Non-existence of Global Solutions to a Wave Equation with Fractional Damping

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***Abstract*—We consider the nonlinear fractional wave-equation**

$$u_{tt}-\Delta u+(-\Delta)^{\beta/2}D^{\alpha}u=|u|^{p},$$

posed in $Q:=(0,\ +\infty)\times \mathbb{R}^N$, where D^{α}, $1<\alpha<2$, is a time fractional derivative, $(-\Delta)^{\beta/2}$, $0<\beta<2$ is fractional power of $-\Delta$, with given initial position and velocity $u(x,0)=u_0(x)$ and $u_t(x,0)=u_1(x)$. We find Fujita's exponent which separates in terms of $p,\alpha,\beta,$ and N, the case of global existence from the one of nonexistence of global solutions. Then we establish necessary conditions on $u_0(x),u_1(x)$ assuring non-existence of local solutions.

***Index Terms*—nonlinear wave equation, fractional power derivative, critical exponent.**

I. Introduction

In this paper, we discuss the nonexistence of weak solutions to the nonlinear fractional wave equation posed in $Q_T=\mathbb{R}^N\times(0,T)$, $0<T\le+\infty$, subject to the initial conditions:

$$\begin{cases} u_{tt}-\Delta u+(-\Delta)^{\frac{\beta}{2}}D_+^{\alpha}u=|u|^{p} \\ u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\mathbb{R}^N. \end{cases} \quad (1)$$

Where $\Delta=\partial_1^2+...+\partial_N^2$ is the usual Laplacian in the space variable x, u_t is the time derivative of u, $(-\Delta)^{\beta/2}$ is $\beta/2$ fractional power of the Laplacian$(0<\beta\le 2)$ which stands for propagation in media with impurities and is defined by

$$(-\Delta)^{\beta/2}v(x)=\mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(v)(\xi)\right)(x),$$

where $\mathcal{F}$ denotes the Fourier transorm and $\mathcal{F}^{-1}$ its inverse; D_+^{α}, is the fractional derivative of order, $1<\alpha<2$, such that $D_+^{\alpha}=D_+^{\alpha-1}u_t$ where $D_+^{\alpha-1}$ is the Caputo fractional derivatives of order $\alpha-1$, $u_0(x)$ and $u_1(x)$ are given initial data. Before we state our results, let us dwell on existing literature concerning equations close to initial value problem (1).

The time fractional derivative has long been found to be very effective means to describe the anomalous attenuation behaviors. For example, Hanyga and Seredynska[13], considered the differential equation

$$D^2u+\gamma D^{\eta+1}u+F(u)=0, \quad (2)$$

where $D^{\eta+1}$,$0<\eta<1$, represents the $(\eta+1)$-order fractional derivative in the sense of Caputo which models the anomalous attenuation, and γ is the thermoviscous coefficient. Recently, Chen and Holm [4], studied the equation

$$\Delta v=\frac{1}{c_0^2}v_{tt}+\frac{2\alpha_0}{c_0^{1-\gamma}}(-\Delta)^{y/2}v_t, \quad (3)$$

which governs the propagation of sound through a viscous fluid, c_0 is the inviscid phase velocity, $2\alpha_0$ is the collective thermoviscous coefficient.

In [5], they extended their study to the wave equation model for frequency dependent lossy media

$$\Delta P=\frac{1}{c_0^2}P_{tt}+\gamma\frac{\partial^{\eta}}{\partial t^{\eta}}(-\Delta)^{s/2}P, \quad (4)$$

$0\le s\le 2$, $0<\eta\le 3$, $\eta\ne 2$, where γ is the viscous constant, s and η can be arbitrary real numbers their range of specification. Equations (3) and (4) can be seen as generalization of the earlier important work of Greenberg, MacCamy and Mizel [6] who considered the equation

$$\rho_0u_{tt}=u_{xx}+\lambda u_{xtx}+g(x,t)$$

with $x\in\mathbb{R},t>0$, and ρ_0,λ are some constants that characterize the medum; $g(x,t)$ is a given function representing an external force. As equations (3) and (4) my be viewed as approximations of nonlinear equations, Eq. (1) contains a nonlinear term that is a prototype of nonlinearities that may occur in practice.

Let's note also that in [3], Cholewa and Carvalho dealt with the equation

$$u_{tt}=\Delta u+(-\Delta)^{\theta}u_t+|u|^{p}$$

which is a special case of problem (1).

If $\beta=0$ and $\alpha=1$ in (1), then we obtain the wave equation with the linear damping u_t.There are many authors treated this case, see for instance Todorova and Yordanov [14], Qi. S. Zhang [15], Mitidieri and Pohozaev[9]. In [9], the authors showed that the Fujita exponentis equal to $1+2/N$. The very interesting article of Todorova and Yordanov is in fact a "complete" study of Eq.(1) when $\beta=0$ and $\alpha=1$. However, when $\beta=0,0<\alpha<1$, Kirane and Tatar [8] showed that the blow-up results for $p<1+\frac{2\alpha}{2+N-2\alpha}$.

The method of proof is rather simple and consists in a judicious choice of a test function. This method developed by Mitidieri and Pohozaev[9], Pohozaev and Tesei [10] for the equation and inequalities with polynomial nonlinearity and then in the paper of Baras and Pierre [1], Baras Kersner [2], Kalashnikov [7] and Qi. S. Zhang [15].

