Connections between Generalized Fuzzy Ideals and Sub-implicative Ideals of BCI-algebras

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***Abstract*—The concept of quasi-coincidence of an interval valued fuzzy set is considered. By using this idea, the notion of interval valued $(\alpha, \beta)-$fuzzy sub-implicative ideals of BCI-algebras is introduced, which is a generalization of a fuzzy sub-implicative ideal. Also some related properties are studied and in particular, the interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideals in a BCI-algebra will be investigated.**

***Index Terms*—BCI-algebra, (sub-implicative) ideal, interval valued $(\alpha, \beta)-$fuzzy (sub-implicative) ideal**

I. Introduction

AFTER the introduction of fuzzy sets by Zadeh [17], there have been a number of generalizations of this fundamental concept. In 1975, Zadeh [18] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of the numbers. Such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm with pneumatic transportation, in a plastic products company and in medicine (see the book [1]). The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, hyperstructures theory, real analysis, measure theory etc (for instance see [4-7], [11], [14], [15], [19]).
The notion of BCK-algebras was proposed by Iami and Iseki in 1966. In the same year, Iseki [8] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. There is a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. For the general development of BCK/BCI-algebras the ideal theory plays an important role. In [10], Jun and Meng considered the fuzzification of $p-$ideals in BCI-algebras. In [13], Liu and Meng introduced the notion of fuzzy positive implicative, and investigate some of their properties. Liu and Meng [10] introduced the notion of sub-implicative ideals in BCI-algebras. Also Jun [9] introduced the notion of fuzzy sub-implicative ideals of BCI-algebras and obtained some related results. A new type of fuzzy subgroups ($(\in, \in \vee q)-$fuzzy subgroups) was introduced in an earlier paper of Bhakat and Das [3] by using the combined notions of belongingness and quasi-coincidence of fuzzy points and fuzzy sets. In fact, $(\in, \in \vee q)-$fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept has been studied further in [2]. The aim of this paper is to introduce and study new sorts of interval valued fuzzy sub-implicative ideals of a BCI-algebra and to investigate the new aspects of related properties. The combined notions of belongingness and quasi-coincidence (in different cases) of interval valued fuzzy points and fuzzy sets were used to introduce these sorts of interval valued fuzzy sub-implicative ideals. Also, the definition of interval valued fuzzy sub-implicative ideals with thresholds was considered and some basic related results are proved.

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II. Preliminaries and notations

By a *BCI-algebra* we mean an algebra X of type $(2, 0)$ satisfying the following conditions for all $x, y, z \in X$:
(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.
If we define a relation $\leq$ on X as follows:

$$x \leq y \text{ if and only if } x * y = 0,$$

then $(X, \leq)$ is a partially ordered set. A BCI-algebra X is said to be *implicative* if $(x * (x * y)) * (y * x) = y * (y * x)$ for all $x, y \in X$.
In any BCI-algebra X, the following hold:
(1) $(x * y) * z = (x * z) * y$,
(2) $x * (x * (x * y)) = x * y$,
(3) $((x * z) * (y * z)) * (x * y) = 0$,
(4) $x * 0 = x$,
(5) $0 * (x * y) = (0 * x) * (0 * y)$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
In what follows, X shall mean a BCI-algebra unless otherwise specified. A non-empty subset A of X is called an *ideal* of X if
(1) $0 \in A$,
(2) $x * y \in A$ and $y \in A$ imply $x \in A$.
We now review some fuzzy logical concepts. A fuzzy set in set X is a function $\mu : X \longrightarrow [0, 1]$. For a fuzzy set μ in X and $t \in [0, 1]$ define μ_t to be the set $\mu_t = \{x \in X | \mu(x) \geq t\}$, which is called a *level set* of μ. A fuzzy set μ in X is said to be a *fuzzy ideal* of X if for all $x, y \in X$
$(I1)$ $\mu(0) \geq \mu(x)$,
$(II1)$ $\mu(x) \geq min\{\mu(x * y), \mu(y)\}$.
For any elements x and y of a BCI-algebra, $x^n * y$ denotes $x * (... * (x * (x * y))...)$, in which x occurs n times.

Definition 2.1. [12] A non-empty subset A of X is called a *sub-implicative ideal* of X if
(1) $0 \in A$,
(2) $((x^2 * y) * (y * x)) * z \in A$ and $z \in A$ imply $y^2 * x \in A$.

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Definition 2.2. [9] A fuzzy set μ in X is called a *sub-implicative ideal* of X if for all $x, y, z \in X$
$(I1)$ $\mu(0) \geq \mu(x)$,
$(III1)$ $\mu(y^2 * x) \geq min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}$.

By an *interval number* $\widetilde{a}$ we mean ([18]) an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval number is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$. For interval numbers $\widetilde{a}_i = [a_i^-, a_i^+] \in D[0, 1], i \in I$, we define

$$\inf \widetilde{a}_i = [\bigwedge_{i\in I} a_i^-, \bigwedge_{i\in I} a_i^+], \quad sup\widetilde{a}_i = [\bigvee_{i\in I} a_i^-, \bigvee_{i\in I} a_i^+]$$

and put
(1) $\widetilde{a_1} \leq \widetilde{a_2} \Longleftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,
(2) $\widetilde{a_1} = \widetilde{a_2} \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,
(3) $\widetilde{a_1} < \widetilde{a_2} \Longleftrightarrow \widetilde{a_1} \leq \widetilde{a_2}$ and $\widetilde{a_1} \neq \widetilde{a_2}$,
(4) $k\tilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.
It is clear that $(D[0, 1], \leq, \bigvee, \bigwedge)$ is a complete lattice with $0 = [0, 0]$ as the least element and $1 = [1, 1]$ as the greatest element.
By an *interval number fuzzy set* F on X we mean ([15]) the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subset of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \widetilde{\mu_F}(x)) \mid x \in X\}$, where $\widetilde{\mu_F} : X \longrightarrow D[0, 1]$.

