

# Stochastic Analysis and Particle Filtering of the Volatility

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*Abstract*—A method for online estimation of the volatility when observing a stock price is proposed. This is based on modeling the volatility dynamics as a stochastic differential equation that is constructed using a technique from the control theory [1]. Identification of the model parameters using the observations is proposed afterwards [2]. It is based on some stochastic calculus. Volatility estimation is then reformulated as a filtering problem. An alternative filter instead of the optimal one is proposed since the latter is not computationally feasible. It is based on samples (or particles) drawn by discretization of the stochastic volatility model. Besides, the main feature that makes online particle filtering possible is analytic resolution of the Fokker-Planck equation for the current return. To the best of our knowledge, such technique for modeling together with online filtering of the volatility are quiet novel. The method is implemented on real data: the Heng Seng index price; this shows a period of relatively high volatility that corresponds obviously to the Asiatic crisis of October 1997.

*Keywords:* stochastic volatility, stochastic differential equations, Fokker-Planck equation, particle filtering.

## 1 Introduction

Let  $S = (S_t)_{t \in \mathbb{R}_+}$  be an  $\mathbb{R}_+$ -valued semimartingale based on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$  which is assumed to be continuous. The process  $S$  is interpreted to model the price of a stock. A basic problem arising in Mathematical Finance is to estimate the price volatility, i.e. the square of the parameter  $\sigma$  in the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where  $W = (W_t)_{t \in \mathbb{R}_+}$  is a Wiener process. It turns out that the assumption of a constant volatility does not hold in practice. Even to the most casual observer of the market, it should be clear that volatility is a random function of time which we denote  $\sigma_t^2$ . Itô's formula for the return  $y_t = \log(S_t/S_0)$  yields

$$dy_t = \left( \mu - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW_t \quad y_0 = 0 \quad (1)$$

The main objective is to estimate in discrete real-time one particular sample path of the volatility process using one observed sample path of the return. As regards the drift  $\mu$ , it is constant but unknown. Under the so-called risk-neutral measure, the drift is a riskless rate which is well known; actually one finds

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that  $\mu$  does not cancel out, for instance, when calculating conditional expectations in a filtering problem. For this argument no change of measure is required, we work directly in the original measure  $\mathbb{P}$ , and  $\mu$  has to be estimated from the observed sample path of the return as well.

## 2 A model for the stochastic volatility

Let  $(z_t)_{t \in \mathbb{R}_+}$  be an arbitrary  $\mathbb{R}$ -valued process; at the moment, this is not the unknown process  $\sigma_t^2$  of instantaneous volatility. Let us assume prior information about the process  $z_t$ : wide sense stationarity and a parametric model for its covariance function

$$\gamma(\tau) = D \exp(-\alpha|\tau|) \quad \tau \in \mathbb{R}$$

for some constants  $D, \alpha > 0$ . Then the spectral density of  $z_t$  is given by the formula

$$\Gamma(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\tau) \exp(-j\omega\tau) d\tau = \frac{1}{2\pi} \frac{2D\alpha}{\omega^2 + \alpha^2}$$

where  $j = \sqrt{-1}$ . The spectral density  $\Gamma(\omega)$  may be rewritten as

$$\Gamma(\omega) = \frac{1}{2\pi} \left| \frac{H(j\omega)}{F(j\omega)} \right|^2 \quad \omega \in \mathbb{R}$$

where  $H(j\omega) = \sqrt{2D\alpha}$  and  $F(j\omega) = j\omega + \alpha$ . Notice now that

$$\Phi(s) = \frac{H(s)}{F(s)} \quad s \in \mathbb{C}$$

represents the transfer function of some temporally homogeneous linear filter; this filter is furthermore stable as the root of  $F(s)$  is in the left half-plane of the complex variable  $s$ . Recalling that  $1/2\pi$  is the spectral density of a white noise with unit intensity, we come to the conclusion that

$$z_t - \mathbb{E}[z_t]$$

may be considered as the response of the filter whose transfer function is  $\Phi(s)$ , to a zero-mean white noise with unit intensity. The differential equation describing such a filter is

$$\dot{u}(t) + \alpha u(t) = \sqrt{2D\alpha} w(t)$$

where  $w(t)$  and  $u(t)$  are respectively the input and the output of the filter. Setting  $m = \mathbb{E}[z_t]$  and  $z_t - m = u(t)$ , the process  $z_t$  solves the following SDE

