Sorting by Prefix Reversals

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Abstract—**The pancake problem, which has attracted considerable attention, concerns the number of prefix reversals needed to sort the elements of an arbitrary permutation. The number of prefix reversals to sort permutations is also the diameter of the often studied n-dimensional Pancake network. We consider restricted pancake problem, when only k of the possible $n-1$ prefix reversals are allowed.**

Keywords: pancake problem, prefix reversals, symmetric group, Cayley networks, diameter of network

1 Introduction

Given the set

$$T_n = \{1, 2, \ldots, n\},$$

a permutation π of T_n is bijective function $\pi : T_n \to T_n$. The symmetric group S_n is the set of all the permutations of T_n. We can view a permutation $\pi \in S_n$ as an ordered arrangement of the elements in T_n where $\pi[i]$ is the element in position i. In this view, the integers $1, 2, \ldots, n$ are used to indicate both positions and elements. The prefix reversal R_i, where

$$1 < i \leq n,$$

is the permutation

$$(i, i-1, \ldots, 1, i+1, i+2, \ldots, n).$$

The pancake problem, which has attracted considerable attention [1], [2], [3], [4], concerns the number of prefix reversals needed to sort the elements of an arbitrary permutation. The number of prefix reversals to sort permutations is also the diameter of the often studied n-dimensional Pancake network [5], [6], [7], [8], [9], [10]. Let $d(n)$ be the maximum number of prefix reversals needed to sort any permutation on n symbols. The best bounds known for $d(n)$ are

$$\frac{15}{14}n \leq d(n) \leq \frac{5n+5}{3}$$

[2], [3]. A related problem, called the burnt pancake problem, [2], concerns the number of prefix reversals needed to sort signed permutations, where each symbol has an attached positive or negative sign and, each time the symbol is involved in a prefix reversal, the sign changes. Let $d_{sign}(n)$ be the maximum number of prefix reversals needed to sort any signed permutation on n symbols. The best bounds known for $d_{sign}(n)$ are

$$\frac{3}{2}n \leq d_{sign}(n) \leq 2n - 3$$

(see [2], [3]).

We consider restricted pancake problems, when only k of the possible $n-1$ prefix reversals are allowed. Suppose we have a group G and a subset X of G. If every element of G can be generated as a finite product of the elements of X, then the elements of X are called generators, and X is called a generating set of G. We also say that X generates G. If G is a group and X generates G, then the Cayley network $C(G, X)$ is a network where the nodes are the elements of G, and the edges are all ordered pairs (a, b) where $b = ac$, for some $c \in G$ and $a \in X$. If the inverse of every generator is again in the generating set, then the Cayley network will be undirected. Cayley networks generated by sets of prefix reversals are always undirected, since every prefix reversal is its own inverse.

Cayley graphs have been extensively studied [10], [11], [12], [13] as bases for interconnection networks, due to their many desirable properties, including regularity, vertex-symmetry and recursive or near-recursive substructure. Recently, a number of Cayley networks of degree $O(1)$ have been proposed [13], [14], [15], [16], [17].

Let $D(G, X)$ be the diameter of the undirected Cayley network on G generated by X. In [13] proved that if $|X| = O(1)$, then

$$D(S_n, X) = \Omega(n \log_2 n).$$

Also in [13] proved that

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$$D(S_n, \{R_{\lfloor \frac{n}{2} \rfloor}, R_{\lceil \frac{n}{2} \rceil}, R_n\}) = \Theta(n \log_2 n)$$

where $n \geq 5$, n is odd, and $(n-1) \bmod 8 \neq 0$.

2 The Main Result

Theorem.

$$D(S_{2n}, \{R_{n-1}, R_n, R_{n+1}, R_{2n}\}) \leq 36n\lceil \log_2 2n \rceil$$

where $n > 1$.

Proof. In order to prove the $O(n \log_2 n)$ bound for the diameter of $C(S_{2n}, \{R_{n-1}, R_n, R_{n+1}, R_{2n}\})$, we describe operations that can be simulated by an $O(1)$ length sequence of $C(S_{2n}, \{R_{n-1}, R_n, R_{n+1}, R_{2n}\})$ generators. We can use these operations as a macro to prove our bound. Let ROL$[i,j,k]$ be defined as a left cyclic shift of elements $\pi[i]$ through $\pi[j]$ by k positions. Let ROR$[i,j,k]$ be defined similarly, but for a right shuffle, and let XCHG$[i]$ denote ROL$[i, i+1, 1]$. If ROL$[i,j,k]$ can be simulated by a given sequence, then ROR$[i,j,k]$ can be simulated by applying that sequence in reverse order.

Let

$$\begin{aligned}
&\text{XCHG}[n] : R_n, R_{n+1}, R_n, R_{n-1};\\
&\text{ROL}[1, 2n, 1] : R_{n+1}, R_{2n}, R_{n-1}, R_n, R_{2n}, R_n;\\
&\text{ROR}[1, 2n, 1] : R_{n-1}, R_{2n}, R_{n+1}, R_n, R_{2n}, R_n;\\
&\text{ROL}[1, n, 1] : R_n, R_{n-1};\\
&\text{ROR}[1, n, 1] : R_{n-1}, R_n;\\
&\text{ROL}[n+1, 2n, 1] : R_{2n}, \text{ROR}[1, n, 1], R_{2n};\\
&\text{ROR}[n+1, 2n, 1] : R_{2n}, \text{ROL}[1, n, 1], R_{2n}.
\end{aligned}$$