III. Interval valued $(\alpha, \beta)-$fuzzy sub-implicative ideals

The concept of quasi-coincidence of a fuzzy point can be extended to the concept of quasi-coincidence of a interval valued fuzzy set. An interval valued fuzzy set F of X of the form

$$\widetilde{\mu_F}(y) = \begin{cases} \widetilde{t}(\neq [0, 0]) & \text{if } y = x, \\ [0, 0] & \text{if } y \neq x, \end{cases}$$

is said to be the *interval valued fuzzy point* with support x and interval valued $\widetilde{t}$ and is denoted by $x_{\widetilde{t}}$. An interval value fuzzy point $x_{\widetilde{t}}$ is said to be *belong to* (resp. be *quasi-coincident with*) an interval valued fuzzy set F, written as $x_{\widetilde{t}} \in F$ (resp. $x_{\widetilde{t}} qF$) if $\widetilde{\mu_F}(x) \geq \widetilde{t}$ (resp. $\widetilde{\mu_F}(x) + \widetilde{t} > [1, 1]$). If $x_{\widetilde{t}} \in F$ or (resp. and) $x_{\widetilde{t}} qF$, then we write $x_{\widetilde{t}} \in \vee qF$ (resp. $x_{\widetilde{t}} \in \wedge qF$). The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.
We use α and β to denote any one of the $\in, q, \in \vee q$ or $\in \wedge q$ unless otherwise specified. We also emphasis that $\widetilde{\mu_F} = [\mu_F^-, \mu_F^+]$ must satisfy the following conditions:
(1) Any two elements of $D[0, 1]$ are comparable,
(2) $[\mu_F^-(x), \mu_F^+(x)] \leq [0.5, 0.5]$ or $[\mu_F^-(x), \mu_F^+(x)] > [0.5, 0.5]$, for all $x \in X$.

Definition 3.1. An interval valued fuzzy set F of X is called an *interval valued* $(\alpha, \beta)-$*fuzzy ideal* of X if for all $t, r \in (0, 1]$ and $x, y \in X$, the following conditions hold:
$(I2)$ $x_{\widetilde{t}} \alpha F$ implies $0_{\widetilde{t}} \beta F$,
$(II2)$ $(x * y)_{\widetilde{t}} \alpha F$, and $y_{\widetilde{r}} \alpha F$ imply $x_{\widetilde{t} \wedge \widetilde{r}} \beta F$.

Definition 3.2. An interval valued fuzzy set F of X is called an *interval valued* $(\alpha, \beta)-$*fuzzy sub-implicative ideal* of X if for all $t, r \in (0, 1]$ and $x, y \in X$, the following conditions hold:
$(I3)$ $x_{\widetilde{t}} \in F$ implies $0_{\widetilde{t}} \beta F$,
$(II3)$ $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \alpha F$, and $z_{\widetilde{r}} \alpha F$ imply $(y^2 * x)_{\widetilde{t} \wedge \widetilde{r}} \beta F$.
Let F be an interval valued fuzzy set of X such that $\widetilde{\mu_F}(x) \leq [0.5, 0.5]$, for all $x \in X$. Suppose that $x \in X$ and $t \in (0, 1]$ such that $x_{\widetilde{t}} \in \wedge qF$. Then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(x) + \widetilde{t} > [1, 1]$. It follows that $[1, 1] < \widetilde{\mu_F}(x) + \widetilde{t} \leq \widetilde{\mu_F}(x) + \widetilde{\mu_F}(x) = 2\widetilde{\mu_F}(x)$, which implies that $\widetilde{\mu_F}(x) > [0.5, 0.5]$. This means that $\{x_{\widetilde{t}} \mid x_{\widetilde{t}} \in \wedge qF\} = \emptyset$. Therefore the case $\alpha = \in \wedge q$ in the Definitions 3.1 and 3.2 can be removed.

Proposition 3.3. *Every interval valued* $(\in \vee q, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X *is an interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X.

Proof. Let F be an interval valued $(\in \vee q, \in \vee q)-$fuzzy sub-implicative ideal of X. Let $x, y, z \in X$ and $t, r \in (0, 1]$ be such that $x_{\widetilde{t}} \in F$, $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in F$ and $z_{\widetilde{r}} \in F$. Then $x_{\widetilde{t}} \in \vee qF$ and $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in \vee qF$ and $z_{\widetilde{r}} \in \vee qF$. It follows that $0_{\widetilde{t}} \in \vee qF$ and $(y^2 * x)_{\widetilde{t} \wedge \widetilde{r}} \in \vee qF$, which completes the proof. For the case of interval valued $(\in, \in \vee q)-$fuzzy ideal the proof is similar.

Proposition 3.4. *Every interval valued* $(\in, \in)-$*fuzzy (sub-implicative) ideal of* X *is an interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X.

Proof. It is clear by considering the definitions.

Lemma 3.5. *Let* I *be a (sub-implicative) ideal of* X*, then* χ_I *(the characteristic function of* I*) is an interval valued* $(\in, \in)-$*fuzzy (sub-implicative) ideal of* X.

Proof. Let $x, y, z \in X$ and $t, r \in (0, 1]$ be such that $x_{\widetilde{t}} \in \chi_I$. Since $0 \in I$, then $\widetilde{\chi_I}(0) = [1, 1] \geq \widetilde{\chi_I}(x) \geq \widetilde{t}$. Thus $0_{\widetilde{t}} \in \chi_I$. Also let $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in \chi_I$ and $z_{\widetilde{r}} \in \chi_I$. Then $\widetilde{\chi_I}(((x^2*y)*(y*x))*z) \geq \widetilde{t} > [0, 0]$ and $\widetilde{\chi_I}(z) \geq \widetilde{r} > [0, 0]$. These imply $\widetilde{\chi_I}(((x^2*y)*(y*x))*z) = \widetilde{\chi_I}(z) = [1, 1]$, and so $(((x^2 * y) * (y * x)) * z) \in I$ and $z \in I$, thus $y^2 * x \in I$. It follows that $\widetilde{\chi_I}(y^2 * x) = [1, 1] \geq \widetilde{t} \wedge \widetilde{r}$, which means $(y^2 * x)_{\widetilde{t} \wedge \widetilde{r}} \in \chi_I$. Therefore χ_I is an interval valued $(\in, \in)-$fuzzy sub-implicative ideal of X. For the case of interval valued $(\in, \in)-$fuzzy ideal, the proof is similar.