$$dz_t = -\alpha(z_t - m) dt + \sqrt{2D\alpha} dW_t$$

Let the  $\mathbb{R}$ -valued process  $(\tilde{z}_t)_{t \in \mathbb{R}_+}$  solve the SDE

$$d\tilde{z}_t = -\alpha\tilde{z}_t dt + \sqrt{2D\alpha} dW_t \quad t > 0$$

with given initial condition  $\tilde{z}_0$ .  $\tilde{z}_t$  is wide sense stationary with zero mean and correlation function

$$r_{\tilde{z}}(\tau) = \mathbb{E} [\tilde{z}(t)\tilde{z}(t - \tau)] = D \exp(-\alpha|\tau|)$$

Consider now the process  $|\tilde{z}_t|$ , written  $x_t$ . Then  $x_t$  is a solution of

$$dx_t = -\alpha x_t dt + \sqrt{2D\alpha} dW_t \quad t > 0 \quad (2)$$

with reflection on the boundary  $\{0\}$  of its state space  $\mathbb{R}_+$ . Starting from any fixed point strictly greater than zero,  $x_t$  reaches this boundary by a predictable stopping time with finite expectation because of the negative sign of the drift. The initial condition  $x_0$  is a random variable with known distribution since  $x_0 = |\tilde{z}_0|$ . It is worthwhile to note the ergodicity of the Markovian process  $x_t$ , with stationary distribution density

$$p(x) = \frac{2}{\sqrt{2\pi D}} \exp\left\{-\frac{x^2}{2D}\right\} \quad x \in \mathbb{R}_+$$

(see pages 55-57 of [3]). This is beyond our expectation in view of the required wide sense stationarity. It follows that for each  $x \in \mathbb{R}_+$ , for any bounded continuous function  $f$  on  $\mathbb{R}_+$

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [f(x_t)] = \int_{\mathbb{R}_+} f(x)p(x) dx$$

and in particular

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [x_t] = \frac{2D}{\sqrt{2\pi D}}$$

Obviously, the second order moment for  $x_t$  coincides with that of  $\tilde{z}_t$ , namely

$$\mathbb{E} [x_t^2] = \mathbb{E} [\tilde{z}_t^2] = D$$

Let us compute  $r_x(\tau)$ , the correlation function of  $x_t$ , for  $\tau \neq 0$

$$\begin{aligned} & \mathbb{E} [x(t)x(t - \tau)] \\ &= \mathbb{E} [|\tilde{z}(t)\tilde{z}(t - \tau)|] \\ &= \mathbb{E} [\tilde{z}(t)\tilde{z}(t - \tau)] \mathbb{P} \{ \text{even passage number by } 0 \} \\ & \quad - \mathbb{E} [\tilde{z}(t)\tilde{z}(t - \tau)] \mathbb{P} \{ \text{odd passage number by } 0 \} \\ &= C \exp(-\alpha|\tau|) \end{aligned}$$

for some constant  $0 < C < D$ . Note the discontinuity of  $r_x(\tau)$  at  $\tau = 0$ . We shall freely call the process  $x_t$  or equivalently the SDE (2) our stochastic volatility model. We just have to denote the Wiener process in (2) differently, say  $\tilde{W}$ , since it is independent of the Wiener process in (1). It should be noted that this type of correlation function:

$$r_x(\tau) = \begin{cases} C \exp(-\alpha|\tau|) & \tau \neq 0 \\ D & \tau = 0 \end{cases} \quad \alpha > 0 \quad 0 < C < D$$

includes short-term or middle-term memory in the volatility.

