Cordial and 3-equitable Labeling for Some Wheel Related Graphs

S K Vaidya , N A Dani , K K Kanani , P L Vihol

***Abstract*—We present here cordial and 3-equitable labeling for the graphs obtained by joining apex vertices of two wheels to a new vertex. We extend these results for k copies of wheels.**

***Index Terms*—Cordial graph, Cordial labeling, 3-equitable graph, 3-equitable labeling**

AMS **Subject classification number(2000):** 05C78.

I. Introduction

We begin with simple, finite and undirected graph $G = (V, E)$. In the present work $W_n = C_n + K_1$ $(n \geq 3)$ denotes the wheel and in W_n vertices correspond to C_n are called rim vertices and vertex which corresponds to K_1 is called an apex vertex. For all other terminology and notations we follow Harary [7]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 Consider two wheels $W_n^{(1)}$ and $W_n^{(2)}$ then $G =< W_n^{(1)} : W_n^{(2)} >$ is the graph obtained by joining apex vertices of wheels to a new vertex x.
Note that G has $2n + 3$ vertices and $4n + 2$ edges.

Definition 1.2 Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \ldots W_n^{(k)}$. Then the $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ is the graph obtained by joining apex vertices of each $W_n^{(p-1)}$ and $W_n^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$.
Note that G has $k(n+2)-1$ vertices and $2k(n+1)-2$ edges.

Definition 1.3 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

According to Hegde [8] most interesting graph labeling problems have following three important characteristics.

1) a set of numbers from which the labels are chosen;
2) a rule that assigns a value to each edge;
3) a condition that these values must satisfy.

The recent survey on graph labeling can be found in Gallian [6]. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde [2] graph labeling serves as a frontier between number theory and structure of graphs.

Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-Ray crystallography, communication network and to determine optimal circuit layouts. A detailed study on variety of applications of graph labeling is carried out in Bloom and Golomb [3].

Definition 1.4 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the $label$ of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e)=|f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$,$e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.5 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [4].

Many researchers have studied cordiality of graphs. e.g.Cahit [4] proved that tree is cordial. In the same paper he proved that K_n is cordial if and only if $n \leq 3$. Ho et al. [9] proved that unicyclic graph is cordial unless it is C_{4k+2} while Andar et al. [1] have discussed cordiality of multiple shells. Vaidya et al. [10], [11], [12] have also discussed the cordiality of various graphs.

Definition 1.6 A vertex labeling of a graph G is called a *3-equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph G is *3-equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labeling was introduced by Cahit [5] and in the same paper he proved that Eulerian graphs with number of edges congruent to $3(mod 6)$ are not 3-equitable. Youssef [17] proved that W_n is 3-equitable for all $n \geq 4$. Several results on 3-equitable labeling for some wheel related graphs in the context of vertex duplication are reported in Vaidya et al. [13].

In the present investigations we prove that graphs $< W_n^{(1)} : W_n^{(2)} >$ and $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ are cordial as well as 3-equitable.

II. Main Results

Theorem-2.1 Graph $< W_n^{(1)} : W_n^{(2)} >$ is cordial.

Proof Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}$ be the rim vertices $W_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}$ be the rim vertices $W_n^{(2)}$. Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$

S K Vaidya is with the Saurashtra University, Rajkot, 360005 INDIA(phone:+919825292539; e-mail: samirkvaidya@yahoo.co.in)

N A Dani is with Government Polytechnic, Junagadh, 362001 INDIA (e-mail: nilesh_a_d@yahoo.co.in)

K K Kanani is with L E College, Morbi, 363642 INDIA (e-mail: kananikkk@yahoo.co.in)

P L Vihol is with Government Polytechnic, Rajkot, 360002 INDIA (e-mail: viholprakash@yahoo.com)

(Advance online publication: 24 May 2011)

respectively and they are adjacent to a new common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. We define binary vertex labeling $f : V(G) \rightarrow \{0,1\}$ as follows.