II. Preliminaries

In this section we present two different definitions of fractional derivatives, and some of their properties. We define the fractional derivative in the Caputo sense (see [11])

$${}^{C}D^{\gamma}u(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-\tau)^{-\gamma}u(\tau)d\tau\ ,\ 0<\gamma<1,$$

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and for higher power,

$$^{C}D^{\gamma}u(t)=\frac{1}{\Gamma\left(m-\gamma\right)}\int_{0}^{t}\left(t-\tau\right)^{m-\gamma-1}u^{(m)}(t),$$
$$m=[\gamma]+1.$$

The fractional derivative in the Riemann-Liouville sense

$$^{RL}D^{\gamma}u(t)=\frac{1}{\Gamma\left(m-\gamma\right)}\frac{d^{m}}{dt^{m}}\int_{0}^{t}\left(t-\tau\right)^{m-\gamma-1}u(t)dt,$$
$$m=[\gamma]+1.$$

The relationship between the two definitions is

$$^{RL}D^{\gamma}u(t)={}^{C}D^{\gamma}u(t)+\sum_{k=0}^{m-1}\frac{t^{k-\gamma}}{\Gamma\left(k-\gamma+1\right)}u^{(k)}\left(0^{+}\right),$$

we have also the formula integration by parts

$$\int_{0}^{T}f(t)(D_{0|t}^{\gamma}g)(t)dt=\int_{0}^{T}g(t)(D_{t|T}^{\gamma}f)(t),\ 0<\gamma<1.$$
$$(\text{see}([12],p.46).$$

III. Non-existence of global solutions

Q_T will denote here the set $Q_T := (0,T)\times\mathbb{R}^N$, $L^p_{loc}\left(Q_T,dtdx\right)$ will denote the space of all functions $v:\mathbb{R}_+\times\mathbb{R}^N\rightarrow\mathbb{R}$ such that $\int_K |v|^p\,dtdx<\infty$ for any compact K in $\mathbb{R}_+\times\mathbb{R}^N$.

Definition 1. A function $u\in L^1_{loc}\left(Q_T\right)$ is a local weak solution of the problem (1) defined on $Q_T,\ 0<T<+\infty$, if $u\in L^p_{loc}\left(Q_T\right)$ such that:

$$\begin{aligned}&\int_{Q_T}\xi_{tt}u-\int_{Q_T}u\Delta\xi-\int_{Q_T}uD_{-}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\xi=\\&\int_{Q_T}|u|^p\,\xi+\int_{\mathbb{R}^N}\xi u_1+\int_{\mathbb{R}^N}\left(-\Delta\right)^{\frac{\beta}{2}}u_0D_{-}^{\alpha-1}\xi\left(0\right),\end{aligned}\tag{5}$$

for any test function $\xi\in C^{2,2}_{x,t}\left(\mathbb{R}^N\times[0,T]\right)$, such that $\xi\geq 0,\ \xi\left(T,x\right)=\xi_t\left(T,x\right)=\xi_t\left(0,x\right)=0$.

Now, we are in position to announce our results.

Theorem 3.1: Assume that

$$\int_{\mathbb{R}^N}\left(-\Delta\right)^{\frac{\beta}{2}}u_0>0,\int_{\mathbb{R}^N}u_1>0,p>1$$

if

$$p\leq p_c=1+\frac{2\alpha}{2+\alpha N-2\alpha},$$

then, problem (1) does not admit global nontrivial solutions in time.

Proof. The proof is by contradiction. So we assume that the solution is global. Let Φ be a decreasing function $C^2_0\left(\mathbb{R}_+\right)$, $0\leq\Phi\leq 1$ such that

$$\Phi\left(y\right):=\begin{cases}1 & \text{if}\quad 0\leq y\leq 1\quad,\\ 0 & \text{if}\quad y\geq 2\quad.\end{cases}$$

We choose

$$\xi\left(x,t\right):=\Phi^{\lambda}\left(\frac{t^{2\alpha}+|x|^4}{R^4}\right),t=R^{\frac{2}{\alpha}}\tau,x=Ry,$$
$$dxdt=R^{\frac{2}{\alpha}+N}dyd\tau$$

Where R is a positive real number and λ is any real greater than p.

The test function ξ is chosen so that

$$\begin{aligned}&\int_{supp\ \xi_{tt}}\xi^{\frac{-p'}{p}}\left|\xi_{tt}\right|^{p'}+\int_{supp\ \Delta\xi}\xi^{\frac{-p'}{p}}\left|\Delta\xi\right|^{p'}\\&+\int_{supp}\xi^{\frac{-p'}{p}}|D_{t|T}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\xi|^{p'}<\infty.\end{aligned}$$

To estimate the right hand side of (5)on, we write by using the ε-Young inequality

$$\int_{Q_T}\xi_{tt}u\leq\varepsilon\int_{supp\ \xi}|u|^p\,\xi+C_{\varepsilon}\int_{supp\ \xi_{tt}}\xi^{\frac{-p'}{p}}\left|\xi_{tt}\right|^{p'},\tag{6}$$

Similarly,

$$\int_{Q_T}u\Delta\xi\leq\varepsilon\int_{supp\xi}|u|^p\,\xi+C_{\varepsilon}\int_{supp\Delta\xi}\left|\Delta\xi\right|^{p'}\xi^{\frac{-p'}{p}},\tag{7}$$

and

$$\begin{aligned}&\int_{Q_T}uD_{-}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\xi\leq\varepsilon\int_{supp\xi}|u|^p\,\xi\\&+C_{\varepsilon}\int_{supp}\xi^{\frac{-p'}{p}}D_{t|T}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\xi^{p'}.\end{aligned}\tag{8}$$