### 3 Filtering

Now we consider the filtering problem associated to the couple  $(x_t, y_t)$ : we have noisy nonlinear observations of  $x_t$ , the  $\mathbb{R}$ -valued discrete-time process of returns  $(y_n)_{n=1,2,\dots}$  indexed at irregularly spaced instants  $t_1, t_2, \dots$ . The observation times are assumed to be rigorously determined. The observations process is related to the state process  $(x_t)_{t \in \mathbb{R}_+}$  via the conditional distribution

$$\mathbb{P} \{y_n \in \Gamma | y_1, \dots, y_{n-1}, (x_t : 0 \leq t \leq t_n)\} \quad n \geq 1$$

for  $\Gamma$  a Borel-measurable set from  $\mathbb{R}$ . For homogeneity of notation we set  $t_0 = 0$  so that  $y_{n=0} = y_{t=t_0} = 0$ . Now look at the distribution above and recall that  $y_n = y(t_n)$  and that the process  $y_t$  solves the SDE

$$dy_t = \left(\mu - \frac{x_t}{2}\right) dt + \sqrt{x_t} dW_t \quad y_0 = 0 \quad (3)$$

This is (1) where  $\sigma_t$  is denoted  $\sqrt{x_t}$ . For  $t \geq t_{n-1}$

$$y_t = y_{n-1} + \int_{t_{n-1}}^t \left(\mu - \frac{x_s}{2}\right) ds + \int_{t_{n-1}}^t \sqrt{x_s} dW_s \quad (4)$$

and thus

$$\begin{aligned} & \mathbb{P} \{y_n \in \Gamma | y_1, \dots, y_{n-1}, (x_t : 0 \leq t \leq t_n)\} = \\ & \mathbb{P} \{y_n \in \Gamma | y_{n-1}, (x_t : t_{n-1} \leq t \leq t_n)\} \end{aligned}$$

Given a sample path of  $(x_t)_{t_{n-1} \leq t \leq t_n}$  and the observation  $y_{n-1}$ ,  $(y_t)_{t_{n-1} \leq t \leq t_n}$  is a Markov process with state space  $\mathbb{R}$  satisfying (4). This leads to the central concept of this section: the Fokker-Planck equation [4][5][6][7]. The domain of the Fokker-Planck operator:

$$\mathcal{L}_{FP} p(y, t) = \left(\frac{x_t}{2} - \mu\right) \frac{\partial p}{\partial y}(y, t) + \frac{x_t}{2} \frac{\partial^2 p}{\partial y^2}(y, t),$$

is the set of distribution densities on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  under  $\mathbb{P}$ . Given a sample path of  $(x_t)_{t_{n-1} \leq t \leq t_n}$  and the observation  $y_{n-1}$ , the distribution density  $p(y, t)$  of  $y_t$  solves the Fokker-Planck equation

$$\frac{\partial p}{\partial t}(y, t) = \mathcal{L}_{FP} p(y, t) \quad t_{n-1} < t \leq t_n \quad (5)$$

with the initial condition  $p(y, t_{n-1}) = \delta(y - y_{n-1})$ . The formal solution of the above partial differential equation is

$$p(y, t) = \exp \{(t - t_{n-1})\mathcal{L}_{FP}\} p(y, t_{n-1})$$

Since  $\mathcal{L}_{FP}$  is a sum of two non commuting operators, the exponential operator  $\exp \{(t - t_{n-1})\mathcal{L}_{FP}\}$  cannot be expressed as simple products of terms involving each of these. Nevertheless, the solution of the Fokker-Planck equation is obtained using the Trotter product formula [8]. For two arbitrary operators  $A$  and  $B$

$$\exp \{t(A + B)\} = \lim_{n \rightarrow \infty} \left( \exp \left\{ \frac{t}{n} A \right\} \exp \left\{ \frac{t}{n} B \right\} \right)^n$$

Then the solution of (5) is the limit as  $n \rightarrow \infty$  of

$$\left( \exp \left\{ \frac{\rho(t-t_{n-1})}{n} \frac{d}{dy} \right\} \exp \left\{ \frac{\varrho(t-t_{n-1})}{n} \frac{d^2}{dy^2} \right\} \right)^n \delta(y-y_{n-1})$$

where

$$\rho = \frac{x_t}{2} - \mu \quad \varrho = \frac{x_t}{2}$$

For algebraic manipulations we use the integral representation of the delta function and write the solution of (5) as

$$p(y, t) = \lim_{n \rightarrow \infty} \Theta^n \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\{-jzy\} \exp\{jzy_{n-1}\} dz$$

where

$$\Theta = \exp \left\{ \frac{\rho(t-t_{n-1})}{n} \frac{d}{dy} \right\} \exp \left\{ \frac{\varrho(t-t_{n-1})}{n} \frac{d^2}{dy^2} \right\}$$

