Theoretical Computation of Lyapunov Exponents for Almost Periodic Hamiltonian Systems

FAROUK CHERIF

Abstract—**Lyapunov exponents are an important concept to describe qualitative properties of dynamical systems. For instance, chaotic systems can be caracterized with the positivity of the largest Lyapunov exponent. In this paper, we use the Iwasawa decomposition of the semisimple Lie group $Sp(n,\mathcal{R})$ and the enlargement of the phase space to give a theoretical computation of Lyapunov exponents of almost periodic Hamiltonian systems. In particular, we obtain the existence of Lyapunov exponents everywhere in the surface of constant energy of the Hamiltonian H. It turns out that, in this context, the Oseledec's assumption is not necessary to guarantee the existence and the finitness of Lyapunov exponents.**

Index Terms—**Almost periodic functions, Hamiltonian systems, Lyapunov exponents.**

I. Introduction

In 1932, Birkhoff proved the individual Ergodic Theorem, which related the time averages of individual orbits to the space average for certain types of dynamical systems. While this theorem is a remarkable result, it is constrained by the type of averages that are used. In 1963, Kingman extended Birkhoff's theorem by proving the subadditive Ergodic Theorem, which allowed more general types of averages for subadditive sequences. On the other hand, the theory of Lyapunov exponents in a form adapted the need of the theory of dynamical systems and of ergodic theory was given only in 1968 in the paper by Oseledec [13]. In his paper, Oseledec proved the Multiplicative Ergodic Theorem, which allowed for the computation of geometric means for similar ergodic process and opened a door to practical analysis of dynamical systems. A direct result of Oseledec's Multiplicative Ergodic Theorem is the existence of Lyapunov exponents. There are as many Lyapunov exponents as there are dimensions in the state space of the system, but the largest is usually the most important. For instance, the largest Lyapunov exponent measures the sensitivity to initial condition in dynamical systems. In particular Lyapunov exponents play a crucial role in analyzing dynamics of evolutionary systems, especially in chaotic and bifurcative systems.

The multiplicative ergodic theorem of Oseledec states that for an invariant measure μ, Lyapunov exponents exist for the orbit of $\mu-$almost every point x. Therefore, Lyapunov exponents exist for the orbit of a randomly chosen point x **with respect to** μ. If the dynamical system is conservative (preserves a measure equivalent to Lebesgue measure), then the multiplicative ergodic theorem does imply that Lyapunov exponents exist for Lebesgue almost every point in the phase space. Nevertheless, the relationship between the existence of such measures and Lyapunov exponents is subtle and sophisticated. However, if a system admits a measure satisfying certain nice properties, then Lyapunov exponents will exist for a large set of points.

Epam Sousse- Avenue du Marechal TITO-4000-Sousse Tel: 216-73223372 Email: faroukcheriff@yahoo.fr

Until now, many analysis and algorithms exist for the computation of Lyapunov exponents of a given system (see for example [1], [2], [7], [8], and [9]). Scientists often compute Lyapunov exponents without checking whether or not the exponents actually exist. If the dynamical system is not conservative, then the existence of Lyapunov exponents is a not an obvious question and one can find a counter-example where the Lyapunov exponents fail to exist [14].

The multiplicative ergodic theorem of Oseledec is at the basis of this paper. More precisely, we focus our attention on the almost periodic Hamiltonian systems. We present here an algorithm for computing the Lyapunov exponents of almost periodic Hamiltonian systems. This algorithm has four major steps. First, we linearize the differential equation $\dot{z} = JH'(z)$ near an almost periodic solution (we suppose that such solution exists). Second, we apply the Iwasawa decomposition of the fundamental matrix of the linearized equation. Later we enlarge the phase space by considering the manifold $\widetilde{X} = \mathcal{R}^{2n} \times (Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R}))$ on which we built the linear differential equation $\dot{Y} = \widetilde{A}(t,x,u)Y$ where the fundamental matrix of this equation is the matricial cocycle $\widetilde{R}(t,x,u)$, the upper triangular matrix of the Iwasawa decomposition. In the last step, we prove that the cocycles $R(t,x)$ and $\widetilde{R}(t,x,u)$ have the same Lyapunov spectra which is the mean value in time of the diagonal elements of $\widetilde{A}(t,x,u)$.

The paper has been organized as follows. In section 2 and section 3, we present the theory of Lyapunov exponents for the case of diffeomorphisms of Riemannian manifold and for the case of Hamiltonian systems respectively. In section 4, we give Theorem 2 which is the heart of this paper and we establish the existence of Lyapunov exponents for the almost periodic Hamiltonian flow and we give a theoretical algorithm for the computation of all of the Lyapunov exponents. Section 5 is devoted to the proof of the main result. One can remark that we obtain the existence of Lyapunov exponents every where in the phase space without using the Oseledec's assumption nor the ergodic theory.