Where $p'=\dfrac{p}{p-1}$. Summing up estimates (6), (7) and (8), with ε small enough, we infer that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(-\Delta\right)^{\frac{\beta}{2}}u_0D_{-}^{\alpha}\xi\left(0\right)+\int_{Q_T}u_1\xi+\int_{Q_T}|u|^p\,\xi\\&\leq C_{\varepsilon}(\int_{supp\xi}\xi^{\frac{-p'}{p}}(\left|\xi_{tt}\right|^{p'}+\left|\Delta\xi\right|^{q}+|D_{t|T}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\xi|^{p'}))\end{aligned}\tag{9}$$

for some positive constant $C_{\varepsilon}=1/p'\left(p\varepsilon\right)^{\frac{p'}{p}}$. At this stage, we introduce the scaled variables

$$t=\tau R^{\frac{2}{\alpha}},\ x=Ry$$

and set $\Omega:=\left\{(\tau,y)\in\mathbb{R}_+\times\mathbb{R}^N;\tau^2+|y|^4\leq 2\right\}$. Therefore, writing

$$\varphi\left(t,x\right)=\varphi\left(\tau R^{\frac{2}{\alpha}},Ry\right):=\chi\left(\tau,y\right)$$

we have

$$\int_{Q_{TR^{\frac{2}{\alpha}}}}\xi^{\frac{-p'}{p}}\left|\xi_{tt}\right|^{p'}=R^{\frac{2}{\alpha}+N-\frac{4}{\alpha}p'}\int_{\Omega}\chi^{\frac{-p'}{p}}\left|\chi_{\tau\tau}\right|^{p'},$$

$$\int_{Q_{TR^{\frac{2}{\alpha}}}}\xi^{\frac{-p'}{p}}\left|\Delta\xi\right|^{p'}\leq R^{\frac{2}{\alpha}+N-2p'}\int_{\Omega}\chi^{\frac{-p'}{p}}\left|\Delta\chi\right|^{p'}$$

and

$$\begin{aligned}&\int_{Q_{TR^{\frac{2}{\alpha}}}}\xi^{\frac{-p'}{p}}D_{t|T}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\xi^{p'}\leq\\&R^{\frac{2}{\alpha}+N-(2+\beta)p'}\int_{\Omega}\chi^{\frac{-p'}{p}}D_{t|T}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\chi^{p'}\end{aligned}$$

Now taking ε small enough, we obtain the estimate

$$\begin{aligned}&\int_{Q_{TR^{\frac{2}{\alpha}}}}\xi\left|u\right|^p\leq C_{\varepsilon}R^{\frac{2}{\alpha}+N-2p'}\times\\&\int_{\Omega}\chi^{\frac{-p'}{p}}\left(\left|\chi_{tt}\right|^{p'}+\left|\Delta\chi\right|^{p'}+|D_{t|T}^{\alpha}\left(-\Delta\right)^{\frac{\beta}{2}}\chi|^{p'}\right)\end{aligned}\tag{10}$$

In the estimate (10),we have to distinguish tow cases:
Either $p<p_c$: In this case, passing to the limit as $R\rightarrow\infty$ in (10) we obtain

$$\lim_{R\rightarrow\infty}\int_{Q_{TR^{\frac{2}{\alpha}}}}\xi\left|u\right|^p=\int_{\mathbb{R}^N\times\mathbb{R}_+}|u|^p=0.$$

Thus $u=0$.
Or $p=p_c$: In this case, we obtain from (10) that

$$\int_{\mathbb{R}^N\times\mathbb{R}_+}|u|^p\le C.$$

So

$$\lim_{R\to\infty}\int_{C_R}|u|^p\,\xi=0, \qquad (11)$$

where $C_R=\left\{(x,t):R^4\le t^{2\alpha}+|x|^4\le 2R^4\right\}$ .we set

$$\Omega_1:=\left\{(\tau,y)\in\mathbb{R}_+\times\mathbb{R}^N;1\le\tau^{2\alpha}+|y|^4\le 2\right\}$$

If instead of using ε-Young inequality, we rather use the Hölder inequality, then than the estimate (9), we find

$$\int|u|^p\,\xi\le$$

$$\left(\int_{C_R}|u|^p\,\xi\right)^{\frac{1}{p}}\left(\int_{\Omega_1}\xi^{\frac{-p'}{p}}|\xi_{tt}|^{p'}\right)^{\frac{1}{p'}}\left(\int_{\Omega_1}\xi^{\frac{-p'}{p}}|\Delta\xi|^{p'}\right)^{\frac{1}{p'}}$$

$$+\left(\int_{\Omega_1}\xi^{\frac{-p'}{p}}|D^\alpha_{t|T}\,(-\Delta)^{\frac{\beta}{2}}\,\xi|^{p'}\right)^{\frac{1}{p'}}$$

Passing to the limit as $R\to\infty$ in (12) and taking into account (11), we obtain

$$\int|u|^p\,\xi=0.$$

Thus $u=0$ a.e. The proof is complete.

Remark 3.1: When $\alpha\to 1$, the critical exponent is $p_c=1+\frac{2}{N}$ (see Todorova-Yordanov [14]).