We claim that

$$\exp \left\{ \frac{\varrho(t-t_{n-1})}{n} \frac{d^2}{dy^2} \right\} \exp\{-jzy\} = \exp \left\{ -\frac{\varrho(t-t_{n-1})}{n} z^2 - jzy \right\}$$

$$\exp \left\{ \frac{\rho(t-t_{n-1})}{n} \frac{d}{dy} \right\} \exp\{-jzy\} = \exp \left\{ -\frac{\rho(t-t_{n-1})}{n} jz - jzy \right\}$$

Therefore

$$\Theta \exp\{-jzy\} = \exp \left\{ -\frac{\varrho(t-t_{n-1})}{n} z^2 - \frac{\rho(t-t_{n-1})}{n} jz - jzy \right\}$$

$$\Theta^n \exp\{-jzy\} = \exp \left\{ -\varrho(t-t_{n-1})z^2 - \rho(t-t_{n-1})jz - jzy \right\}$$

and thus

$$p(y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\{-\varrho(t-t_{n-1})z^2 + jz[-y+y_{n-1}-\rho(t-t_{n-1})]\} dz$$

Let  $Z$  be a Gaussian random variable and  $\psi(u)$ ,  $u \in \mathbb{R}$ , be its characteristic function:

$$\begin{aligned} \psi(u) &= \mathbb{E}[\exp\{juZ\}] \\ &= (2\pi \text{Var}[Z])^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp\{juz\} \exp\left\{-\frac{(z-\mathbb{E}[Z])^2}{2\text{Var}[Z]}\right\} dz \\ &= \exp\left\{ju \mathbb{E}[Z] - \frac{u^2}{2} \text{Var}[Z]\right\} \end{aligned}$$

Then

$$p(y, t) = \frac{1}{2\sqrt{\pi\varrho(t-t_{n-1})}} \psi(-y+y_{n-1}-\rho(t-t_{n-1}))$$

with

$$\mathbb{E}[Z] = 0 \quad \text{Var}[Z] = \frac{1}{2\varrho(t-t_{n-1})}$$

and hence we obtain for  $t_{n-1} \leq t \leq t_n$

$$\begin{aligned} p(y, t) &= \frac{1}{\sqrt{2\pi x_t(t-t_{n-1})}} \\ &\times \exp \left\{ -\frac{[-y+y_{n-1}+(\mu-\frac{x_t}{2})(t-t_{n-1})]^2}{2x_t(t-t_{n-1})} \right\} \end{aligned}$$

### 3.1 Conditional density characterization: the optimal filter

The optimal estimate—in a sense of the mean square—of  $f(x_t)$  given the observations  $y_1, \dots, y_{n-1}$  up to time  $t$  is the conditional expectation

$$\mathbb{E}[f(x_t)|y_1, \dots, y_{n-1}] \quad t_{n-1} \leq t < t_n \quad n \geq 1$$

for all reasonable functions  $f$  on  $\mathbb{R}_+$ . We assume that  $\mathbb{P}\{x_t \leq x|y_1, \dots, y_{n-1}\}$  possesses a density with respect to the Lebesgue measure  $\lambda$  on  $\mathbb{R}_+$ :

$$\Pi_{x_t|y_1, \dots, y_{n-1}}(x) = \frac{d\mathbb{P}\{x_t \leq x|y_1, \dots, y_{n-1}\}}{\lambda(dx)}$$

Now look at the SDE (2), the Fokker-Planck operator for  $x_t$  is

$$\mathcal{L}_{FP}p(x) = \alpha p(x) + \alpha(x-m)p'(x) + D\alpha p''(x)$$

The domain of this operator is the set of distribution densities  $p(x)$  on  $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ , under  $\mathbb{P}$ , satisfying

$$mp(0) - Dp'(0) = 0$$

This is due to the reflection of the process  $x_t$  on the boundary  $\{0\}$  of its state space  $\mathbb{R}_+$ .