II. The Oseledec theory and Lyapunov exponents

As starting point we consider the $n-$dimensional autonomous dynamical system:

$$\dot{x} = f(x) \tag{1}$$

where x denotes a point in an $n-$dimensional phase space $\subset \mathcal{R}^n$, $f(x)$ is an $n-$dimensional continuously differentiable vector field and the overdot denotes the derivate with respect to time t. The solution of this dynamical system for a fixed initial value x_0 is given by the trajectories $\varphi^t(x_0)$ (or

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$\varphi(x_0,t)$). We refer to φ as a flow. Assume there exists a compact subset $M \subset \mathcal{R}^n$ such that:

$$\varphi^t(M) \subset M, \text{ for all } t \geq 0.$$

We study the flow on M. For $x \in M$, we denote $\Phi_x(\cdot,\cdot)$ the scalar product on the tangent space T_xM and corresponding norm $\|\cdot\|$. Let $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. For $x \in M$ and $t \in \mathcal{R}$, $T_x(\varphi^t) = T(\varphi^t(x))$ denotes the tangent map of φ^t at the point x. Furthermore, we have:

$$\begin{array}{ccc} T_xM & \longrightarrow & T_{\varphi^s(x)}M \\ \downarrow & & \downarrow \\ T_{\varphi^{t+t}(x)}M & \longrightarrow & T_{\varphi^t(\varphi^s(x))}M \end{array}$$

This diagram express the notion of Oseledec's cocycle i.e.:

$$T_x\left(\varphi^{t+s}\right) = T_{\varphi^s(x)}\left(\varphi^t\right) \circ T_x\left(\varphi^s\right).$$

For $x \in M$, we denote $L_x(M)$ the set of all ordained basis of the tangent space T_xM. Let

$$L(M) = \bigcup_{x\in M} L_xM.$$

Let us consider the map:

$$\Pi : L(M) \longrightarrow M$$

such that the fiber of x in M under Π is L_xM. Let $S(M)$ be set of C^∞ sections of $L(M)$ i.e.:

$$s : M \longrightarrow L(M)$$

such that for all $x \in M$:

$$s(x) = (s_1(x), s_2(x), \cdots, s_n(x))$$

is a basis of $T_x(M)$. Hence, for $s \in S(M)$, $x \in M$ and $t, s \in \mathcal{R}$, we can construct a matricial cocycle from $T_x(\varphi^t)$ by the relation :

$$R(t,x) = s^{-1}(y)\, T_x\left(\varphi^t\right) s(x) \in GL_n(\mathcal{R}). \quad (2)$$

where $y = \varphi^t(x)$ and $GL_n(\mathcal{R})$ the set of invertible matrices of order n.

Otherwise, if $z(t) = \varphi^t(x)$ is a solution of (2.1), we can linearize near z obtaining the linear system

$$\dot{y} = f'(\varphi^t(x))y \quad (3)$$

Consequently the fundamental matrix of (3) is a matricial cocycle i.e. for $x \in M$ and $t, s \in \mathcal{R}$

$$R(t+s,x) = R(t,\varphi^s(x))R(s,x) \quad (4)$$

To see this is enough to prove that the both sides of the above equality are solutions of (2.3) with the same initial condition. We denote *(H)* the Oseledec's assumption:

$$\int_M \sup_{-1\leq\theta\leq 1} \ln^+ \left\|T_x\varphi^\theta(x)\right\| d\mu(x) < +\infty \quad (5)$$

where $\ln^+ = \sup(\ln, 0)$ and μ is a measure on M satisfying $\mu(M) = 1$.

There are many incarnations of Oseledec's Multiplicative Ergodic Theorem for various types of dynamical systems. For the case of diffeomorphisms of Riemannian manifold, its general form is:

Theorem 1 (**OSELEDEC**) *[13] Let $\varphi^t : M \longrightarrow M$ be a C^1-diffeomorphism of a compact manifold of dimension n and let μ an φ^t-invariant measure satisfying (5). Then one can find:*

i. *real numbers :*

$$\chi_1 > \chi_2 > \cdots > \chi_k, \text{ where } k \leq n$$

such that there exists;

ii. *positive integers $n_1, n_2, \cdots, n_k$ such that:*

$$n = n_1 + n_2 + \cdots + n_k$$

that invoke;

iii. *a measurable splitting*

$$T_xM = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^k$$

with $\dim\left(E_x^i\right) = n_i$ and $T_x\varphi^t\left(E_x^i\right) = E^i_{\varphi^t(x)}$; such that for $\mu - a.e. x \in M$

$$\lim_{t\to+\infty} \frac{1}{t} \ln\left\|T_x\varphi^t v\right\| = \chi_l(x,v) \quad (6)$$

where l is a unique integer satisfying:

$$v \in E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^l$$

and:

$$v \notin E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^{l-1}.$$

Remark 1 We will call the numbers $\chi_k(x,v)$ the Lyapunov exponents for the measure μ. Furthermore, we will call the set of points for which the Oseledec's Multiplicative Ergodic Theorem hold for a particular measure μ, $\mu-$regular.