IV. Necessary conditions for local and global existence

Theorem 4.1: Let u, be a local solution to (WE)-(1) where $T<+\infty$ and $p>1$. Then there exist constants C_1,C_2 and C_3 such that

$$C_1T^{-\alpha}\inf_{|x|>R}\left((-\Delta)^{\frac{\beta}{2}}\,u_0\right)+C_2\inf_{|x|>R}u_1(x)\le C_3T^{-2q}$$

Proof. Let us consider the following test function

$$\xi(x,t)=\Phi\left(\frac{x}{R}\right)\left(1-\frac{t^2}{T^2}\right)^{2q}$$

where $\Phi\in W_{1,\infty}\left(\mathbb{R}^N\right)$ is nonnegative with *supp*$\Phi\subset\left\{x\in\mathbb{R}^N/1<|x|<2\right\}$ (*supp* stands for support), and satisfies $|\Delta\Phi|\le k\Phi$ and $\left|(-\Delta)^{\frac{\beta}{2}}\,\Phi\right|\le k'\Phi$ for some positive constants k,k'. From (9), we have

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{\beta}{2}}\,u_0D_-^{\alpha-1}\xi(0)+\int_{Q_T}u_1\xi\le$$

$$C\left(\int_{Q_T}\xi^{\frac{-q}{p}}|\xi_{tt}|^q+|\Delta\xi|^q+\left|D^\alpha_{t|T|}\,(-\Delta)^{\frac{\beta}{2}}\,\xi\right|^q\right)$$

for some positive constant positive C.

It is clear, from our choice of ξ that the requirements

$$\xi(x,T)=\xi_t(x,T)=\xi_t(x,0)=0$$

are satisfied. Now, we estimate the right hand side in terms of T and R. First, if we set $t=\tau T$, we find

$$\int_{Q_T}\xi^{\frac{-q}{p}}|\xi_{tt}|^q\,dxdt\le\frac{C_1}{T^{2q-1}}\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right)dx, \qquad (12)$$

for some $C_1>0$ and

$$\int_{Q_T}\xi^{\frac{-q}{p}}|\Delta\xi|^q\le C_2k^qR^{-2q}T\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right)dx \qquad (13)$$

For the last term, we compute $D^\alpha_-\xi$

$$\Gamma(2-\alpha)\,D^\alpha_{t|T}\,\xi=\int_t^T\frac{\varphi"(\sigma)}{(\sigma-t)^{\alpha-1}}d\sigma=$$

$$\int_t^T\frac{\left[\left(1-\frac{\sigma^2}{T^2}\right)^{2q}\right]''}{(\sigma-t)^{\alpha-1}}d\sigma$$

$$=\frac{-4q}{T^2}\int_t^T\frac{\left[\sigma\left(1-\frac{\sigma^2}{T^2}\right)^{2q-1}\right]'}{(\sigma-t)^{\alpha-1}}d\sigma$$

$$=\frac{-4q}{T^2}\int_t^T\left[\left(1-\frac{\sigma^2}{T^2}\right)^{2q-1}-2\frac{\sigma^2}{T^2}(2q-1)\left(1-\frac{\sigma^2}{T^2}\right)^{2q-2}\right]$$

$$\times(\sigma-t)^{1-\alpha}\,d\sigma$$

then

$$\Gamma(2-\alpha)\,D^\alpha_{t|T}\,\xi=$$

$$-4qT^{-4q}\int_t^T\left(T^2-\sigma^2\right)^{2q-1}(\sigma-t)^{1-\alpha}\,d\sigma$$

$$+8q(2q-1)\,T^{-4q}\int_t^T\sigma^2\left(T^2-\sigma^2\right)^{2q-2}(\sigma-t)^{1-\alpha}\,d\sigma$$

$$\equiv I+J$$

Using the Euler's change of variables

$$y=\frac{\sigma-t}{T-t}\Rightarrow\sigma-t=(T-t)\,y$$

we see that

$$y=\frac{\sigma-t}{T-t},\quad 1-y=\frac{T-\sigma}{T-t}$$

and

$$1-y^2=\frac{T^2-\sigma^2}{(T-t)^2}-2t\frac{1-y}{T-t}.$$

and

$$T^2-\sigma^2=\left(1-y^2\right)(T-t)^2+2t\,(1-y)\,(T-t)$$

Therefore

$$I=-4qT^{-4q}\int_t^T\left(T^2-\sigma^2\right)^{2q-1}(\sigma-t)^{1-\alpha}\,d\sigma$$

$$=-4qT^{-4q}\,(T-t)^{1-\alpha+2q}\times$$

$$\int_0^1(1-y)^{2q-1}\left((T-t)\,(1+y)+2t\right)^{2q-1}y^{1-\alpha}dy$$

since we have

$$(T-t)\,(1+y)+2t=(T+t)+y\,(T-t)$$

and as

$$y\,(T-t)<(T-t)\le(T+t)\,,\quad\text{for}\quad y<1$$

then one can apply the Binomial formula for non integer power to

$$I=-4qT^{-4q}\,(T-t)^{1-\alpha+2q}\times$$

$$\int_0^1(1-y)^{2q-1}\left((T+t)+y\,(T-t)\right)^{2q-1}y^{1-\alpha}dy$$

or

$$I=-4qT^{-4q}\sum_{k=0}^{\infty}C_k^{2q-1}\left(T-t\right)^{1-\alpha+2q+k}\left(T+t\right)^{2q-k-1}\times\int_0^1\left(1-y\right)^{2q-1}y^{1-\alpha+k}dy$$

where

$$C_k^{2q-1}=\frac{(2q-1)!}{k!\,(2q-1-k)!}$$

Using the formula

$$\int_0^1\left(1-\tau\right)^{u-1}\tau^{v-1}d\tau=\frac{\Gamma\left(u\right)\Gamma\left(v\right)}{\Gamma\left(u+v\right)},\quad u,v>0$$