For all $\tau \in S_{2n}$, let

$$\text{Left}_\tau \cup \text{Right}_\tau = \{\tau[m_1], \tau[m_1+1], \ldots, \tau[m_3]\},$$

$$\text{Left}_\tau \cap \text{Right}_\tau = \emptyset,$$

$$a < b, a \in \text{Left}_\tau, b \in \text{Right}_\tau,$$

$$|\text{Left}_\tau| = m_2 - m_1 + 1,$$

$$|\text{Right}_\tau| = m_3 - m_2$$

where

$$1 \leq m_1 < m_2 < m_3 \leq 2n.$$

Consider the procedure SORT(τ, m_1, m_2, m_3).

```
procedure SORT(τ, m1, m2, m3)
//Step1
i := 0;
n1 := m1;
n2 := m3;
if m2 > n
   then
      begin
         j := m2 − n;
         n1 := m1 − j;
         n2 := m3 − j;
         for i := 1 to j do
            τ := ROL[1, 2n, 1](τ);
      end
   else
      if m2 < n
         then
            begin
               j := n − m2;
               n1 := m1 + j;
               n2 := m3 + j;
               for i := 1 to j do
                  τ := ROR[1, 2n, 1](τ);
            end;
//Step2
l := 0;
r := 0;
while {τ[n + 1], τ[n + 2], . . . , τ[n2]} ∩ Left_τ ≠ ∅ do
   begin
      while τ[n + 1] ∉ Left_τ do
         begin
            r := r + 1;
            τ := ROL[n + 1, 2n, 1](τ);
         end;
      while τ[n] ∉ Right_τ do
         begin
            l := l + 1;
            τ := ROR[1, n, 1](τ);
         end;
      τ := XCHG[n](τ);
   end;
if r > 0 then
   for i := 1 to r do
      τ := ROR[n + 1, 2n, 1](τ);
```

```
if l > 0 then
   for i := 1 to l do
      τ := ROL[1, n, 1](τ);
τ' := τ;
//Step3
if m_2 > n
   then
      for i := 1 to j do
         τ := ROR[1, 2n, 1](τ)
   else
      if m_2 < n
         then
            for i := 1 to j do
               τ := ROL[1, 2n, 1](τ);
```

It is easy to check that if

$$0 < n_1 \leq n_2 \leq 2n,$$

then

$$\tau'[i] = \tau[i],$$
$$\tau'[j] < \tau[k]',$$
$$\tau'[j] \in \mathrm{Left}_\tau,$$
$$\tau'[k] \in \mathrm{Right}_\tau,$$

where

$$i \in \{1, 2, \ldots, m_1 - 1, m_3 + 1, m_3 + 2, \ldots, 2n\},$$
$$j \in \{m_1, m_1 + 1, \ldots, m_2\},$$
$$k \in \{m_2 + 1, m_2 + 2, \ldots, m_3\}.$$

Clearly, using a binary tree and procedure $\mathrm{SORT}(\tau, m_1, m_2, m_3)$, we can sort any permutation. This strategy takes at most

$$2(2n)^2\lceil \log_2 2n\rceil$$

operations $\mathrm{ROL}[1, 2n, 1]$ and $\mathrm{ROR}[1, 2n, 1]$;

$$4n\lceil \log_2 2n\rceil$$

operations $\mathrm{ROL}[1, n, 1]$, $\mathrm{ROR}[1, n, 1]$, $\mathrm{ROL}[n + 1, 2n, 1]$, and $\mathrm{ROR}[n + 1, 2n, 1]$;

$$2n\lceil \log_2 2n\rceil$$

operations $\mathrm{XCHG}[n]$. But if we use global shuffle instead Step 1 and Step 3, then only $2n\lceil \log_2 2n\rceil$ operations $\mathrm{ROL}[1, 2n, 1]$ and $\mathrm{ROR}[1, 2n, 1]$ needed. Note that operations $\mathrm{ROL}[1, 2n, 1]$ and $\mathrm{ROR}[1, 2n, 1]$ take 6 prefix reversals. Operations $\mathrm{XCHG}[n]$, $\mathrm{ROL}[1, n, 1]$, $\mathrm{ROR}[1, n, 1]$, $\mathrm{ROL}[n + 1, 2n, 1]$, and $\mathrm{ROR}[n + 1, 2n, 1]$ takes at most 4 prefix reversals. Therefore, sorting procedure takes at most

$$36n\lceil \log_2 2n\rceil$$

prefix reversals. □

3 A status of the symmetric group

For each generating set X of a finite semigroup S the integer $\Delta(X)$ is defined as the least n for which every element of S is expressible as a product of at most n elements of X. The status $\mathrm{Stat}(S)$ of S is defined as the least value of $|X|\Delta(X)$ among generating sets of X. In [18] proved that

$$\mathrm{Stat}(S_n) \leq \lfloor \frac{3}{2}(n-1)\rfloor (n-1)$$

where $n \geq 3$. Note that

$$\mathrm{XCHG}[n], \mathrm{ROL}[1, 2n, 1], \mathrm{ROR}[1, 2n, 1], \mathrm{ROL}[1, n, 1],$$
$$\mathrm{ROR}[1, n, 1], \mathrm{ROL}[n + 1, 2n, 1], \mathrm{ROR}[n + 1, 2n, 1] \in S_{2n}.$$

Corollary.

$$\mathrm{Stat}(S_{2n}) \leq 56n\lceil \log_2 2n\rceil$$

where $n > 1$.

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