Remark 2 From relation (2) we can deduce that the condition *(H)*

$$\int_M \sup_{-1\leq\theta\leq 1} \ln^+ \left\|T_x\varphi^\theta(x)\right\| d\mu(x) < +\infty$$

is equivalent to:

$$\int_M \sup_{-1\leq\theta\leq 1} \ln^+ \|R(\theta,x)\|\, d\mu(x) < +\infty. \quad (7)$$

Hence we can give the Oseledec's Multiplicative Ergodic Theorem for matricial cocycles.

III. The almost periodic and Hamiltonian case

It is well known, that for periodic systems the Lyapunov exponents are the real part of Floquet exponents [6]. We turn our attention on the almost periodic Hamiltonian systems. Let us consider a Hamiltonian function:

$$H \in \mathcal{C}^2\left(\mathcal{R}^{2n}, \mathcal{R}\right), \ (n \in \mathcal{N}^*) \quad (8)$$

and the Hamiltonian system:

$$(S_H) : \begin{cases} \dot{x}_i = \frac{\partial H}{\partial p_i}(x,p) \\ \dot{p}_i = \frac{-\partial H}{\partial x_i}(x,p) \end{cases}, \ 1 \leq i \leq n.$$

By setting:

$$z(t) = (x(t), p(t)) = (x_1, \cdots, x_n, p_1, \cdots, p_n)$$

and if we define:

$$H' = \nabla H = \left(\frac{\partial H}{\partial x_1}, \cdots, \frac{\partial H}{\partial x_n}, \frac{\partial H}{\partial p_1}, \cdots \frac{\partial H}{\partial p_n}\right)$$

the gradient vector of H en z for the standard inner product of $\mathcal{R}^{2n}$, then the Hamiltonian system (S_H) can be written as a Hamiltonian vector equation:

$$\dot{z} = JH'(z) \tag{9}$$

where J is the standard symplectic matrix:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & \end{pmatrix}$$

with I_n being the $n \times n$ identity matrix. We denote $\{\varphi^t\}_t$ the Hamiltonian flow of (9), and x the initial point.

Throughout this section, we suppose that for any solution $z(t) = (x(t), p(t))$ of (9) is defined for all $t \in \mathcal{R}$, so that the Hamiltonian flow $\{\varphi^t\}_t$ is well defined and is of class C^1. The surface of constant energy of the Hamiltonian H:

$$\sum\nolimits_H = \left\{ z \in \mathcal{R}^{2n} \quad / \quad H(z) = e \right\},$$

is $\{\varphi^t\}$ $-$invariant. Furthermore, we suppose:

$$\sum\nolimits_H \text{ is non empty and compact} \tag{10}$$

$$\forall z \in \sum\nolimits_H, \; H'(z) \neq 0 \tag{11}$$

$$H \text{ is strictly convex on } \mathcal{R}^{2n} \tag{12}$$

According to the assumption (11), $\sum_H$ is a $\mathcal{C}^2$ submanifold of $\mathcal{R}^{2n}$ of codimension 1. The standard inner product of $\mathcal{R}^{2n}$ induced on $\sum_H$ a Riemannian submanifold structure. It is well-known that for all $t \in \mathcal{R}$, $H \circ \varphi^t = H$ and we can define a canonical Liouville measure $\mu_L$on $\sum_H$ which is φ^t- invariant on $\sum_H$ and given by the formula:

$$\mu_M = \frac{1}{\alpha} \int \frac{d\sigma}{\|\nabla H\|}$$

where σ is the Lebesgue measure on $\sum_H$ and α is a normalization constant (see for example [11]). Hence for this measure, conditions (5) and (7) are automatically satisfied.