For any $n \in N - \{1,2\}$ and $i = 1,2,\dots n$ where N is set of natural numbers.
In this case we define labeling as follows

$$f(v_i^{(1)}) = 1;$$
$$f(c_1) = 0;$$
$$f(v_i^{(2)}) = 0;$$
$$f(c_2) = 1;$$
$$f(x) = 1;$$

Thus rim vertices of $W_n^{(1)}$ and $W_n^{(2)}$ are labeled with the sequences $1,1,1,\dots,1$ and $0,0,\dots,0$ respectively. The common vertex x is labeled with 1 and apex vertices with 0 and 1 respectively.

The labeling pattern defined above covers all possible arrangement of vertices. The graph G satisfies the vertex condition $v_f(0) + 1 = v_f(1)$ and edge condition $e_f(0) = e_f(1)$. i.e. G admits cordial labeling.

Illustration 2.2 Consider $G =< W_6^{(1)} : W_6^{(2)} >$. Here $n = 6$. The cordial labeling is as shown in Figure 1.

Theorem 2.3 Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \dots : W_n^{(k)} >$ is cordial.

Proof Let $W_n^{(j)}$ be k copies of wheel W_n, $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ and c_j be the apex vertex of $W_n^{(j)}$ (here $i = 1,2,\dots n$ and $j = 1,2,\dots\ k$).Let $x_1, x_2 \dots\ x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. Consider $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \dots : W_n^{(k)} >$. To define binary vertex labeling $f : V(G) \rightarrow \{0,1\}$ we consider following cases.

Case 1: $n \in N - \{1,2\}$ and even k where $k \in N - \{1,2\}$.
In this case we define labeling function f as
For $i = 1,2,\dots n$ and $j = 1,2,\dots k$

$$f(v_i^{(j)}) = 0; \text{ if } j \text{ even.}$$
$$= 1; \text{ if } j \text{ odd.}$$
$$f(c_j) = 1; \text{ if } j \text{ even.}$$
$$= 0; \text{ if } j \text{ odd.}$$
$$f(x_j) = 1; \text{ if } j \text{ even, } j \neq k.$$
$$= 0; \text{ if } j \text{ odd, } j \neq k.$$

Case 2: $n \in N - \{1,2\}$ and odd k where $k \in N - \{1,2\}$.
In this case we define labeling function f for first $k-1$ wheels as

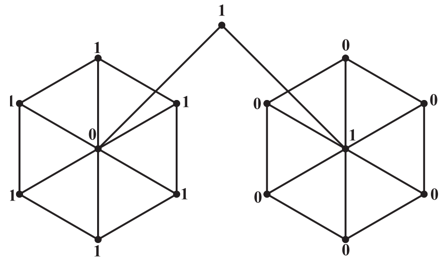

Fig. 1. Cordial labeling of graph G.

For $i = 1,2,\dots n$ and $j = 1,2,\dots k-1$

$$f(v_i^{(j)}) = 0; \text{ if } j \text{ even.}$$
$$= 1; \text{ if } j \text{ odd.}$$
$$f(c_j) = 1; \text{ if } j \text{ even.}$$
$$= 0; \text{ if } j \text{ odd.}$$
$$f(x_j) = 1; \text{ if } j \text{ even.}$$
$$= 0; \text{ if } j \text{ odd.}$$

To define labeling function f for k^{th} copy of wheel we consider following subcases

Subcase 1: If $n \equiv 3(mod4)$.
For $1 \leq i \leq n-1$

$$f(v_i^{(k)}) = 0; \text{ if } i \equiv 0,1(mod4).$$
$$= 1; \text{ if } i \equiv 2,3(mod4).$$
$$f(v_n^{(k)}) = 0;$$
$$f(c_k) = 1;$$

Subcase 2: If $n \equiv 0,2(mod4)$.

$$f(v_i^{(k)}) = 0; \text{ if } i \equiv 0,1(mod4).$$
$$= 1; \text{ if } i \equiv 2,3(mod4).$$
$$f(c_k) = 0; \ n \equiv 0(mod4)$$
$$f(c_k) = 1; \ n \equiv 2(mod4)$$

Subcase 3: If $n \equiv 1(mod4)$.

$$f(v_i^{(k)}) = 0; \text{ if } i \equiv 0,3(mod4).$$
$$= 1; \text{ if } i \equiv 1,2(mod4).$$
$$f(c_k) = 0;$$

The labeling pattern defined above exhaust all the possibilities and in each one the graph G under consideration satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 1. i.e. G admits cordial labeling.