we obtain

$$I=-4qT^{-4q}\times\sum_{k=0}^{\infty}C_k^{2q-1}\left(T-t\right)^{1-\alpha+2q+k}\left(T+t\right)^{2q-k-1}\tfrac{\Gamma(2q)\Gamma(2-\alpha+k)}{\Gamma(2q+2-\alpha+k)}.$$

the same thing for J

$$J=8q(2q-1)T^{-4q}\int_t^T\sigma^2(T^2-\sigma^2)^{2q-2}(\sigma-t)^{1-\alpha}d\sigma=8q(2q-1)T^{-4q}\times$$

$$\int_0^1(t+(T-t)y)^2left(1-y)^{2q-2}((T+t)+y(T-t))^{2q-2}(T-t)^{-\alpha+2q}y^{1-\alpha}dy$$

then

$$\begin{aligned}J&=8q\left(2q-1\right)T^{-4q}\sum_{k=0}^{\infty}C_k^{2q-2}\int_0^1\left(t+\left(T-t\right)y\right)^2\times\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q}\left(1-y\right)^{2q-2}y^{k+1-\alpha}dy\\&=8q\left(2q-1\right)T^{-4q}\times\sum_{k=0}^{+\infty}C_k^{2q-2}\Big[t^2\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q}\times\int_0^1\left(1-y\right)^{2q-2}y^{k+1-\alpha}dy\\&+2t\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q+1}\times\int_0^1\left(1-y\right)^{2q-2}y^{k+2-\alpha}dy\\&+\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q+2}\times\int_0^1\left(1-y\right)^{2q-2}y^{k+3-\alpha}dy\Big]\end{aligned}$$

Using the formula in the above

$$\begin{aligned}J&=8q\left(2q-1\right)T^{-4q}\times\sum_{k=0}^{\infty}C_k^{2q-2}\Big[t^2\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q}\tfrac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}\\&+2t\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q+1}\tfrac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\\&+\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q+2}\tfrac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\Big]\end{aligned}$$

Hence

$$\begin{aligned}D_{t|T}^{\alpha}(1-\tfrac{t^2}{T^2})^{2q}&=\tfrac{-4qT^{-4q}}{\Gamma(2-\alpha)}\times\Big[\sum_{k=0}^{\infty}C_k^{2q-1}(T-t)^{1-\alpha+2q+k}\times(T+t)^{2q-k-1}\tfrac{\Gamma(l)\Gamma(2-\alpha+r)}{\Gamma(l+2-\alpha+r)}\Big]+\tfrac{8q(2q-1)}{\Gamma(2-\alpha)}T^{-4q}\times\\&\sum_{k=0}^{\infty}C_k^{2q-2}\big[t^2(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q}\tfrac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}\\&.+2t(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+1}\tfrac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\\&+(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+2}\tfrac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\Big]\end{aligned}$$

if we set $t=\tau T$ we find

$$\begin{aligned}D_{t|T}^{\alpha}(1-\tfrac{t^2}{T^2})^{2q}&=\tfrac{-4qT^{-\alpha}}{\Gamma(2-\alpha)}\times\Big[\sum_{k=0}^{\infty}C_k^{2q-1}(1-\tau)^{1-\alpha+2q+k}\times(1+\tau)^{2q-k-1}\tfrac{\Gamma(l)\Gamma(2-\alpha+r)}{\Gamma(l+2-\alpha+r)}\Big]\\&+\tfrac{8q(2q-1)}{\Gamma(2-\alpha)}\times\sum_{k=0}^{\infty}C_k^{2q-2}\big[\tau^2(1+\tau)^{2q-2-k}(1-\tau)^{k-\alpha+2q}\tfrac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}.\\&+2\tau(1+\tau)^{2q-2-k}(1-\tau)^{k-\alpha+2q+1}\tfrac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\\&+(1+\tau)^{2q-2-k}(1-\tau)^{k-\alpha+2q+2}\tfrac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\Big]\end{aligned}$$

Thus

$$D_{t|T}^{\alpha}\left(1-\frac{t^2}{T^2}\right)^{2q}\leq\frac{C_{q,\alpha}}{\Gamma\left(2-\alpha\right)}T^{-\alpha}\tag{14}$$

where $C_{q,\alpha}$ is constant depending of q and α.

Substituting expression (15) into the following integral

$$\int_Q\xi^{1-\frac{p}{p-1}}\left|D_-^{\alpha}\xi\right|^{\frac{p}{p-1}}=\int_{Q_T}\Phi\left(\frac{x}{R}\right)\left(1-\frac{t^2}{T^2}\right)^{2q\left(1-\frac{p}{p-1}\right)}\left|D_-^{\alpha}\xi\right|^{\frac{p}{p-1}}$$

We have the estimate

$$\int_Q\left(h\xi\right)^{1-q}\left|\left(-\Delta\right)^{\frac{\beta}{2}}D_-^{\alpha}\xi\right|^q\leq\frac{T^{1-\alpha q}}{\Gamma\left(2-\alpha\right)\times}\int_0^1\left(1-\tau\right)^{2q(1-q)}\tau^{2q(1-q)}d\tau\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right)$$

Then we have

$$\int_Q\left(h\xi\right)^{1-q}\left|\left(-\Delta\right)^{\frac{\beta}{2}}D_-^{\alpha}\xi\right|^q\leq\frac{C_{\alpha,q}\Gamma\left(2q\left(1-q\right)+1\right)^2}{\Gamma\left(2-\alpha\right)\Gamma\left(4q\left(1-q\right)+2\right)}T^{1-\alpha q}\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right).\tag{15}$$