Throughout the rest of this paper, we suppose that the hypothesis (8), (10), (11) and (12) are satisfied. If $z(t) = (\varphi^t(x))$ is an almost periodic solution of (9), (we suppose that such solution exists as the case of Hénon-Heiles system) one can linearize (9) near z obtaining the almost periodic linear Hamiltonian system:

$$\dot{y} = JH''(\varphi^t(x))y. \tag{13}$$

Let us denote $R(t,x)$ the fundamental matrix of (13). Let $Sp(n,\mathcal{R})$ the symplectic group and $sp(n,\mathcal{R})$ his Lie algebra i.e.:

$$Sp(n,\mathcal{R}) = \left\{ A \in \mathcal{SL}(2n,\mathcal{R}), {}^t AJA = J \right\}$$

where $\mathcal{SL}(2n,\mathcal{R})$ denotes the group of $2n \times 2n$ matrices with unit determinant, and:

$$\begin{aligned} sp(n,\mathcal{R}) &= \{A \in \mathcal{M}(2n,\mathcal{R}), \exp(tA) \in Sp(n,\mathcal{R})\} \\ &= \left\{ A \in \mathcal{M}(2n,\mathcal{R}), {}^t AJ + JA = 0 \right\}. \end{aligned}$$

By applying the Oseledec's Multiplicative Ergodic Theorem we obtain for $\mu_M - a.e. x \in M$:

$$\chi_1(x) \geq \chi_2(x) \geq \cdots \geq \chi_{2n}(x),$$

and due to the symplectic properties we get (see for example [1]):

$$\chi_{2n-i+1}(x) = -\chi_i(x), \qquad 1 \leq i \leq n.$$

IV. The Main result

Since the Lie group $Sp(n,\mathcal{R})$ is semisimple the Iwasawa decomposition (see for example [10] and/or [12]) tells us that:

$$\begin{array}{ccc} Sp(n,\mathcal{R}) & \longrightarrow & \mathcal{KAN} \\ S & \longmapsto & KAN \end{array}$$

is an analytic diffeomorphism where $\mathcal{K}$ to be the orthogonal matrices, $\mathcal{A}$ to be the positive diagonal matrices, and $\mathcal{N}$ to be the unipotent subgroup of $Sp(n,\mathcal{R})$ consisting of upper triangular matrices with 1s on the diagonal. Now, let $u \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$. By the Iwasawa decomposition we obtain the following: $\exists! \Delta(t,x,u) \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$ and $\exists! \widetilde{R}(t,x,u)$ an upper triangular matrix satisfying:

$$R(t,x)u = \Delta(t,x,u)\widetilde{R}(t,x,u).$$

We consider the manifold $\widetilde{X}$ given by:

$$\widetilde{X} = \sum\nolimits_H \times (Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R}))$$

One can see that:

$$\begin{array}{ccc} U(n) & \longrightarrow & Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R}) \\ A + iB & \longmapsto & \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \end{array}$$

is an isomorphism of Lie groups, where:

$$U(n) = \left\{ A \in \mathcal{GL}(n,\mathcal{C}), {}^t A\, A = I_n \right\}.$$

Hence the manifold:

$$\widetilde{X} = \sum\nolimits_H \times (Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R}))$$

can be identified to the analytic manifold

$$\widetilde{X} \simeq \sum\nolimits_H \times U(n)$$

and can be seen as a "complexification" of $\sum_H$. Define the function $\widetilde{\varphi^t}(\cdot,\cdot)$ on $\widetilde{X}$ by taking:

$$\widetilde{\varphi^t}(x,u) = \left(\varphi^t(x), \Delta(t,x,u)\right)$$

On the manifold $\widetilde{X}$ we define the matrix $\widetilde{A}$ by:

$$\begin{aligned} \widetilde{A}(t,x,u) &= \left(\dot{\Delta}(t,x,u)\right)^{-1} \Delta(t,x,u) \\ &\quad + \Delta(t,x,u) A(t,x) \left(\Delta(t,x,u)\right)^{-1}, \end{aligned}$$

where:

$$A(t,x) = JH''(\varphi^t(x))$$

and

$$\dot{\Delta}(t,x,u) = \frac{\partial}{\partial t}\left(\Delta(t,x,u)\right).$$

We built the linear differential equation:

$$\dot{Y} = \widetilde{A}(t,x,u)\,Y. \tag{14}$$

The main result of this paper is the following:

Theorem 2 *Suppose that the conditions (8), (10), (11) and (12) are satisfied and* $t \longmapsto \varphi^t(x)$ *is an almost periodic function then the Lyapunov exponents of (13) exist for all* $x \in \sum_H$*. Furthermore:*

$$\begin{aligned} &\left\{ \chi\left(\widetilde{R}(t,x,u), v\right) : v \in \mathcal{R}^{2n} \right\} \\ =\; &\left\{ \mathcal{M}\left\{ a_{ii}(t,x,u) \right\}_t, 1 \leq i \leq 2n \right\} \end{aligned}$$

where $a_{ii}(s,x,u)$ *are the diagonal elements of* $\widetilde{A}(t,x,u)$ *and:*

$$\mathcal{M}\{f\}_t = \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(s)\,ds.$$