Theorem 3.6. *For any subset* I *of* X*,* χ_I *is an interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X *if and only if* I *is a (sub-implicative) ideal of* X.

Proof. Let χ_I be an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X. If $x \in I$ then $x_{[1,1]} \in \chi_I$. This implies $0_{[1,1]} \in \vee q\chi_I$, hence $\widetilde{\chi_I}(0) > [0, 0]$, so $0 \in I$. Also if $((x^2 * y) * (y * x)) * z \in I$ and $z \in I$, then $(((x^2 * y) * (y * x)) * z)_{[1,1]} \in \chi_I$ and $z_{[1,1]} \in \chi_I$. These imply $(y^2*x)_{[1,1]} \in \vee q\chi_I$, hence $\chi_I(y^2*x) > [0, 0]$, so $y^2*x \in I$. Conversely, if I is a sub-implicative ideal of X, then χ_I is an interval valued $(\in, \in)-$fuzzy sub-implicative ideal of X by Lemma 3.5. Therefore χ_I is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X by Proposition 3.4. For the case of interval valued $(\in, \in \vee q)-$fuzzy ideal the proof is similar.

Theorem 3.7. *Let* F *be a non-zero interval valued* $(\alpha, \beta)-$*fuzzy (sub-implicative) ideal of* X*. Then the set*

$supp(\widetilde{\mu_F}) = \{x \in X \mid \widetilde{\mu_F}(x) > [0,0]\}$ *is a (sub-implicative) ideal of* X.

Proof. Let $x \in supp(\widetilde{\mu_F})$ then $\widetilde{\mu_F}(x) > [0,0]$. Now, we assume that $\widetilde{\mu_F}(0) = [0,0]$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\widetilde{\mu_F(x)}}\alpha F$, but $(0)_{\widetilde{\mu_F(x)}}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Also $x_{[1,1]}qF$ but $(0)_{[1,1]}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence $\widetilde{\mu_F}(0) > [0,0]$, that is $0 \in supp(\widetilde{\mu_F})$. Also let $((x^2 * y) * (y * x)) * z \in supp(\widetilde{\mu_F})$ and $z \in \widetilde{\mu_F}$, then $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) > [0,0]$ and $\widetilde{\mu_F}(z) > [0,0]$. Now, we assume that $\widetilde{\mu_F}(y^2 * x) = [0,0]$. If $\alpha \in \{\in, \in \vee q\}$, then $(((x^2 * y) * (y * x)) * z)_{\widetilde{\mu_F(((x^2*y)*(y*x))*z)}}\alpha F$ and $z_{\widetilde{\mu_F(z)}}\alpha F$, but $(y^2 * x)_{\widetilde{\mu_F(((x^2*y)*(y*x))*z)} \wedge \widetilde{\mu_F(z)}}\overline{\beta}F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Also $(((x^2 * y) * (y * x)) * z)_{[1,1]}qF$ and $z_{[1,1]}qF$, but $(y^2 * x)_{[1,1]}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence $\widetilde{\mu_F}(y^2 * x) > [0,0]$, that is $y^2 * x \in supp(\widetilde{\mu_F})$. Therefore $supp(\widetilde{\mu_F})$ is a sub-implicative ideal of X. Similarly, we can prove that $supp(\widetilde{\mu_F})$ is an ideal of X.

Let F be an interval valued fuzzy set. Then, for every $t \in [0,1]$, the set $F_{\widetilde{t}} = \{x \in X \mid \widetilde{\mu_F}(x) \geq \widetilde{t}\}$ is called the *interval valued level subset* of F. An interval valued fuzzy set F of X is called *proper* if ImF contains at least two elements. Two interval valued fuzzy sets are said to be *equivalent* if they have same family of interval valued level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.8. *Suppose that* X *contains some proper sub-implicative ideals. Then a proper interval valued* $(\in, \in)-$*fuzzy (sub-implicative) ideal* F *of* X *with* $|ImF| \geq 3$ *can be expressed as the union of two proper non-equivalent interval valued* $(\in, \in)-$*fuzzy (sub-implicative) ideal of* X.

Proof. Let F be a proper interval valued $(\in, \in)-$fuzzy sub-implicative ideal of X with $ImF = \{\widetilde{t_0}, \widetilde{t_1}, ..., \widetilde{t_n}\}$, where $\widetilde{t_0} > \widetilde{t_1} > ... > \widetilde{t_n}$ and $n \geq 2$. Then $F_{\widetilde{t_0}} \subseteq F_{\widetilde{t_1}} \subseteq ... \subseteq F_{\widetilde{t_n}} = X$ is the chain of interval valued $\in -$level sub-implicative ideals of F. Define two interval valued fuzzy sets G and H in X by

$$\widetilde{\mu_G}(x) = \begin{cases} \widetilde{r_1} & \text{if } x \in F_{\widetilde{t_1}}, \\ \widetilde{t_k} & \text{if } x \in F_{\widetilde{t_k}} \setminus F_{\widetilde{t_{k-1}}} \text{ and } 2 \leq k \leq n \end{cases}$$

$$\widetilde{\mu_H}(x) = \begin{cases} \widetilde{t_0} & \text{if } x \in F_{\widetilde{t_1}}, \\ \widetilde{t_1} & \text{if } x \in F_{\widetilde{t_1}} \setminus F_{\widetilde{t_0}}, \\ \widetilde{r_2} & \text{if } x \in F_{\widetilde{t_3}} \setminus F_{\widetilde{t_1}}, \\ \widetilde{t_k} & \text{if } x \in F_{\widetilde{t_k}} \setminus F_{\widetilde{t_{k-1}}} \text{ and } 4 \leq k \leq n \end{cases}$$