(In Table 1 $n = 4a + b$ and $a \in N \cup \{0\}$)
Let us understand the labeling pattern with some examples given below.

Illustrations 2.4
Example 1: Consider $G =< W_7^{(1)} : W_7^{(2)} : W_7^{(3)} : W_7^{(4)} >$. Here $n = 7$ and $k = 4$ i.e k is even. The cordial labeling is as shown in Figure 2.

Example 2: Consider $G =< W_5^{(1)} : W_5^{(2)} : W_5^{(3)} >$. Here $n = 5$ i.e $n \equiv 1(mod4)$ and $k = 3$ i.e k is odd. The cordial labeling is as shown in Figure 3.

Theorem 2.5 Graph $< W_n^{(1)} : W_n^{(2)} >$ is 3-equitable.

Proof Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots v_n^{(1)}$ be the rim vertices $W_n^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots v_n^{(2)}$ be the rim vertices $W_n^{(2)}$.

TABLE I
VERTEX AND EDGE CONDITIONS FOR f

k	*b*	Vertex Condition	Edge Condition
even	0,1,2,3	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
odd	0	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
	1,3	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$
	2	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$

(Advance online publication: 24 May 2011)

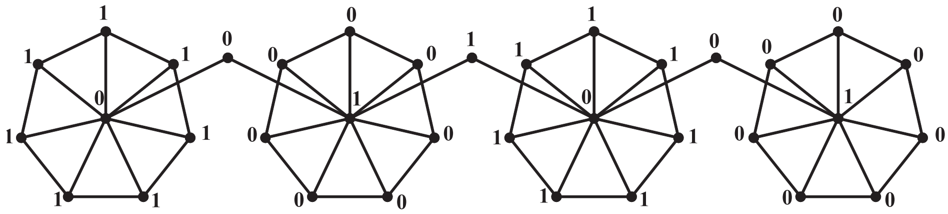

Fig. 2. Cordial labeling of graph G.

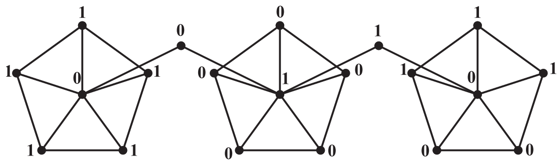

Fig. 3. Cordial labeling of graph G.

Let c_1 and c_2 be the apex vertices of $W_n^{(1)}$ and $W_n^{(2)}$ respectively and they are adjacent to a new common vertex x. Let $G =< W_n^{(1)} : W_n^{(2)} >$. To define vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ we consider the following cases.

Case 1: $n \equiv 0(mod6)$
In this case we define labeling f as:

$$\begin{aligned} f(v_i^{(1)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 2, 3(mod6) \\ &= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n \\ f(c_1) &= 2; \\ f(v_i^{(2)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 2;\ i \equiv 2, 3(mod6) \\ &= 1;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n-3 \\ &= 1;\ i \geq n-2 \\ f(c_2) &= 0; \\ f(x) &= 0; \end{aligned}$$

Case 2: $n \equiv 1(mod6)$
In this case we define labeling f as:

$$\begin{aligned} f(v_i^{(1)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 2, 3(mod6) \\ &= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n \\ f(c_1) &= 2; \\ f(v_i^{(2)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 2, 3(mod6) \\ &= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n \\ f(c_2) &= 2; \\ f(x) &= 1; \end{aligned}$$

Case 3: $n \equiv 2(mod6)$
In this case we define labeling f as:

$$\begin{aligned} f(v_i^{(1)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 0, 5(mod6) \\ &= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n-2 \\ &= 1;\ i \geq n-1 \\ f(c_1) &= 0; \\ f(v_i^{(2)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 0, 5(mod6) \\ &= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n-2 \\ &= 2;\ i \geq n-1 \\ f(c_2) &= 0; \\ f(x) &= 1; \end{aligned}$$