Before proving our main result we have several remarks. A basic property of the multiplicative cocycles $R(t,x)$ is their regularity, since Theorem 1 guarantees the existence and the finiteness of Lyapunov exponents for regular multiplicative cocycles. Thus, it is important to determine specific conditions that multiplicative cocycles should fulfill in order to be regular. In particular, we have to assume the Oseledec's assumption:

$$\int_M \sup_{-1\le\theta\le1} \ln^+ \|R(\theta,x)\|\,d\mu(x) < +\infty \ .$$

Since our presentation is mainly focused on autonomous almost periodic Hamiltonian systems we will also state the Multiplicative Ergodic Theorem for the symplectic cocycle $R(t,x)$ and hence we do not consider the assumption above. Second, with our approach and due to almost periodicity we obtain the existence of Lyapunov exponents every where in surface $\sum_H$ of constant energy of the Hamiltonian H without using ergodic theory.

V. The proof of the main result

In order to prove Theorem 2, we need some preliminary observations:

Lemma 1 *For* $x \in \sum_H$ *and* $t \in \mathcal{R}$ *:*

$$R(t,x) \in Sp(n,\mathcal{R}).$$

Proof It is clear that:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[{}^tR(t,x)JR(t,x)\right]\\
=\;& \frac{\partial}{\partial t}\left[{}^tR(t,x)\right]JR(t,x)\\
&+ {}^tR(t,x)\;J\frac{\partial}{\partial t}R(t,x)\\
=\;& {}^t\left(JH^{"}(\varphi^t(x))R(t,x)\right)\;JR(t,x)\\
&+{}^tR(t,x)JJH^{"}(\varphi^t(x))R(t,x)\\
=\;& {}^tR(t,x)\;H^{"}(\varphi^t(x))\;{}^tJ\;JR(t,x)\\
&-{}^tR(t,x)\;H^{"}(\varphi^t(x))R(t,x)\\
=\;& 0.
\end{aligned}$$

Thus:

$$t \longmapsto {}^tR(t,x)\;JR(t,x)$$

is constant. Since for $t=0$ one has:

$${}^tR(0,x)JR(0,x) = J,$$

and the result follows.

Lemma 2 *Let* $x \in \sum_H, u \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$ *and* $t \in \mathcal{R}$*. Then :*

$$\Delta(t,x,u) = \Delta(0,\varphi^t(x),u).$$

Proof From Iwasawa decomposition [10], we obtain respectively:

$$R(t,x)u = \Delta(t,x,u)\widetilde{R}(t,x,u)$$

and:

$$R(0,\varphi^t(x))u = \Delta(0,\varphi^t(x),u)\widetilde{R}\left(0,\varphi^t(x),u\right).$$

Since $R(t,x)$ and $R(0,\varphi^t(x))$ solve the same differential equation with the same initial condition:

$$R(t,x)u = R(0,\varphi^t(x))u.$$

Hence the result follows from the fact that the Iwasawa decomposition is a diffeomorphism.

Lemma 3 *Let* $x \in \sum_H, u \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$ *and* $t \in \mathcal{R}$*. Then the function:*

$$t \longmapsto \frac{\partial}{\partial t}\left(\Delta(t,x,u)\right)$$

is almost periodic.

Proof Clearly the function:

$$t \longmapsto R(t,x)$$

is C^1 on $\mathcal{R}$. Since the Iwasawa decomposition is a diffeomorphism, the function:

$$(t,x,u) \longmapsto \Delta(t,x,u)$$

is also C^1 on $\mathcal{R}\times\widetilde{X}$. Let Γ be the function defined by:

$$\Gamma\left(\varphi^t(x)\right) = \Delta(0,\varphi^t(x),u).$$

Then:

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma\left(\varphi^t(x)\right) &= \frac{\partial}{\partial t}\left(\Delta(t,x,u)\right)\\
&= \partial_2\Delta(0,\varphi^t(x),u)\left(\frac{\partial\varphi^t(x)}{\partial t}\right).
\end{aligned}$$