such that $\widetilde{t_2} < \widetilde{r_1} < \widetilde{t_1}$ and $\widetilde{t_4} < \widetilde{r_2} < \widetilde{t_2}$. Then G and H are interval valued $(\in, \in)-$fuzzy sub-implicative ideals of X, where $F_{\widetilde{t_1}} \subseteq F_{\widetilde{t_2}} \subseteq ... \subseteq F_{\widetilde{t_n}} = X$, and $F_{\widetilde{t_0}} \subseteq F_{\widetilde{t_1}} \subseteq ... \subseteq F_{\widetilde{t_n}} = X$ are respectively the chain of interval valued $\in -$level sub-implicative ideals of X, and $G, H \leq F$. Thus G and H are non-equivalent, and it is obvious that $G \cup H = F$. Therefore F can be expressed as the union of two proper non-equivalent interval valued $(\in, \in)-$fuzzy sub-implicative ideal of X. For the case of interval valued $(\in, \in)-$fuzzy ideals the proof is similar.

IV. Interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideals

Definition 4.1. An interval valued fuzzy set F of X is called an *interval valued fuzzy (sub-implicative) ideal* of X if for all $x, y, z \in X$, it satisfies the following conditions:
$(I4)$ $\widetilde{\mu_F}(0) \geq \widetilde{\mu_F}(x)$,
$(II4)$ $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y)$,
$(\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z))$.

Theorem 4.2. *An interval valued fuzzy set* F *of* X *is an interval valued fuzzy (sub-implicative) ideal of* X *if and only if for any* $[0,0] < \widetilde{t} \leq [1,1], F_{\widetilde{t}}(\neq \emptyset)$ *is a (sub-implicative) ideal of* X.

Proof. Let F be an interval valued fuzzy sub-implicative ideal of X and $[0,0] < \widetilde{t} \leq [1,1]$ such that $F_{\widetilde{t}}(\neq \emptyset)$. Also let $x \in F_{\widetilde{t}}$, then $\widetilde{\mu_F}(x) \geq \widetilde{t}$. So $\widetilde{\mu_F}(0) \geq \widetilde{\mu_F}(x) \geq \widetilde{t}$ and hence $0 \in F_{\widetilde{t}}$. Also if $((x^2 * y) * (y * x)) * z \in F_{\widetilde{t}}$ and $z \in F_{\widetilde{t}}$, then $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \geq \widetilde{t}$ and $\widetilde{\mu_F}(z) \geq \widetilde{t}$. So $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \geq \widetilde{t}$ and hence $y^2 * x \in F_{\widetilde{t}}$. Therefore $F_{\widetilde{t}}$ is a sub-implicative ideal of X. Similarly we can prove that $F_{\widetilde{t}}$ is an ideal of X.
Conversely, suppose for any $[0,0] < \widetilde{t} \leq [1,1], F_{\widetilde{t}}(\neq \emptyset)$ is a sub-implicative ideal of X. Let $x, y, z \in X$ and $\widetilde{\mu_F}(x) = \widetilde{t_1}$, $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) = \widetilde{t_2}$ and $\widetilde{\mu_F}(z) = \widetilde{t_3}$. Then $0 \in F_{\widetilde{t_1}}$, thus $\widetilde{\mu_F}(0) \geq \widetilde{t_1} = \widetilde{\mu_F}(x)$. Also $((x^2 * y) * (y * x)) * z \in F_{\widetilde{t_2} \wedge \widetilde{t_3}}$ and $z \in F_{\widetilde{t_2} \wedge \widetilde{t_3}}$. Then $y^2 * x \in F_{\widetilde{t_2} \wedge \widetilde{t_3}}$, thus $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{t_2} \wedge \widetilde{t_3} = \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$. Therefore F is an interval valued fuzzy sub-implicative ideal of X. Similarly, we can show that F is an interval valued fuzzy ideal of X.

Example 4.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following table.

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Define an interval valued fuzzy set F by $\widetilde{\mu_F}(0) = [0.8, 0.9]$ and $\widetilde{\mu_F}(x) = [0.1, 0.2]$ for all $x \neq 0$. Then F is an interval valued fuzzy ideal of X, but it is not an interval valued fuzzy sub-implicative ideal of X because

$$\widetilde{\mu_F}(a^2 * b) \ngeq \widetilde{\mu_F}(((b^2 * a) * (a * b)) * 0) \wedge \widetilde{\mu_F}(0).$$

Definition 4.4. An interval valued fuzzy set F of X is said to be an *interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal* of X if for all $[0,0] < \widetilde{t}, \widetilde{r} \leq [1,1]$ and $x, y, z \in X$, the following conditions hold:
$(I5)$ $x_{\widetilde{t}} \in F$ implies that $0_{\widetilde{t}} \in \vee qF$,
$(II5)$ $(x * y)_{\widetilde{t}} \in F$ and $y_{\widetilde{r}} \in F$ imply that $x_{\widetilde{t} \wedge \widetilde{r}} \in \vee qF$,
$((((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in F$ and $z_{\widetilde{r}} \in F$ imply that $(y^2 * x)_{\widetilde{t} \wedge \widetilde{r}} \in \vee qF)$.

Example 4.5. Consider the BCI-algebra of Example 4.3. Define an interval valued fuzzy set F by $\widetilde{\mu_F}(0) = [0.7, 0.8]$ and $\widetilde{\mu_F}(x) = [0.2, 0.3]$ for all $x \neq 0$. It is easy to verify that F is an interval valued $(\in, \in \vee q)-$fuzzy ideal of X.

Example 4.6. Consider a BCI-algebra $X = \{0, 1, 2\}$ with the following table.