Case 4: $n \equiv 3(mod6)$
Subcase 1: $n \neq 3$
In this case we define labeling f as:

$$\begin{aligned} f(v_i^{(1)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 0, 5(mod6) \\ &= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n \\ f(c_1) &= 0; \\ f(v_i^{(2)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 2, 3(mod6) \\ &= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n-3 \\ &= 1;\ i \geq n-2 \\ f(c_2) &= 0; \\ f(x) &= 2; \end{aligned}$$

Subcase 2: $n = 3$

$$\begin{aligned} f(v_1^{(1)}) &= f(v_1^{(2)}) = f(c_2) = 0; \\ f(v_2^{(1)}) &= f(v_3^{(1)}) = f(c_1) = 1; \\ f(v_2^{(2)}) &= f(v_3^{(2)}) = f(x) = 2; \end{aligned}$$

Case 5: $n \equiv 4(mod6)$
In this case we define labeling f as:

$$\begin{aligned} f(v_i^{(1)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 0, 5(mod6) \\ &= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n-3 \\ &= 1;\ i = n-2, n-1 \\ &= 0;\ i = n \\ f(c_1) &= 2; \\ f(v_i^{(2)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 0, 5(mod6) \\ &= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n \\ f(c_2) &= 2;\ f(x) = 1. \end{aligned}$$

Case 6: $n \equiv 5(mod6)$
In this case we define labeling f as:

$$\begin{aligned} f(v_i^{(1)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 2, 3(mod6) \\ &= 2;\ i \equiv 0, 5(mod6),\ 1 \leq i \leq n-5 \\ &= 1;\ i = n-4, n-3 \\ &= 2;\ i = n-2, n \\ &= 0;\ i = n-1 \\ f(c_1) &= 2; \\ f(v_i^{(2)}) &= 0;\ i \equiv 1, 4(mod6) \\ &= 1;\ i \equiv 0, 5(mod6) \\ &= 2;\ i \equiv 2, 3(mod6),\ 1 \leq i \leq n-5 \\ &= 0;\ i = n-4, n-1 \end{aligned}$$

$= 1;\ i = n-3, n-2$
$= 2;\ i = n$
$f(c_2) = 0;$
$f(x) = 0;$

The labeling pattern defined above covers all the possible arrangement of vertices and in each case the resulting labeling satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in Table 2. i.e. G admits 3-equitable labeling.

(In Table 2 $n = 6a + b$ and $a \in N \cup \{0\}$)

Let us understand the labeling pattern defined in Theorem 2.5 by means of following Illustration 2.6.

Illustration 2.6 Consider a graph $G =< W_5^{(1)} : W_5^{(2)} >$ Here $n = 5$ i.e $n \equiv 5(mod6)$. The corresponding 3-equitable labeling is shown in Figure 4.

Theorem 2.7 Graph $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ is 3-equitable.

Proof Let $W_n^{(j)}$ be k copies of wheel W_n, $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ where $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots k$. Let c_j be the apex vertex of $W_n^{(j)}$. Consider $G =< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \ldots : W_n^{(k)} >$ and vertices $x_1, x_2, \ldots x_{k-1}$ as stated in Theorem 2.3. To define vertex labeling $f : V(G) \to \{0, 1, 2\}$ we consider following cases.

Case 1: For $n \equiv 0(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 1, 2(mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$, $i \leq n-3$.
$f(v_i^{(j)}) = 1;$ if $i \geq n-2$.
$f(c_j) = 0$.
$f(x_j) = 2;$ if $j \equiv 1(mod3)$.
$= 0;$ if $j \equiv 2(mod3)$.
For $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1, 4(mod6)$.