It is clear that $\partial_2\Delta(0,\varphi^t(x),u)$ is continuous from $\sum_H$ onto $\mathcal{L}\left(\sum_H,\mathcal{M}(2n,\mathcal{R})\right)$, also:

$$t \longmapsto \Delta(0,\varphi^t(x),u)$$

is almost periodic. Then the function:

$$t \longmapsto \left(\partial_2\Delta(0,\varphi^t(x),u),\left(\frac{\partial\varphi^t(x)}{\partial t}\right)\right)$$

is almost periodic since each of components (vectorial) is almost periodic [4]. Finally, the function:

$$\mathcal{B}: \mathcal{L}\left(\sum\nolimits_H,\mathcal{M}(2n,\mathcal{R})\right)\times\sum\nolimits_H \longrightarrow \mathcal{M}(2n,\mathcal{R})$$

defined by $\mathcal{B}(L,v) := L(v)$ is bilinear and continuous. Then we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\Delta(t,x,u)\right) &= \frac{\partial}{\partial t}\Gamma\left(\varphi^t(x)\right)\\
&= \mathcal{B}\left(\partial_2\Delta(0,\varphi^t(x),u),\frac{\partial\varphi^t(x)}{\partial t}\right).
\end{aligned}$$

Therefore:

$$t \longmapsto \frac{\partial}{\partial t}\Gamma\left(\varphi^t(x)\right)$$

is almost periodic as the composition of an uniformly continuous function and an almost periodic function [5].

Lemma 4 *Let $x \in \sum_H, u \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$ and $t \in \mathcal{R}$. Then:*

$$\widetilde{A}(t,x,u) \in Sp(n,\mathcal{R})$$

and there exists a symmetric matrix $\widetilde{B}(t,x,u)$ such that:

$$\widetilde{A}(t,x,u) = J\widetilde{B}(t,x,u).$$

Proof First, we have to prove that:

$${}^t\left(\widetilde{A}(t,x,u)\right)J + J\widetilde{A}(t,x,u) = 0.$$

Let us introduce the matrices:

$$B(t,x,u) := \left(\dot{\Delta}(t,x,u)\right)^{-1}\Delta(t,x,u)$$

and:
$C(t,x,u) := \Delta(t,x,u)JH"(\varphi^t(x))(\Delta(t,x,u))^{-1}$
Then we obtain:

$$\begin{aligned} & JB(t,x,u) +^t B(t,x,u)J \\ = \ & J\left(\dot{\Delta}(t,x,u)\right)^{-1}\Delta(t,x,u) \\ & + (\Delta(t,x,u))^{-1}\left(\dot{\Delta}(t,x,u)\right)J \\ = \ & {}^t\left(JH"(\varphi^t(x))R(t,x)\right) JR(t,x) \\ & + {}^tR(t,x)\ JJH"(\varphi^t(x))R(t,x) \\ = \ & {}^tR(t,x)\ H"(\varphi^t(x))^tJJR(t,x) \\ & - {}^tR(t,x)\ H"(\varphi^t(x))R(t,x) \\ = \ & 0. \end{aligned}$$

In a similar manner we obtain:

$$JC(t,x,u) +^t C(t,x,u)J = 0$$

Since

$$\widetilde{A}(t,x,u) = B(t,x,u) + C(t,x,u)$$

the equality holds. We set:

$$\widetilde{B}(t,x,u) = J\left(\dot{\Delta}(t,x,u)\right)^{-1}\Delta(t,x,u) + J\Delta(t,x,u)A(t,x)(\Delta(t,x,u))^{-1},$$

then $\widetilde{B}(t,x,u)$ is symmetric and:

$$\widetilde{A}(t,x,u) = J\widetilde{B}(t,x,u).$$

Lemma 5 *$\left\{\widetilde{\varphi^t}\right\}_t$ is a flow on the manifold $\widetilde{X}$.*

Proof. For $t,s \in \mathcal{R}$, one get:

$$\begin{aligned} & \widetilde{\varphi^{t+s}}(x,u) \\ = \ & \left(\varphi^{t+s}(x), \Delta(t,x,u)\right) \\ = \ & \left(\varphi^t(\varphi^s(x)), \Delta(0,\varphi^{t+s}(x),u)\right) \\ = \ & \left(\varphi^t(\varphi^s(x)), \Delta(0,\varphi^t(\varphi^s(x)),u)\right) \\ = \ & \left(\varphi^t(\varphi^s(x)), \Delta(t,\varphi^s(x),u)\right) \\ = \ & \widetilde{\varphi^t}\circ\widetilde{\varphi^s}(x,u) \end{aligned}$$

Clearly, if we replace s by $-t$, we deduce that:

$$\widetilde{\varphi^{-t}} = \left(\widetilde{\varphi^t}\right)^{-1}.$$

Lemma 6 *Let $x \in \sum_H$, $u \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$ and $t \in \mathcal{R}$ then $\widetilde{R}(t,x,u)$ is multiplicative cocycle on $\widetilde{X}$ related to the dynamical systems $\left\{\widetilde{\varphi^t}\right\}_t$ and $\widetilde{R}(t,x,u)$ is the fundamental matrix of* (14)