It follows that the posterior distribution density  $\Pi_{x_t|y_1, \dots, y_{n-1}}(x)$  for  $t_{n-1} \leq t < t_n$ ,  $n \geq 1$ , solves the Fokker-Planck equation

$$\frac{\partial p}{\partial t}(x, t) = \mathcal{L}_{FP}p(x, t) \quad t_{n-1} < t < t_n$$

i.e.

$$\frac{\partial p}{\partial t}(x, t) = \alpha p(x, t) + \alpha(x-m)\frac{\partial p}{\partial x}(x, t) + D\alpha\frac{\partial^2 p}{\partial x^2}(x, t) \quad (6)$$

with the initial condition

$$p(x, t_{n-1}) = \Pi_{x(t_{n-1})|y_1, \dots, y_{n-1}}(x) \quad (7)$$

and the boundary condition

$$mp(0, t) - D\frac{\partial p}{\partial x}(0, t) = 0 \quad (8)$$

This is a static relation for  $x = 0$ , i.e., it holds for any  $t \in [t_{n-1}, t_n]$ .

At each observation instant  $t_n$ ,  $n \geq 1$ ,  $\Pi_{x(t_n)|y_1, \dots, y_n}(x)$  solves the Bayes rule

$$\Pi_{x(t_n)|y_1, \dots, y_n}(x) \propto \Pi_{x(t_n^-)|y_1, \dots, y_{n-1}}(x) \Pi_{y_n|y_1, \dots, y_{n-1}, x(t_n)=x}(y_n) \quad (9)$$

where

$$\begin{aligned} \Pi_{y_n|y_1, \dots, y_{n-1}, x(t_n)=x}(y_n) &= \frac{1}{\sqrt{2\pi x(t_n-t_{n-1})}} \\ &\times \exp \left\{ -\frac{[-y_n+y_{n-1}+(\mu-\frac{x}{2})(t_n-t_{n-1})]^2}{2x(t_n-t_{n-1})} \right\} \end{aligned}$$

and  $\Pi_{x(t_n^-)|y_1, \dots, y_{n-1}}(x)$  is the solution of (6-8) as  $t \uparrow t_n$ .

### 4 Identification

It follows from (3) that the variation process  $[y]_t$  of  $y_t$  is given by

$$[y]_t = \int_0^t x_s ds$$

thus

$$[y]_{t_n} - [y]_{t_{n-1}} = \int_{t_{n-1}}^{t_n} x_s ds \quad n = 1, 2, \dots$$

On the other hand, so long as every duration between two successive observations is small, the following approximation holds

$$[y]_{t_n} \approx \sum_{i=1}^n (y_i - y_{i-1})^2$$

Thus

$$\int_{t_{n-1}}^{t_n} x_s ds \approx (y_n - y_{n-1})^2$$

i.e., the couple of series below coincide approximatively

$$S = \left\{ \int_{t_{n-1}}^{t_n} x_s ds \right\}_{n=1,2,\dots} \quad S' = \left\{ (y_n - y_{n-1})^2 \right\}_{n=1,2,\dots}$$

and so do their first and second order moments. The following is the computation of the mean and the correlation function for the series  $S$  of aggregations of the instantaneous volatility on the observation intervals. To do this we need to have  $t_n - t_{n-1} = \delta$  for each  $n = 1, 2, \dots$  and as mentioned above  $\delta$  must be small (we set  $\delta = 1$  time unit). Then

$$\mathbb{E} \left[ \int_{t_{n-1}}^{t_n} x_s ds \right] = \frac{2D\delta}{\sqrt{2\pi D}}$$

and for  $k = 1, 2, \dots$

$$\mathbb{E} \left[ \int_{t_{n-1}}^{t_n} x_u du, \int_{t_{n-k-1}}^{t_{n-k}} x_v dv \right] = \int_{t_{n-1}}^{t_n} \int_{t_{n-k-1}}^{t_{n-k}} r_x(u-v) du dv$$

If we replace  $r_x(u-v)$  by its expression, we obtain the following formula for  $k = 1, 2, \dots$

$$\mathbb{E} \left[ \int_{t_{n-1}}^{t_n} x_u du, \int_{t_{n-k-1}}^{t_{n-k}} x_v dv \right] = \frac{C}{\alpha^2} (\exp\{-\alpha\delta(k-1)\} - 2 \exp\{-\alpha\delta k\} + \exp\{-\alpha\delta(k+1)\}) \tag{10}$$

It follows that  $C$  and  $\alpha$  may be obtained by least squares of the difference between the correlation function of  $S'$ , estimated from the observations, and the correlation function given by formula (10).