TABLE II
VERTEX AND EDGE CONDITIONS FOR f

b	Vertex Condition	Edge Condition
0	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
1,4	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
2	$v_f(0)+1=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
3	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)$
5	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$

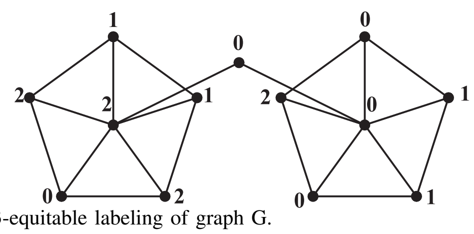

Fig. 4. 3-equitable labeling of graph G.

$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$.
$f(c_j) = 2$.
$f(x_j) = 0,\ j \neq k$.

Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$.
$f(c_1) = 2$.
$f(x_1) = 0$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
$f(v_i^{(1)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$.
$f(c_1) = 0$.
$f(x_1) = 2$.
$f(v_i^{(2)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$, $i \leq n-3$.
$f(v_i^{(2)}) = 1;$ if $i \geq n-2$.
$f(c_2) = 0$.
$f(x_2) = 0$.
For remaining vertices take $j = k-2$ and label them as in subcase 1.
Case 2: For $n \equiv 1(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
$f(v_i^{(j)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$, $i \leq n-1$.
$f(v_n^{(j)}) = 0;$ if $j \equiv 1(mod3)$.
$f(v_n^{(j)}) = 1;$ if $j \equiv 0, 2(mod3)$.
$f(c_j) = 2;$ if $j \equiv 1(mod3)$.
$f(c_j) = 0;$ if $j \equiv 0, 2(mod3)$.
$f(x_j) = 1;$ if $j \equiv 1(mod3)$.
$= 2;$ if $j \equiv 0, 2(mod3)$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$, $i \leq n-1$.
$f(v_n^{(1)}) = 1;$
$f(c_1) = 2$.
$f(x_1) = 0$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0;$ if $i \equiv 1, 4(mod6)$.
$= 1;$ if $i \equiv 0, 5(mod6)$.
$= 2;$ if $i \equiv 2, 3(mod6)$, $i \leq n-1$.
$f(v_n^{(j)}) = 1;$
$f(c_1) = 0$.
$f(c_2) = 2$.
$f(x_1) = 2$.
$f(x_2) = 0$.
For remaining vertices take $j = k-2$ and label them as in subcase 1.

Case 3: For $n \equiv 2(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 1, 2(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-4$.
$f(v_{n-3}^{(j)}) = 2$.
$f(v_i^{(j)}) = 1$; if $i \geq n-2$.
$f(c_j) = 0$; if $j \equiv 1(mod3)$.
$f(c_j) = 2$; if $j \equiv 2(mod3)$.
$f(x_j) = 0$.
For $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$, $i \leq n-2$.
$f(v_i^{(j)}) = 1$; if $i \geq n-1$.
$f(c_j) = 2$.
$f(x_j) = 0$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-2$.
$f(v_{n-1}^{(1)}) = 2$.
$f(v_n^{(1)}) = 0$.
$f(c_1) = 0$.
$f(x_1) = 1$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-4$.
$f(v_{n-3}^{(j)}) = 2$;
$f(v_i^{(j)}) = 1$; if $i \geq n-2$.
$f(c_j) = 0$.
$f(x_1) = 1$.
$f(x_2) = 0$.
For remaining vertices take $j = k-2$ and label them as in subcase 1.
Case 4: For $n \equiv 3(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-3$.
If $j \equiv 1(mod3)$
$f(v_i^{(j)}) = 1$; if $i \geq n-2$.
$f(c_j) = 0$.
$f(x_j) = 1$.
If $j \equiv 2(mod3)$
$f(v_{n-2}^{(j)}) = 0$.
$f(v_{n-1}^{(j)}) = 2$.
$f(v_n^{(j)}) = 1$.
$f(c_j) = 0$.
$f(x_j) = 2$.
If $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0$; $if j = n-1, n-2$.
$f(v_n^{(j)}) = 2$.
$f(c_j) = 2$.
$f(x_j) = 2$, $j \neq k$.

Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$, $i \leq n-3$.
$f(v_i^{(1)}) = 2$; if $i \geq n-2$.
$f(c_1) = 0$.
$f(x_1) = 1$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-3$.
$f(v_i^{(1)}) = 1$; if $i = n-1, n-2$.
$f(v_n^{(1)}) = 0$.
$f(v_i^{(2)}) = 2$; if $i \geq n-2$.
$f(c_j) = 0$.
$f(x_1) = 1$.
$f(x_2) = 2$.
For $n = 3$ label rim vertices of $W_n^{(1)}$ by $0, 1, 0$ and apex vertex by 1.
For remaining vertices take $j = k-2$ and label them as in subcase 1.
Case 5: For $n \equiv 4(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 0, 1, 2(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n-4$.
$f(v_{n-3}^{(j)}) = 0$; if $j \equiv 0, 1(mod3)$.
$f(v_{n-3}^{(j)}) = 2$; if $j \equiv 2(mod3)$.
$f(v_i^{(j)}) = 1$; if $j \equiv 1, 2(mod3)$, $i \geq n-2$.
$f(v_i^{(j)}) = 2$; if $j \equiv 0(mod3)$, $i \geq n-2$.
$f(c_j) = 2$, $j \equiv 1, 2(mod3)$.
$f(c_j) = 0$, $j \equiv 0(mod3)$.
$f(x_j) = 0$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$.
$f(c_1) = 0$.
$f(x_1) = 1$.
For remaining vertices take $j = k-1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$.
$f(v_i^{(2)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$.
$f(c_1) = 2$.

$f(c_2) = 0.$
$f(x_1) = 1.$
$f(x_2) = 2.$
For remaining vertices take $j = k - 2$ and label them as in subcase 1.

Case 6: For $n \equiv 5(mod6)$.
In this case we define labeling function f as follows
Subcase 1: For $k \equiv 0(mod3)$.
For $j \equiv 1, 2(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 2, 3(mod6)$.
$= 2$; if $i \equiv 0, 5(mod6)$, $i \leq n - 2$.
$f(v_{n-1}^{(j)}) = 1.$
$f(v_n^{(j)}) = 2$; if $j \equiv 1(mod3)$.
$f(v_n^{(j)}) = 0$; if $j \equiv 2(mod3)$.
$f(c_j) = 2$; if $j \equiv 1(mod3)$.
$f(c_j) = 0$; if $j \equiv 2(mod3)$.
$f(x_j) = 1$; if $j \equiv 1(mod3)$.
$f(x_j) = 2$; if $j \equiv 2(mod3)$.
For $j \equiv 0(mod3)$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$ $i \leq n - 1$.
$f(v_n^{(j)}) = 2.$
$f(c_j) = 0.$
$f(x_j) = 2$, $j \neq k$.
Subcase 2: For $k \equiv 1(mod3)$.
$f(v_i^{(1)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n - 2$.
$f(v_i^{(1)}) = 1$; if $i \geq n - 1$.
$f(c_1) = 0.$
$f(x_1) = 2.$
For remaining vertices take $j = k - 1$ and label them as in subcase 1.
Subcase 3: For $k \equiv 2(mod3)$.
For $j = 1, 2$
$f(v_i^{(j)}) = 0$; if $i \equiv 1, 4(mod6)$.
$= 1$; if $i \equiv 0, 5(mod6)$.
$= 2$; if $i \equiv 2, 3(mod6)$, $i \leq n - 2$.
$f(v_i^{(j)}) = 1$, $i \geq n - 1$.
$f(c_1) = 0.$
$f(c_2) = 2.$
$f(x_j) = 0.$
For remaining vertices take $j = k - 2$ and label them as in subcase 1.

The labeling pattern defined above covers all possible arrangement of vertices. In each case, the graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ as shown in Table 3. i.e. G admits 3-equitable labeling.

(In Table 3 $n = 6a + b$ and $k = 3c + d$ where $a \in N \cup \{0\}$,$c \in N$)

The labeling pattern defined above is demonstrated by means of following Illustration 2.8.

Illustration 2.8 Consider a graph $G = < W_6^{(1)} : W_6^{(2)} : W_6^{(3)} : W_6^{(4)} >$. Here $n = 6$ and $k = 4$. The corresponding 3-equitable labeling is as shown in Figure 5.