Now we compute $D_{t|T}^{\alpha-1}\left(1-\frac{t^2}{T^2}\right)^{2q}$

$$\Gamma\left(2-\alpha\right)D_{t|T}^{\alpha-1}\left(1-\tfrac{t^2}{T^2}\right)^{2q}=-\int_t^T\left(\sigma-t\right)^{1-\alpha}\left(\left(\frac{T^2-\sigma^2}{T^2}\right)^{2q}\right)'d\sigma$$

Using the above Euler's change of variable to compute

$$I:=\int_t^T\sigma\left(T^2-\sigma^2\right)^{2q-1}\left(\sigma-t\right)^{1-\alpha}d\sigma$$

or

$$I = (T-t)^{2q-\alpha+1} \times \int_t^T ((T-t)\,y+t)\,(1-y)^{2q-1}$$
$$((T+t)+y\,(T-t))^{2q-1}\,y^{1-\alpha}dy$$

By using the generalized binomial formula we may write

$$I = (T-t)^{2q-\alpha+1} \int_0^1 ((T-t)\,y+t)\,(1-y)^{2q-1} \times$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-1-k}\,(T-t)^k\,y^{k+1-\alpha}dy$$

Using the formula

$$\int_0^1 (1-\tau)^{u-1}\,\tau^{v-1}d\tau = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}, \quad u,v>0$$

we obtain

$$I = (T-t)^{2q-\alpha+2} \sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-1-k}\,(T-t)^k \times$$
$$\int_0^1 y^{k+2-\alpha}\,(1-y)^{2q-1}\,dy$$
$$+\,(T-t)^{2q-\alpha+1}\,t \sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-1-k}\,(T-t)^k \times$$
$$\int_0^1 y^{k+1-\alpha}\,(1-y)^{2q-1}\,dy,$$

hence

$$D_{t|T}^{\alpha-1}\left(1-\tfrac{t^2}{T^2}\right)^{2q} = \tfrac{4qT^{-4q}}{\Gamma(2-\alpha)}((T-t)^{2q-\alpha+2} \times$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-1-k}\,(T-t)^k\,\tfrac{\Gamma(k+3-\alpha)\Gamma(2q)}{\Gamma(k+3-\alpha+2q)}+$$
$$(T-t)^{2q-\alpha+1}\,t\times \tag{16}$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-1-k}\,(T-t)^k\,\tfrac{\Gamma(k+2-\alpha)\Gamma(2q)}{\Gamma(k+2-\alpha+2q)}).$$

In particular we have

$$D_{t|T}^{\alpha-1}\xi\,(0) = \tfrac{4qT^{-\alpha+1}}{\Gamma(2-\alpha)} \sum_{k=0}^{\infty} \left(\ C_{2q-1}^k\ \right) \tfrac{\Gamma(k+3-\alpha)\Gamma(2q)}{\Gamma(k+3-\alpha+2q)}. \tag{17}$$

Substituting the expression of $D_{t|T}^{\alpha-1}\xi$, in the following term,

$$\int_{Q_T} u_1 D_-^{\alpha-1}\xi =$$
$$\tfrac{1}{\Gamma(2-\alpha)} \int_{\mathbb{R}^N} u_1(x)\,\Phi\left(\frac{x}{R}\right) \int_0^T [(T-t)^{2q-\alpha+2l} \times$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-k}\,(T-t) \int_0^1 y^{k+2-\alpha}\,(1-y)^{2q-1}\,dy$$
$$+\,(T-t)^{2q-\alpha+1}\,t\times$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(T+t)^{2q-k}\,(T-t)^k \times$$
$$\int_0^1 y^{k+1-\alpha}\,(1-y)^{2q-1}\,dy]dt$$

it easy to see that

$$\int_{Q_T} u_1 D_-^{\alpha-1}\xi = \frac{T^{-\alpha+2}}{\Gamma(2-\alpha)} \times$$
$$\int_{\mathbb{R}^N} u_1(x)\,\Phi\left(\frac{x}{R}\right) \int_0^1 [(1-\tau)^{2q-\alpha+2} \times$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(1+\tau)^{2q-k}\,(1-\tau)^k$$
$$\int_0^1 y^{k+2-\alpha}\,(1-y)^{2q-1}\,dy+$$
$$(1-\tau)^{2q-\alpha+1}\,\tau\times$$
$$\sum_{k=0}^{\infty} C_{2q-1}^k\,(1+\tau)^{2q-k}\,(1-\tau)^k \times$$
$$\int_0^1 y^{k+1-\alpha}\,(1-y)^{2q-1}\,dy]d\tau$$

therefore

$$\int_{Q_T} u_1 D_-^{\alpha-1}\xi = \frac{C_{\alpha,q}T^{-\alpha+2}}{\Gamma(2-\alpha)} \int_{\mathbb{R}^N} u_1(x)\,\Phi\left(\frac{x}{R}\right)dx. \tag{18}$$

We have,

$$\int_{Q_T} \xi^{\frac{-q}{p}}\,|D_{t|T}^{\alpha}(-\Delta)^{\frac{\beta}{2}}\xi|q \leq$$
$$k'^q R^{-\beta q} \int_Q \Phi\left(\frac{x}{R}\right) \left| D_-^{\alpha}\left(1-\frac{t^2}{T^2}\right)^{2q} \right|^q$$