The following gives an approximation for the drift parameter  $\mu$  in (1). We have

$$y_n - y_{n-1} = \int_{t_{n-1}}^{t_n} \left( \mu - \frac{x_s}{2} \right) ds + \int_{t_{n-1}}^{t_n} \sqrt{x_s} dW_s$$

Then

$$\begin{aligned} \mathbb{E}[y_n - y_{n-1}] &= \mu\delta - \frac{1}{2} \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} x_s ds \right] \\ &= \mu\delta - \frac{D\delta}{\sqrt{2\pi D}} \end{aligned}$$

and thus

$$\mu = \frac{1}{\delta} \mathbb{E}[y_n - y_{n-1}] + \frac{D}{\sqrt{2\pi D}}$$

The Hang Seng index price of the market of Hong Kong is observed during 3191 successive trading days from 1995 to 2007. This is plotted in Figure 1. Figure 2 shows the daily returns

$$y_n - y_{n-1} = \log \left( \frac{S_{t_n}}{S_{t_{n-1}}} \right) \quad n = 1, \dots, 3190$$

The empirical mean of  $S'$  yields an approximation for the second order moment  $D$  of  $1.0977e-007$ . This approximation together with the empirical mean of the daily returns yield an approximation for the drift  $\mu$  of  $5.4008e-004$ . The constant  $C$  and the rate  $\alpha$  that give a good fitting between the correlation function of  $S$  and its approximation are  $3.5926e-007$  and  $0.0857$  respectively. The model for the stochastic volatility of the stock is thus calibrated, and we now go back to filtering.

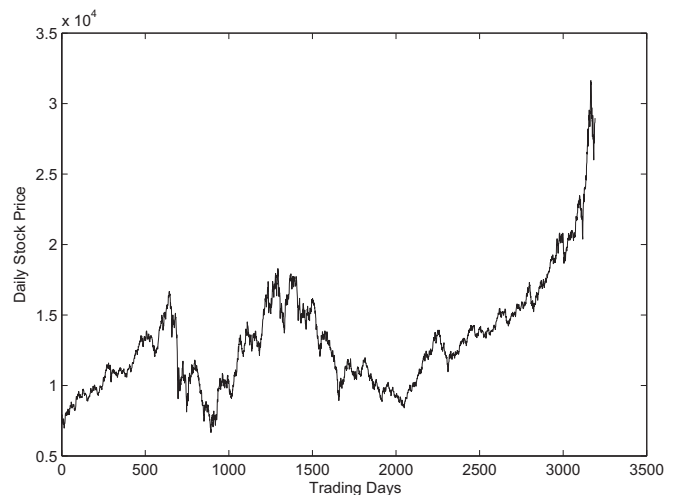


Figure 1: The observed sample path for the daily price of the Hang Seng index.

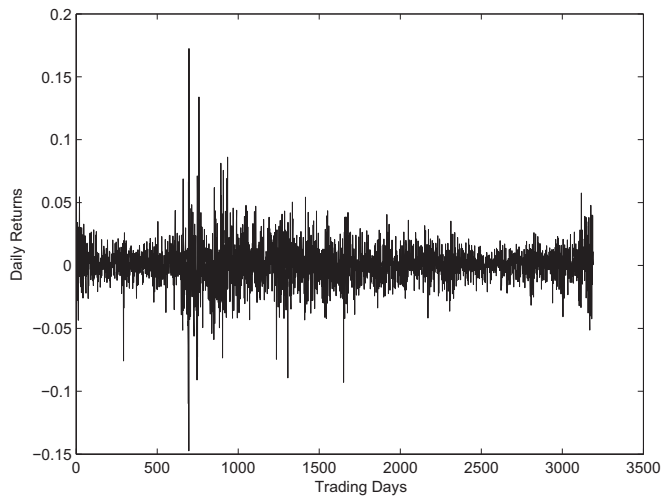


Figure 2: The observed sample path for the daily return.