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Let F be an interval valued fuzzy set in X defined by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(1) = [0.6, 0.7]$ and $\widetilde{\mu_F}(2) = [0.2, 0.3]$. It is easy to verify that F is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X.

Theorem 4.7. *The conditions* $(I5)$ *and* $(II5)$ *in Definition 4.4, are equivalent to the following conditions, respectively for all* $x, y, z \in X$
$(I6)$ $\widetilde{\mu_F}(x) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(0)$,
$(II6)$ $\widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(x)$,
$(\widetilde{\mu_F}(((x^2*y)*(y*x))*z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(y^2*x))$.

Proof. $(I5) \Longrightarrow (I6)$: Suppose that $x \in X$. Then we consider the following cases:
(a) $\widetilde{\mu_F}(x) \leq [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(0) < \widetilde{\mu_F}(x) \wedge [0.5, 0.5]$. Then, it implies that $\widetilde{\mu_F}(0) < \widetilde{\mu_F}(x)$. Choose $\widetilde{t}$ such that $\widetilde{\mu_F}(0) < \widetilde{t} < \widetilde{\mu_F}(x)$. Then $x_{\widetilde{t}} \in F$ but $0_{\widetilde{t}} \overline{\in \vee q} F$, which contradicts $(I5)$.
(b) $\widetilde{\mu_F}(x) > [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(0) < [0.5, 0.5]$ Then $x_{[0.5,0.5]} \in F$ but $0_{[0.5,0.5]} \overline{\in \vee q} F$, which is a contradiction. Hence $(I6)$ holds.
$(II5) \Longrightarrow (II6)$: Suppose that $x, y, z \in X$. Then we can consider the following cases:
(a) $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \leq [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(y^2 * x) < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5]$. Then, it implies that $\widetilde{\mu_F}(y^2 * x) < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$. Choose $\widetilde{t}$ such that $\widetilde{\mu_F}(y^2 * x) < \widetilde{t} < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$. Then $(((x^2*y)*(y*x))*z)_{\widetilde{t}} \in F$ and $z_{\widetilde{t}} \in F$, but $(y^2*x)_{\widetilde{t}} \overline{\in \vee q} F$, which contradicts $(II5)$.
(b) $\widetilde{\mu_F}(((x^2*y)*(y*x))*z) \wedge \widetilde{\mu_F}(z) > [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(y^2*x) < [0.5, 0.5]$. Then $(((x^2*y)*(y*x))*z)_{[0.5,0.5]} \in F$ and $z_{[0.5,0.5]} \in F$, but $(y^2 * x)_{[0.5,0.5]} \overline{\in \vee q} F$, which is a contradiction.
Similarly we can prove that $\widetilde{\mu_F}(x*y) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(x)$. Therefore $(II6)$ holds.
$(I6) \Longrightarrow (I5)$: Straightforward.
$(II6) \Longrightarrow (II5)$: Let $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in F$ and $z_{\widetilde{r}} \in F$. Then $\widetilde{\mu_F}(((x^2*y)*(y*x))*z) \geq \widetilde{t}$ and $\widetilde{\mu_F}(z) \geq \widetilde{r}$. We have $\widetilde{\mu_F}(y^2*x) \geq \widetilde{\mu_F}(((x^2*y)*(y*x))*z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5] \geq \widetilde{t} \wedge \widetilde{r} \wedge [0.5, 0.5]$. We can consider two following cases:
(a) $\widetilde{t} \wedge \widetilde{r} > [0.5, 0.5]$, then $\widetilde{\mu_F}(y^2 * x) \geq [0.5, 0.5]$, which implies that $\widetilde{\mu_F}(y^2 * x) + (\widetilde{t} \wedge \widetilde{r}) > [1, 1]$, or equivalently $(y^2 * x)_{\widetilde{t}\wedge\widetilde{r}} qF$. Thus $(y^2 * x)_{\widetilde{t}\wedge\widetilde{r}} \in \vee qF$.
(b) $\widetilde{t} \wedge \widetilde{r} \leq [0.5, 0.5]$, then $\widetilde{\mu_F}(y^2*x) \geq \widetilde{t} \wedge \widetilde{r}$, or equivalently $(y^2 * x)_{\widetilde{t}\wedge\widetilde{r}} \in F$. Thus $(y^2 * x)_{\widetilde{t}\wedge\widetilde{r}} \in \vee qF$.
Similarly, we can prove $(x * y)_{\widetilde{t}} \in F$ and $y_{\widetilde{r}} \in F$ imply that $x_{\widetilde{t}\wedge\widetilde{r}} \in \vee qF$. Therefore $(II5)$ holds.

Corollary 4.8. *An interval valued fuzzy set* F *of* X *is an interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X *if and only if conditions* $(I6)$ *and* $(II6)$ *in Theorem 4.7 hold.*

Theorem 4.9. *Let* F *be an interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X. *Then for all* $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, $F_{\widetilde{t}} = \emptyset$ *or* $F_{\widetilde{t}}$ *is a (sub-implicative) ideal of* X.
Conversely , if F *is an interval valued fuzzy set of* X *such that* $F_{\widetilde{t}}(\neq \emptyset)$ *is a (sub-implicative) ideal of* X *for all* $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, *then* F *is an interval valued* $(\in, \in \vee q)-$*fuzzy (sub-implicative) ideal of* X.

