.1053	.0378	-.4012	-1.469	-.4896	-.0211	.0660	n/a	.6066
$[0, t_2]$	n/a	.1699	-.2539	-.9413	-1.734	-1.153	-.4581	.0563	n/a	.8886
$[0, t_3]$.1326	.2187	-.3053	-.8942	-1.449	-1.286	-.7612	-.0711	.0988	.7678
$[0, t_4]$.2345	.2856	-.1747	-.5494	-.8598	-.9098	-.7168	-.2008	.1712	.5395
$[0, t_5]$.3123	.3255	.0711	-.0900	-.2296	-.3090	-.3160	-.1722	.2040	.2440
$[0, t_6]$.2895	.3191	.2110	.1419	.0791	.0296	.0015	-.0064	.1799	.1794
$[0, t_7]$.1821	.2110	.1825	.1600	.1410	.1170	.1005	.0811	.1259	.1501
$[0, t_8]$.0048	-.0087	-.0007	.0005	.0081	.0013	.0008	-.0064	.0015	.0049

Table 3: Estimated parameters through sequential calibration to implied probability densities.

	C_n	C_p	G	M	Y_n	Y_p	L^2 -dist.	rmse
$[0, t_1]$.01554	.009194	3.642	3.893	1.793	1.090	0	.0085
$(t_1, t_2]$.01837	.01979	2.419	3.802	1.723	.5679	.0004	.0091
$(t_2, t_3]$.01874	.01135	1.994	3.937	1.602	1.343	.0001	.0131
$(t_3, t_4]$.06310	.00970	2.950	4.452	1.025	1.608	.0006	.0148
$(t_4, t_5]$.09185	.02842	2.858	4.160	.9331	1.092	.0006	.0110
$(t_5, t_6]$.3549	.06817	3.585	4.290	.1570	.7200	.0009	.0070
$(t_6, t_7]$.7611	.1172	3.768	4.443	-.5122	.5164	.0012	.0074
$(t_7, t_8]$	1.436	.1980	4.043	4.475	-1.044	.1663	.0015	.0070

Table 4: Estimated parameters through sequential calibration to model premiums of implied distribution.

	C_n	C_p	G	M	Y_n	Y_p	H_1	rmse
$[0, t_1]$.01554	.009194	3.642	3.893	1.793	1.090	0	.0085
$(t_1, t_2]$.01880	.02358	2.430	3.744	1.719	.4642	6.40e-6	.0080
$(t_2, t_3]$.02288	.008743	2.298	4.302	1.530	1.537	2.39e-5	.0124
$(t_3, t_4]$.07601	.01096	3.101	5.205	.9133	1.613	4.17e-5	.0138
$(t_4, t_5]$.09787	.03634	2.915	4.532	.9029	1.024	1.88e-5	.0101
$(t_5, t_6]$.3424	.07654	3.511	4.540	.1521	.7087	2.57e-5	.0061
$(t_6, t_7]$.9179	.1088	3.936	4.516	-.6630	.6100	4.10e-5	.0044
$(t_7, t_8]$	1.974	.1942	4.328	4.568	-1.301	.2523	6.31e-5	.0051

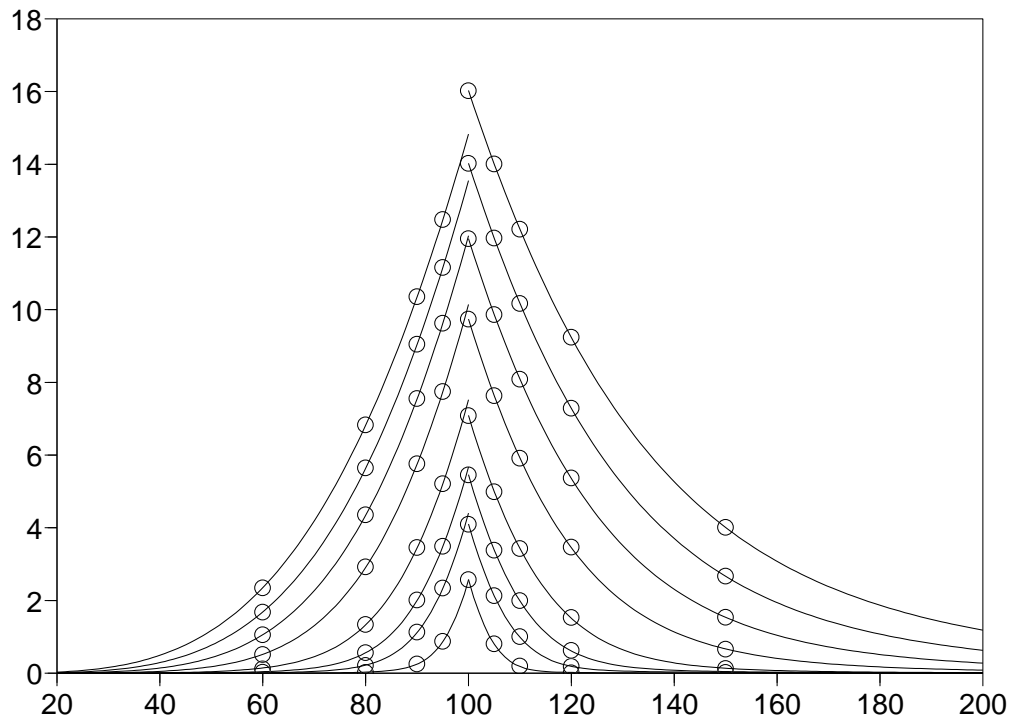


Figure 1: Market quotes (“o”), and model option premiums (“—” through sequential calibration to model premiums of implied distribution).

Table 5: Estimated parameters through sequential calibration to market quotes.

	C_n	C_p	G	M	Y_n	Y_p	rmse
$[0, t_1]$.01554	.009194	3.642	3.893	1.793	1.090	.0085
$(t_1, t_2]$.04146	.006598	3.920	4.199	1.511	1.658	.0063
$(t_2, t_3]$.1024	.01019	4.449	5.062	.9928	1.647	.0076
$(t_3, t_4]$.08356	.01773	3.415	5.295	.9674	1.443	.0101
$(t_4, t_5]$.1477	.03298	3.295	4.685	.6863	1.148	.0083
$(t_5, t_6]$.5569	.08893	4.064	4.680	-.07991	.6549	.0050
$(t_6, t_7]$	1.242	.1186	4.239	4.626	-.8392	.5802	.0033
$(t_7, t_8]$	2.108	.2010	4.386	4.612	-1.348	.2388	.0042