Proof. Let p_2 the projection defined by:

$$\begin{array}{cccc} p_2: & \widetilde{X} & \longrightarrow & Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R}) \\ & (x,u) & \longmapsto & u \end{array}$$

Hence:

$$\begin{aligned} \widetilde{R}(t+s,x,u) &= (\Delta(t,x,u))^{-1}R(t+s,x)u \\ &= p_2\left(\widetilde{\varphi^{t+s}}(x,u)\right)^{-1}R(t+s,x)u \\ &= p_2\left(\widetilde{\varphi^t}\circ\widetilde{\varphi^s}(x,u)\right)^{-1}R(t,\varphi^s(x)) \\ &\quad \times R(s,x)p_2(x,u) \\ &= p_2\left(\widetilde{\varphi^t}\circ\widetilde{\varphi^s}(x,u)\right)^{-1}R(t,\varphi^s(x)) \\ &\quad \times p_2\left(\widetilde{\varphi^s}(x,u)\right)p_2\left(\widetilde{\varphi^s}(x,u)\right)^{-1} \\ &\quad \times R(s,x)p_2(x,u) \\ &= \widetilde{R}(t,\varphi^s(x),u)\widetilde{R}(s,x,u), \end{aligned}$$

which proves that $\widetilde{R}(t,x,u)$ is multiplicative cocycle on $\widetilde{X}$ related to the flow $\left\{\widetilde{\varphi^t}\right\}_t$.
Now, we have to prove that $\widetilde{R}(t,x,u)$ is the fundamental matrix of (14). First, we have:

$$\begin{aligned} \widetilde{R}(0,x,u) &= p_2\left(\widetilde{\varphi^0}(x,u)\right)^{-1}R(0,x)u \\ &= p_2((x,u))^{-1}R(0,x)u \\ &= I_{2n}. \end{aligned}$$

On the other hand:

$$\begin{aligned} \dot{\widetilde{R}}(t,x,u) &= \left(\dot{\Delta}(t,x,u)\right)^{-1}R(t,x)u \\ &\quad + (\Delta(t,x,u))^{-1}\dot{R}(t,x)u \\ &= \left(\dot{\Delta}(t,x,u)\right)^{-1}R(t,x)u \\ &\quad + (\Delta(t,x,u))^{-1}A(t,x)R(t,x)u \\ &= \left(\dot{\Delta}(t,x,u)\right)^{-1}\Delta(t,x,u)(\Delta(t,x,u))^{-1} \\ &\quad \times R(t,x)u + (\Delta(t,x,u))^{-1}A(t,x) \\ &\quad \times \Delta(t,x,u)\Delta^{-1}(t,x,u)R(t,x)u \\ &= \left(\left(\dot{\Delta}(t,x,u)\right)^{-1} + \Delta^{-1}(t,x,u)A(t,x)\right) \\ &\quad \times R(t,x)u \\ &= \widetilde{A}(t,x,u)\widetilde{R}(t,x,u), \end{aligned}$$

which proves that $\widetilde{R}(t,x,u)$ is the fundamental matrix of (14) and consequently proves lemma 6.

Lemma 7 *Let $x \in \sum_H, u \in Sp(n,\mathcal{R}) \cap O(2n,\mathcal{R})$ and $t \in \mathcal{R}$ then the cocycles $\widetilde{R}(t,x,u)$ and $R(t,x)$ have the same Lyapunov exponents.*

Proof. By Iwasawa decomposition we get:

$$\begin{aligned} R(t,x) &= \Delta(t,x,u)\widetilde{R}(t,x,u)u^{-1} \\ &= p_2(x,\Delta(t,x,u))\widetilde{R}(t,x,u)(p_2(x,u))^{-1} \\ &= \Delta(t,x,u)\widetilde{R}(t,x,u)(\Delta(0,x,u))^{-1}. \end{aligned}$$

Let $(v_1, v_2, \cdots, v_{2n})$ an orthonormal basis of $\mathcal{R}^{2n}$. Then for every $1 \leq i \leq 2n$, since $\Delta(t,x,u)$ is an isometry, one has:

$$\begin{aligned} \|R(t,x)v_i\| &= \left\|\Delta(t,x,u)\widetilde{R}(t,x,u)\left(\Delta(0,x,u)\right)^{-1} v_i\right\| \\ &= \left\|\widetilde{R}(t,x,u)\left(\Delta(0,x,u)\right)^{-1} v_i\right\|. \end{aligned}$$