Proof. Let F be an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X and $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. It is easy to verify that $0 \in F_{\widetilde{t}}$. If $((x^2*y)*(y*x))*z \in F_{\widetilde{t}}$ and $z \in F_{\widetilde{t}}$, then $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \geq \widetilde{t}$ and $\widetilde{\mu_F}(z) \geq \widetilde{t}$. Hence $\widetilde{\mu_F}((y^2*x)) \geq \widetilde{\mu_F}(((x^2*y)*(y*x))*z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5] \geq \widetilde{t} \wedge [0.5, 0.5] = \widetilde{t}$. Thus $y^2 * x \in F_{\widetilde{t}}$. Similarly, $x * y \in F_{\widetilde{t}}$ and $y \in F_{\widetilde{t}}$ imply that $x \in F_{\widetilde{t}}$.
Conversely , Let F be an interval valued set of X such that $F_{\widetilde{t}}(\neq \emptyset)$ is a sub-implicative ideal of X, for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. Let $x, y, z \in X$. It is easy to verify that $\widetilde{\mu_F}(0) \geq \widetilde{\mu_F}(x) \wedge [0.5, 0.5]$. Also we can say that $\widetilde{\mu_F}(((x^2*y)*(y*x))*z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5] = \widetilde{t_0}$ and $\widetilde{\mu_F}(z) \geq \widetilde{\mu_F}(((x^2 * y) * (y * x))*z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5] = \widetilde{t_0}$. Hence $((x^2*y)*(y*x))*z \in F_{\widetilde{t_0}}$ and $z \in F_{\widetilde{t_0}}$, so $y^2 * x \in F_{\widetilde{t_0}}$. Similarly, $x * y \in F_{\widetilde{t}}$ and $y \in F_{\widetilde{t}}$ imply that $x \in F_{\widetilde{t}}$.

Theorem 4.10. *Let* F *be an interval valued fuzzy set of* X. *Then* $F_{\widetilde{t}}(\neq \emptyset)$ *is a (sub-implicative) ideal of* X *for all* $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$ *if and only if for all* $x, y, z \in X$ *the following conditions hold:*
$(I7)$ $\widetilde{\mu_F}(0) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$,
$(II7)$ $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y)$,
$(\widetilde{\mu_F}(y^2*x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(((x^2*y)*(y*x))*z) \wedge \widetilde{\mu_F}(z))$.

Proof. Assume that $F_{\widetilde{t}}(\neq \emptyset)$ is a sub-implicative ideal of X for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$. It is easy to verify that $\widetilde{\mu_F}(0) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$. If there exist $x, y, z \in X$ such that $\widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5] < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) = \widetilde{t}$, then we have $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$, $\widetilde{\mu_F}(y^2*x) < \widetilde{t}$ and $((x^2*y)*(y*x))*z \in F_{\widetilde{t}}$ and $z \in F_{\widetilde{t}}$. So $y^2*x \in F_{\widetilde{t}}$, which implies that $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{t}$. This is a contradiction. Thus $\widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$ for all $x, y, z \in X$. Similarly we can show that $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y)$. Therefore $(I7)$ and $(II7)$ hold.
Conversely , suppose that conditions $(I7)$ and $(II7)$ hold. Let $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$. It is easy to see that $0 \in F_{\widetilde{t}}$. Also let $((x^2 * y) * (y * x)) * z \in F_{\widetilde{t}}$ and $z \in F_{\widetilde{t}}$. We have $[0.5, 0.5] < \widetilde{t} \leq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \leq \widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5]$, which implies that $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{t}$. Thus $y^2 * x \in F_{\widetilde{t}}$. Similarly, $x * y \in F_{\widetilde{t}}$ and $y \in F_{\widetilde{t}}$ imply that $x \in F_{\widetilde{t}}$.

By Theorem 4.2, it is well known that an interval valued fuzzy set F of X is an interval valued fuzzy (sub-implicative) ideal if and only if $F_{\widetilde{t}}(\neq \emptyset)$ is a (sub-implicative) ideal of X for all $[0, 0] < \widetilde{t} \leq [1, 1]$. In Theorem 4.9, we prove that F is an interval valued $(\in, \in \vee q)-$fuzzy (sub-implicative) ideal of X if and only if the set $F_{\widetilde{t}}(\neq \emptyset)$ is a (sub-implicative) ideal of X for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. Naturally, a corresponding result should be considered when $F_{\widetilde{t}}$ is a (sub-implicative) ideal of X for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$.

Definition 4.11. An interval valued fuzzy set F of X is said to be an *interval valued* $(\overline{\in}, \overline{\in \wedge q})-$*fuzzy (sub-implicative)*

ideal if for all $[0,0] < \widetilde{t}, \widetilde{r} \leq [1,1]$ and $x, y, z \in X$ the following conditions hold:
$(I8)$ $0_{\widetilde{t}} \overline{\in} F$ implies $x_{\widetilde{t}} \overline{\in \wedge q} F$,
$(II8)$ $x_{\widetilde{t} \wedge \widetilde{r}} \overline{\in} F$ implies $(x * y)_{\widetilde{t}} \overline{\in \wedge q} F$ or $y_{\widetilde{r}} \overline{\in \wedge q} F$,
$((y^2 * x)_{\widetilde{t} \wedge \widetilde{r}} \overline{\in} F$ implies $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \overline{\in \wedge q} F$ or $z_{\widetilde{r}} \overline{\in \wedge q} F)$.

Theorem 4.12. *Let F be an interval valued fuzzy set of X. Then F is an interval valued $(\overline{\in}, \overline{\in \wedge q})-$fuzzy (sub-implicative) ideal of X if and only if for all $x, y, z \in X$ the following conditions hold:*
$(I9)$ $\widetilde{\mu_F}(0) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$,
$(II9)$ $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y)$,
$(\widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z))$.