Using (15), we find

$$\int_{Q_T} \xi^{\frac{-q}{p}} \left| D_{t|T}^{\alpha}\,(-\Delta)^{\frac{\beta}{2}}\,\xi \right|^q \leq$$
$$C_3 R^{-\beta q} T^{-\alpha q+1} \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right). \tag{19}$$

Gathering the estimates (19), and (20), we obtain

$$C_1 T^{-\alpha+1} \int_{\mathbb{R}^n} \Phi\,(-\Delta)^{\frac{\beta}{2}}\,u_0 + C_2 T \int_{\mathbb{R}^N} \Phi u_1 \leq$$
$$\left(\tfrac{C_1}{T^{2q-1}} + C_2 k^q R^{-2q} T + C_3 R^{-\beta q} T^{-\alpha q+1}\right) \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right)dx \tag{20}$$

On the other hand we have

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{\beta}{2}}\,u_0 \Phi\left(\frac{x}{R}\right) \geq$$
$$\inf_{|x|>R}\left((-\Delta)^{\frac{\beta}{2}}\,u_0\right) \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) \int_{\mathbb{R}^N} u_1(x)\,\Phi\left(\frac{x}{R}\right) \geq$$
$$\inf_{|x|>R} u_1(x) \int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right) \tag{21}$$

Taking into account the estimate (22), the inequality (17) dividing by the term $\int_{\mathbb{R}^N} \Phi\left(\frac{x}{R}\right)$, imply that

$$C_1 T^{-\alpha} \inf_{|x|>R}\left((-\Delta)^{\frac{\beta}{2}}\,u_0\right) + C_2 \inf_{|x|>R} u_1(x) \leq$$
$$(C_3 T^{-2q} + C_4 k^q R^{-2q} + C_5 R^{-\beta q} T^{-\alpha q}). \tag{22}$$

and Passing to the limit as $R \to +\infty$, we get

$$C_1 T^{-\alpha} \inf_{|x|>R}\left((-\Delta)^{\frac{\beta}{2}}\,u_0\right) + C_2 \inf_{|x|>R} u_1(x) \leq C_3 T^{-2q} \tag{23}$$

Corollary *Assume that problem has a nontrivial global weak solution. Then one at least of the following is satisfied* $\lim_{|x|\to+\infty} \inf\left((-\Delta)^{\frac{\beta}{2}}\,u_0\right) = 0$, $\lim_{|x|\to+\infty} \inf u_1(x) = 0$.

Corollary *If one of the following limits is infinite*

$\liminf_{|x|\to+\infty}\left((-\Delta)^{\frac{\beta}{2}}u_0\right) > 0, \liminf_{|x|\to+\infty} u_1(x) > 0$,then problem (1), cannot have any local weak solution.

If $A = \lim_{|x|\to+\infty}\inf(-\Delta)^{\frac{\beta}{2}}u_0 > 0$ and $B = \lim_{|x|\to+\infty}\inf u_1(x) > 0$. Then $T \le \min\left\{\frac{C_6}{A^{\frac{1}{2q-\alpha}}}, \frac{C_7}{B^{\frac{1}{2q}}}\right\}$.

Theorem 4.2: Suppose the problem (1) has a nontrivial global weak solution. Then, there are two positive constants k_1 and k_2 such that

$$\lim_{|x|\to+\infty}\inf\left((-\Delta)^{\frac{\beta}{2}}u_0\,|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\right) \le K_1,$$

and $\lim_{|x|\to+\infty}\inf\left(|x|^{\min\{2q,(\alpha+\beta)\}}u_1(x)\right) \le K_2$,

Proof. In the relation

$$C_1T^{-\alpha}\int_{\mathbb{R}^N}\Phi(-\Delta)^{\frac{\beta}{2}}u_0 + C_2\int_{\mathbb{R}^N}u_1(x)\,\Phi\left(\tfrac{x}{R}\right) \le$$

$$\left(C_3T^{-2q} + C_4k^qR^{-2q} + C_5R^{-\beta q}T^{-\alpha q}\right)\int_{\mathbb{R}^N}\Phi\left(\tfrac{x}{R}\right).$$

as $\beta < 2$ and $\alpha - 2q < 0$, we have

$$C_1\int_{\mathbb{R}^N}\Phi(-\Delta)^{\frac{\beta}{2}}u_0 \le$$

$$\left(C_3T^{\alpha-2q} + C_4k^qR^{-2q}T^{\alpha} + C_5R^{-\beta q}T^{\alpha-\alpha q}\right)\int_{\mathbb{R}^N}\Phi\left(\tfrac{x}{R}\right).$$

Taking in the right hand side $T = R$, we obtain

$$C_1\int_{\mathbb{R}^N}\Phi(-\Delta)^{\frac{\beta}{2}}u_0 \le$$

$$\left(C_3R^{\alpha-2q} + C_4k^qR^{\alpha-2q} + C_5R^{-\beta q+\alpha(1-q)}\right)\int_{\mathbb{R}^N}\Phi\left(\tfrac{x}{R}\right).$$

Then

$$\int_{\mathbb{R}^N}\Phi(-\Delta)^{\frac{\beta}{2}}u_0 \le C_8R^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right). \tag{24}$$

Now, using assumptions on Φ (namely, $R < |x| < 2R$)

$$\int_{\mathbb{R}^N}\Phi(-\Delta)^{\frac{\beta}{2}}u_0 \le C_8\int_{\mathbb{R}^N}|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\Phi\left(\frac{x}{R}\right)$$

we see that

$$\inf_{|x|>R}\left((-\Delta)^{\frac{\beta}{2}}u_0\,|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\right)\times\int_{\mathbb{R}^N}|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\Phi \le C_8\int_{\mathbb{R}^N}|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\Phi\left(\frac{x}{R}\right). \tag{25}$$