In other respects, the matrix $(\Delta(0,x,u))^{-1}$ transforms the orthonormal basis $(v_i)_i$ to the orthonormal basis $(h_i)_i$. Thus:

$$\ln\|R(t,x)v_i\| = \ln\left\|\widetilde{R}(t,x,u)h_i\right\|.$$

which imply for all $1 \leq i \leq 2n$:

$$\lim_{t\to+\infty}\frac{1}{t}\ln\|R(t,x)v_i\| = \lim_{t\to+\infty}\frac{1}{t}\ln\left\|\widetilde{R}(t,x,u)h_i\right\|.$$

Now; we are able to achieve the proof of theorem 2. **Proof.** By lemma 6, $\widetilde{R}(t,x,u)$ is the fundamental matrix of (4.1). Let's denote $\widetilde{a}_{ii}(t,x,u)$, $1 \leq i \leq 2n$, the diagonal elements of the matricial cocycle $\widetilde{R}(t,x,u)$. Then one has

$$\widetilde{a}_{ii}(t,x,u) = \exp\int_0^t a_{ii}(s,x,u)\,ds$$

where $a_{ii}(s,x,u)$ are the diagonal elements of $\widetilde{A}(t,x,u)$. Furthermore, the matrix:

$$A(t,x) = JH''(\varphi^t(x))$$

is almost periodic. Otherwise, by lemma 2:

$$\Delta(t,x,u) = \Delta(0,\varphi^t(x),u),$$

thus $\Delta(t,x,u)$ and $(\Delta(t,x,u))^{-1}$have almost periodic coefficients in t. On the other hand, by lemma 3, the function:

$$t \longmapsto \frac{\partial}{\partial t}\left(\Delta(t,x,u)\right)$$

is almost periodic. Consequently,

$$t \longmapsto \widetilde{A}(t,x,u) \text{ is an almost periodic function,}$$

and so:

$$t \longmapsto a_{ii}(t,x,u) \text{ is an almost periodic function,}$$

which imply that for all $1 \leq i \leq 2n$:

$$\mathcal{M}\{a_{ii}(t,x,u)\} = \lim_{t\to+\infty}\frac{1}{t}\int_0^t a_{ii}(s,x,u)\,ds \in \mathcal{R}.$$

Finally, let $(e_i)_{1\leq i\leq 2n}$ the canonical basis of $\mathcal{R}^{2n}$ then one has:

$$\begin{aligned} &\left\{\chi\left(\widetilde{R}(t,x,u),v\right)/v\in\mathcal{R}^{2n}\right\} \\ =& \left\{\lim_{t\to+\infty}\frac{1}{t}\ln\left\|\widetilde{R}(t,x,u)e_i\right\| \quad 1\leq i\leq 2n.\right\} \\ =& \left\{\limsup_{t\to+\infty}\frac{1}{t}\ln\left|\widetilde{a}_{ii}(t,x,u)\right|, 1\leq i\leq 2n\right\} \\ =& \left\{\limsup_{t\to+\infty}\frac{1}{t}\int_0^t a_{ii}(s,x,u)\,ds, 1\leq i\leq 2n\right\} \\ =& \left\{\lim_{t\to+\infty}\frac{1}{t}\int_0^t a_{ii}(s,x,u)\,ds, 1\leq i\leq 2n\right\} \\ =& \left\{\mathcal{M}\{a_{ii}(t,x,u)\}_t, 1\leq i\leq 2n\right\}. \end{aligned}$$

and our theorem holds.

VI. Conclusion

The key feature of our approach is the use of the Iwasawa decomposition of the semi-simple Lie group $Sp(n,\mathcal{R})$. This factorization is an analytic diffeomorphism of the Lie group $Sp(n,\mathcal{R})$ onto the manifold $\mathcal{KAN}$ where $\mathcal{K}$ to be the orthogonal matrices, $\mathcal{A}$ to be the positive diagonal matrices, and $\mathcal{N}$ to be the unipotent subgroup of $Sp(n,\mathcal{R})$ consisting of upper triangular matrices with 1s on the diagonal. The algorithm presented here is different from [1] and [7] since the factors in [1] and in the QR factorization are not symplectic, in general. On the other hand, to obtain Lyapunov exponents from observed data, Eckmann and Ruelle [8] and [9] proposed a method based on non parametric regression which is known as the Jacobian method.

Hence, with our approach we obtain the existence of Lyapunov exponents every where in surface of constant energy of the Hamiltonian H without using the Oseledec's assumption nor the ergodic theory. Finally, the computation of the Iwasawa decomposition of a symplectic matrix $\widetilde{R}(t,x,u)$ can be done by an algorithm given by Benzi and Razouk [3].

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