Proof. Let F be an interval valued $(\overline{\in}, \overline{\in \wedge q})-$fuzzy sub-implicative ideal of X. If there exists $x \in X$ such that $\widetilde{\mu_F}(0) \vee [0.5, 0.5] < \widetilde{\mu_F}(x) = \widetilde{t}$, then $[0.5, 0.5] < \widetilde{t} \leq [1,1]$, $0_{\widetilde{t}} \overline{\in} F$ and $x_{\widetilde{t}} \in F$. It follows that $x_{\widetilde{t}} \overline{q} F$. Then $\widetilde{\mu_F}(x) + \widetilde{t} \leq [1,1]$. So $\widetilde{t} \leq [0.5, 0.5]$, which is a contradiction. Hence $(I9)$ holds. Also if there exist $x, y, z \in X$ such that $\widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5] < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) = \widetilde{t}$, then $[0.5, 0.5] < \widetilde{t} \leq [1,1]$, $(y^2 * x)_{\widetilde{t}} \overline{\in} F$, $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in F$ and $z_{\widetilde{t}} \in F$. It follows that $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \overline{q} F$ or $z_{\widetilde{t}} \overline{q} F$. Then $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) + \widetilde{t} \leq [1,1]$ or $\widetilde{\mu_F}(z) + \widetilde{t} \leq [1,1]$. So $\widetilde{t} \leq [0.5, 0.5]$, which is a contradiction. Hence $\widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$. Similarly, we can prove that $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y)$. Therefore $(II9)$ holds.
Conversely , let $(I9)$ and $(II9)$ hold. Also let $x, y, z \in X$ and $[0,0] < \widetilde{t}, \widetilde{r} \leq [1,1]$. If $0_{\widetilde{t}} \overline{\in} F$, then it is easy to verify that $x_{\widetilde{t}} \overline{\in \wedge q} F$. Now if $(y^2 * x)_{\widetilde{t} \wedge \widetilde{r}} \overline{\in} F$, then $\widetilde{\mu_F}(y^2 * x) < \widetilde{t} \wedge \widetilde{r}$. Then we have the following cases:
(a) If $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$, then $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) < \widetilde{t} \wedge \widetilde{r}$, and so $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) < \widetilde{t}$ or $\widetilde{\mu_F}(z) < \widetilde{r}$. It follows that $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \overline{\in} F$ or $z_{\widetilde{r}} \overline{\in} F$, which implies that $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \overline{\in \wedge q} F$ or $z_{\widetilde{r}} \overline{\in \wedge q} F$.
(b) If $\widetilde{\mu_F}(y^2 * x) < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$, then we have $[0.5, 0.5] \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$ (since $\widetilde{\mu_F}(y^2 * x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z)$). Now if $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \in F$ and $z_{\widetilde{r}} \in F$ then $\widetilde{t} \leq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \leq [0.5, 0.5]$ or $\widetilde{r} \leq \widetilde{\mu_F}(z) \leq [0.5, 0.5]$. It follows that $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \overline{q} F$ or $z_{\widetilde{r}} \overline{q} F$, which implies that $(((x^2 * y) * (y * x)) * z)_{\widetilde{t}} \overline{\in \wedge q} F$ or $z_{\widetilde{r}} \overline{\in \wedge q} F$. Similarly, $x_{\widetilde{t} \wedge \widetilde{r}} \overline{\in} F$ implies that $(x * y)_{\widetilde{t}} \overline{\in \wedge q} F$ or $y_{\widetilde{r}} \overline{\in \wedge q} F$. Therefore the proof is completed.

Theorem 4.13. *An interval valued fuzzy set F of X is an $(\overline{\in}, \overline{\in \wedge q})-$fuzzy (sub-implicative) ideal of X if and only if $F_{\widetilde{t}} (\neq \emptyset)$ is a (sub-implicative) ideal of X for all* $[0.5, 0.5] < \widetilde{t} \leq [1,1]$.

Proof. The result is immediately followed by Theorem 4.10 and 4.12.

In [13], Yuan et al. gave the definition of fuzzy subgroup with thresholds which is a generalization of the fuzzy subgroup of Rosenfeld and also the fuzzy subgroup of Bhakat and Das. Based on [13], we can extend the concept of a fuzzy subgroup with thresholds to the concept of interval valued (sub-implicative) ideal with thresholds in the following way.

Definition 4.14. Let $[0,0] \leq \widetilde{s} < \widetilde{t} \leq [1,1]$. Then an interval valued fuzzy set F of X is called an *interval valued fuzzy (sub-implicative) ideal with thresholds* $(\widetilde{s}, \widetilde{t})$ of X if for all $x, y, z \in X$ the following conditions hold:
$(I10)$ $\widetilde{\mu_F}(0) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{t}$,
$(II10)$ $\widetilde{\mu_F}(x) \vee \widetilde{s} \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y) \wedge \widetilde{t}$,
$(\widetilde{\mu_F}(y^2 * x) \vee \widetilde{s} \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge \widetilde{t})$.

Remark. If F is an interval valued fuzzy (sub-implicative) ideal with thresholds of X, then we can conclude that F is an ordinary interval valued fuzzy (sub-implicative) ideal when $\widetilde{s} = [0,0]$ and $\widetilde{t} = [1,1]$. Also F is an interval valued $(\in, \in \vee q)-$fuzzy (resp. $(\overline{\in}, \overline{\in \wedge q})-$fuzzy) (sub-implicative) ideal when $\widetilde{s} = [0,0]$ and $\widetilde{t} = [0.5, 0.5]$ (resp. $\widetilde{s} = [0.5, 0.5]$ and $\widetilde{t} = [1,1]$).

Theorem 4.15. *An interval valued fuzzy set F of X is an interval valued fuzzy (sub-implicative) ideal with thresholds $(\widetilde{s}, \widetilde{t})$ of X if and only if $F_{\alpha} (\neq \emptyset)$ is a (sub-implicative) ideal of X for all $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$.*