To conclude it suffices to take the sup with respect to t of (26) and divide by $\int_{\mathbb{R}^N}|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\Phi\left(\frac{x}{R}\right)$.we get

$$\lim_{|x|\to+\infty}\inf\left((-\Delta)^{\frac{\beta}{2}}u_0\,|x|^{-\min\{2q-\alpha,\beta q-\alpha(1-q)\}}\right) \le C_8$$

With similar argument we have,

$$\int_{\mathbb{R}^N}\Phi u_1 \le \left(\tfrac{C_1}{T^{2q}} + C_2k^qR^{-2q} + C_3R^{-\beta q}T^{-\alpha q}\right)\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right)dx$$

We obtain,

$$\int_{\mathbb{R}^N}\Phi u_1 \le C_9R^{-\min\{2q,(\alpha+\beta)\}}\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right)dx,$$

hence

$$\int_{\mathbb{R}^N}\Phi u_1 \le C_9\int_{\mathbb{R}^N}|x|^{-\min\{2q,(\alpha+\beta)\}}\Phi\left(\frac{x}{R}\right)dx.$$

Thus

$$\inf_{|x|>R}\left(|x|^{\min\{2q,(\alpha+\beta)q\}}u_1(x)\right)\int_{\mathbb{R}^N}|x|^{-\min\{2q,(\alpha+\beta)q\}}\Phi \le C_9\int_{\mathbb{R}^N}|x|^{-\min\{2q,(\alpha+\beta)q\}}\Phi\left(\frac{x}{R}\right)dx.$$

Finally passing to the *sup* , and deviding both sides of the resulting relation by the expression $\int_{\mathbb{R}^N}|x|^{-\min\{2q,(\alpha+\beta)q\}}\Phi\left(\frac{x}{R}\right)dx$ we obtain

$$\lim_{|x|\to+\infty}\inf\left(|x|^{\min\{2q,(\alpha+\beta)q\}}u_1(x)\right) \le C_{10}$$

V. Conclusion

Our analysis is certainly robust for more general wave equations. It can surely be used, for equations wich interpolates heat equation and the wave equation.

References

[1] Baras, P. and R. Kersner: *Local and global solvability of a class of semil-inear parabolic equations.* J. Diff. Eqs., 68 (2) (1987), 238-252.

[2] Baras, P. and M. Pierre: *Critères d'existence de solutions positives pour des équations semi-linéaires non monotones.* Ann. Inst. H. Poincaré, Anal. Non linéaire, 2 (1985), 185-212.

[3] A. N. Carvalho and J. W. Cholewa: *Attractors for strongly damped wave equations with critical nonlinearities,* Pacific J. of Math., 207 (2) (2002), 287-310.

[4] W. Chen and S. Holm: *Physical interpretation of fractional diffusion-Wave equation via lossy media obeying frequency power law*. Preprint.

[5] W. Chen and S. Holm: *Fractional Laplacian, Levy stable distribution and time-space models for linear and nonlinear frequency dependent lossy media.* Preprint.

[6] J. M. Greenberg, R. MacCamy and V. J. Mizel: *On the existence, uniqueness, and stability of solutions of the equation $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$.* J. Math. Mech., 17 (1967), 707-728.

[7] A. S. Kalashnikov: *On a heat conduction equation for a medium with non uniformly distributed non-linear heat source or absorbers*, Bull. Univ. Moscou Math. Mech.3 (1983), 20-24.

[8] M. Kirane and N.-e. Tatar: *Nonexistence of Solutions to a Hyperbolic Equation with a Time Fractional Damping*, J. Math. Anal. Appl.,

[9] E. Mitidieri and S. I. Pohozaev: *A priori estimates, and blow-up of solutions to nonlinear partial differential equations and inequalities.* Proc. Stekolov Inst. Math., v. 234, 1-383.

[10] S. I. Pohozaev and A. Tesei: *Blow-up of nonnegative solutions to quasilinear parabolic inequalities,* Lincei CI. Sci. Fis. Mat. Natur. Rend. Lincei (9) Math. Appl. vol.11(2), pp.99-109 (2000).

[11] I. Podlubny: Fractional Differential Equations. Mathematics in Science and Ingineering, vol.198. New York/London: Springer; 1999.

[12] S. G. Samko, A. A. Kilbas and O. I. Marichev: Fractional integrals andderivatives, Theory and Applications, Gordon and Beach Sciences Publishers, 1987.

[13] A. Hanyga, M.Seredynska: *Nonlinear Hamiltonian equations with fractional damping.* J. Math. Phys,vol.41 (2000), pp.2135-2156.

[14] Todorova, G. and B, Yordanov: *Critical exponent for nonlinear wave equations with damping.* J. Diff. Eqs,vol.174 (2001), pp.464-489.

[15] Zhang, Q. S.: A *blow-up result for a nonlinear wane equation with damping: the critical case.* C. R. Math. Acad. Sci. Paris,vol.333(2), (2001), pp.109-114.

[16] Jonathan M. Blackledge: *Application of the fractional diffusion equation for predicting market behavior* . IAENG International Journal of Applied Mathematics, vol.40, issue 3.

[17] Z. Odibat, A. El-ajou: *Construction of analytical solution to fractional differential equation using homotopy analysis method* . IAENG International Journal of Applied Mathematics, vol.40, issue 2.