### 5 A Monte-Carlo particle filter

The true filter (6-9) which is optimal in a mean square sense involves a resolution of the Fokker-Planck equation. Both analytic and numerical solutions for this partial differential equation are computationally intractable. This drives us to an alternative Monte-Carlo filter [9]. We wish to approximate the posterior distribution as a weighted sum of random Dirac measures: for  $\Gamma$  a Borel-measurable set from  $\mathbb{R}_+$

$$\mathbb{P} \{x_t \in \Gamma | y_1, \dots, y_{n-1}\} \approx \sum_{k=1}^K w_k \epsilon_{\xi_k}(\Gamma) \quad t_{n-1} \leq t < t_n \quad n \geq 1$$

where the particles  $\xi_k$  are independent identically distributed random variables with “the same” law as  $x_t$ ; these particles are indeed samples drawn from the Euler discretization of the SDE (2). Here we use the well known Euler scheme since there isn't a significant gain with more sophisticated discretization schemes. Then, for any function  $f$  on  $\mathbb{R}_+$

$$\mathbb{E} [f(x_t) | y_1, \dots, y_{n-1}] \approx \sum_{k=1}^K w_k f(\xi_k) \quad t_{n-1} \leq t < t_n \quad n \geq 1$$

The weights  $\{w_k\}_{k=1, \dots, K}$  are updated only as and when an observation  $y_n$  proceeds, each one according to the likelihood of its corresponding particle, i.e., at each observation time  $t_n$

$$w_k = \frac{\prod_{y_n | y_1, \dots, y_{n-1}, x(t_n) = \xi_k} (y_n)}{\sum_{\ell=1}^K \prod_{y_n | y_1, \dots, y_{n-1}, x(t_n) = \xi_\ell} (y_n)}$$

where  $\{\xi_k\}_{k=1, \dots, K}$  are samples with the same law as  $x(t_n)$ .

Besides sampling, there may be (importance) resampling at each observation time: the set of particles is updated for removing particles with small weights and duplicating those

with important weights. We simulate  $K$  new iid random variables according to the distribution

$$\sum_{k=1}^K w_k \epsilon_{\xi_k}$$

Obviously, the new particles have new weights and thus give a new approximation for the posterior distribution. On the other hand, these new particles are used to initialize the Euler discretization scheme for the next sampling.

The following is the remainder of implementation details of the Monte-Carlo particle filter.

- Number of particles:  $K = 1000$
- Time step of the Euler discretization: 0.01 time unit
- In practice the distribution for the initial volatility  $x_0$  is not available, here we take a uniform distribution on  $[\epsilon, 1]$  ( $\epsilon > 0$  must be small); its density satisfies the imposed condition (8).

The sample path of the square root volatility—in percent—is displayed in Figure 3. This sample path exhibits relatively high volatilities that are clustered together round the 697th trading day; this corresponds to the Asian financial crisis of October 1997.

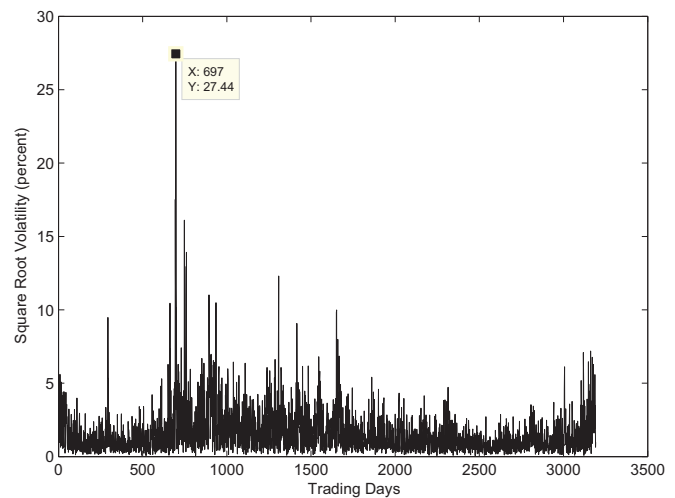


Figure 3: The estimated sample path for the volatility of the Hang Seng index.

## 6 Conclusion

Probabilistic management of uncertainty in dynamical systems can be illustrated with a financial engineering application: volatility estimation. We treat volatility as a stochastic process and construct a filter that is recursive and pathwise in observations. These two aspects are designated with the term online, or real-time, filtering. The filter output is one particular sample path of the volatility process. The main feature that makes online particle filtering possible is analytic resolution of a Fokker-Planck equation. Our method does not require data transformation, such as removing seasonality. The conformity between the implementation result—within a low simulation cost—and some practical issues prove the performance of the method to my satisfaction.

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