Proof. Let F be an interval valued fuzzy sub-implicative ideal with thresholds $(\widetilde{s}, \widetilde{t})$ of X and $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$. It is easy to verify that $0 \in F_{\widetilde{\alpha}}$. Let $((x^2 * y) * (y * x)) * z \in F_{\widetilde{\alpha}}$ and $z \in F_{\widetilde{\alpha}}$, then $\widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \geq \widetilde{\alpha}$ and $\widetilde{\mu_F}(z) \geq \widetilde{\alpha}$. Now we have

$$\widetilde{\mu_F}(y^2 * x) \vee \widetilde{s} \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge \widetilde{t} \geq \widetilde{\alpha} \wedge \widetilde{t} \geq \widetilde{\alpha} > \widetilde{s},$$

which implies that $\widetilde{\mu_F}(y^2 * x) > \widetilde{\alpha}$, and so $y^2 * x \in F_{\widetilde{\alpha}}$. Similarly, $x * y \in F_{\widetilde{\alpha}}$ and $y \in F_{\widetilde{\alpha}}$ imply that $x \in F_{\widetilde{\alpha}}$.
Conversely , let F be an interval valued fuzzy set of X such that $F_{\widetilde{\alpha}} (\neq \emptyset)$ is a sub-implicative ideal of X for all $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$. It is easy to verify that $\widetilde{\mu_F}(0) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{t}$ for all $x \in X$. If there exist $x, y, z \in X$ such that $\widetilde{\mu_F}(y^2 * x) \vee \widetilde{s} < \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge \widetilde{t} = \widetilde{\alpha}$, then $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$, $\widetilde{\mu_F}(y^2 * x) < \widetilde{\alpha}$ and $((x^2 * y) * (y * x)) * z \in F_{\widetilde{\alpha}}$ and $z \in F_{\widetilde{\alpha}}$. So $y^2 * x \in F_{\widetilde{\alpha}}$, which implies that $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\alpha}$. This contradicts $\widetilde{\mu_F}(y^2 * x) < \widetilde{\alpha}$. Thus $\widetilde{\mu_F}(y^2 * x) \vee \widetilde{s} \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge \widetilde{t}$, for all $x, y, z \in X$. Similarly, we can prove that $\widetilde{\mu_F}(x) \vee \widetilde{s} \geq \widetilde{\mu_F}(x * y) \wedge \widetilde{\mu_F}(y) \wedge \widetilde{t}$, which completes the proof.

V. Properties of interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideals

Theorem 5.1. *If F is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X, then the following inequality holds:*
$(I11)$ $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) \wedge [0.5, 0.5]$.

Proof. If F is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X, then by taking $z = 0$ in $(II6)$ of Theorem 4.7 and using $(I6)$ of Theorem 4.7, we have:

$$\begin{aligned}\widetilde{\mu_F}(y^2 * x) &\geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * 0) \wedge \widetilde{\mu_F}(0) \wedge [0.5, 0.5] \\ &= \widetilde{\mu_F}((x^2 * y) * (y * x)) \wedge \widetilde{\mu_F}(0) \wedge [0.5, 0.5] \\ &= \widetilde{\mu_F}((x^2 * y) * (y * x)) \wedge [0.5, 0.5],\end{aligned}$$

which completes the proof.

Theorem 5.2. *Every interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X is an interval valued $(\in, \in \vee q)-$fuzzy ideal of X.*

Proof. Let F be an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X. Putting $y = x$ in $(II6)$ of Theorem 4.7, we obtain for all $x, z \in X$

$$\widetilde{\mu_F}(x) = \widetilde{\mu_F}(x^2 * x)$$
$$\geq \widetilde{\mu_F}(((x^2 * x) * (x * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5]$$
$$= \widetilde{\mu_F}(x * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5].$$

Therefore F is an interval valued $(\in, \in \vee q)-$fuzzy ideal of X.

The following example shows that the converse of Theorem 5.2 may not be true.

Example 5.3. Consider the BCI-algebra X of Example 4.3. Define an interval valued fuzzy set F of X by $\widetilde{\mu_F}(0) = [0.72, 0.78]$ and $\widetilde{\mu_F}(x) = [0.22, 0.28]$ for all $x \neq 0$. Then it is easy to verify that F is an interval valued $(\in, \in \vee q)-$fuzzy ideal of X, but it is not an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X. Because $\widetilde{\mu_F}(a^2 * b) = [0.22, 0.28] \ngeq [0.5, 0.5] = \widetilde{\mu_F}(((b^2 * a) * (a * b)) * 0) \wedge \widetilde{\mu_F}(0) \wedge [0.5, 0.5]$.

Theorem 5.4. *Every interval valued $(\in, \in \vee q)-$fuzzy ideal F of X satisfying the condition $(I11)$ of the Theorem 5.1 is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X.*

Proof. Let F be an interval valued $(\in, \in \vee q)-$fuzzy ideal of X. For any x, y, z
inX, by conditions $(I11)$ of Theorem 5.1 and $(II6)$ of Theorem 4.7, we have

$$\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\mu_F}((x^2 * y) * (y * x)) \wedge [0.5, 0.5]$$
$$\geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5] \wedge [0.5, 0.5]$$
$$\geq \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5].$$

Therefore F is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X.

Example 5.5. Consider a BCI-algebra $X = \{0, a, 1, 2, 3\}$ with the following table

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Define an interval valued fuzzy set F in X by $\widetilde{\mu_F}(0) = [0.7, 0.8]$, $\widetilde{\mu_F}(a) = [0.5, 0.6]$ and $\widetilde{\mu_F}(1) = \widetilde{\mu_F}(2) = \widetilde{\mu_F}(3) = [0.2, 0.3]$. Then F is an interval valued $(\in, \in \vee q)-$fuzzy ideal of X such that the inequality $\widetilde{\mu_F}(y^2 * x) \geq \widetilde{\mu_F}((x^2 * y) * (y * x))$ holds for all $x, y \in X$. Therefore by Theorem 5.4, F is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X.

Theorem 5.6. *In an implicative BCI-algebra X every interval valued $(\in, \in \vee q)-$fuzzy ideal of X is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X.*

Proof. Let X be an implicative BCI-algebra and F be an interval valued $(\in, \in \vee q)-$fuzzy ideal of X. We have

$$\widetilde{\mu_F}(y^2 * x) = \widetilde{\mu_F}(y * (y * x))$$
$$\geq \widetilde{\mu_F}((y * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5]$$
$$= \widetilde{\mu_F}(((x^2 * y) * (y * x)) * z) \wedge \widetilde{\mu_F}(z) \wedge [0.5, 0.5].$$

Therefore F is an interval valued $(\in, \in \vee q)-$fuzzy sub-implicative ideal of X.

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