TABLE III
VERTEX AND EDGE CONDITIONS FOR f

b	d	Vertex Condition	Edge Condition
0	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)+1$
1	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)=e_f(1)=e_f(2)$
2	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	2	$v_f(0)+1=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
3	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)+1=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
	2	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
4	0	$v_f(0)+1=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)+1=e_f(2)$
	2	$v_f(0)=v_f(1)+1=v_f(2)$	$e_f(0)=e_f(1)=e_f(2)$
5	0	$v_f(0)=v_f(1)=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	1	$v_f(0)=v_f(1)=v_f(2)$	$e_f(0)+1=e_f(1)=e_f(2)+1$
	2	$v_f(0)=v_f(1)+1=v_f(2)+1$	$e_f(0)+1=e_f(1)=e_f(2)+1$

III. CONCLUDING REMARKS

Cordial and 3-equitable labeling of some star and shell related graphs are reported in Vaidya et al. [14], [15] while the present work corresponds to cordial and 3-equitable labeling of some wheel related graphs. Here we provide cordial and 3-equitable labeling for the larger graphs constructed from the standard graph.

Further scope of research

- Similar investigations can be carried out in the context of different graph labeling techniques and for various standard graphs.
- Cordial labeling in the context of arbitrary super subdivision of graphs is discussed in Vaidya and Kanani [16]. In likeway all the results reported here can be discussed in the context of arbitrary super subdivision of graphs.

REFERENCES

[1] M Andar, S Boxwala and N B Limaye: "A Note on cordial labeling of multiple shells", *Trends Math.*, pp. 77-80, 2002.
[2] L W Beineke and S M Hegde, "Strongly Multiplicative graphs", Discuss.*Math. Graph Theory*,**21**, pp.63-75, 2001. (http://www.discuss.wmie.uz.zgora.pl/gt/index.php)
[3] G S Bloom and S W Golomb, "Application of numbered undirected graphs", *Proceedings of IEEE*, **65(4)**, pp. 562-570, 1977. (http://ieeexplore.ieee.org/xpl/tocresult.jsp?isnumber=31252)
[4] I Cahit, Cordial Graphs: "A weaker version of graceful and harmonious Graphs", *Ars Combinatoria*, **23**, pp. 201-207, 1987.
[5] I Cahit, "On cordial and 3-equitable labelings of graphs", *Util. Math.*, **37**, pp. 189-198, 1990.
[6] J A Gallian, A dynamic survey of graph labeling, *The Electronics Journal of Combinatorics*, **17** $\sharp DS6$, 2010. (http://www.combinatorics.org/Surveys/index.html)
[7] F Harary, *Graph theory*, Addison Wesley, Reading, Massachusetts, 1972.
[8] S M Hegde, "On Multiplicative Labelings of a Graph", *Labeling of Discrete Structures and applications*, Narosa Publishing House, New Delhi, pp. 83-96, 2008.

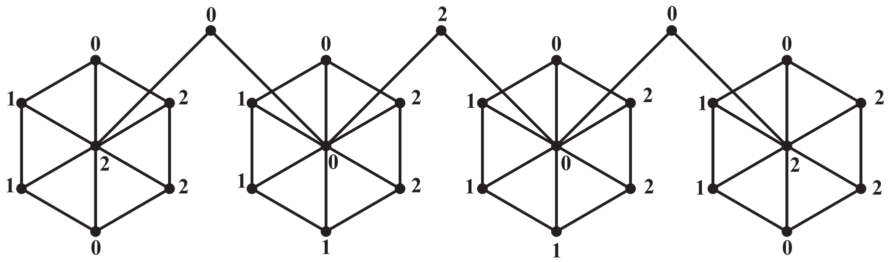

Fig. 5. 3-equitable labeling of graph G.

[9] Y S Ho, S M Lee and S C Shee, "Cordial labeling of unicyclic graphs and generalized Petersen graphs", *Congress. Numer.*,**68**, pp. 109-122, 1989.

[10] S K Vaidya, G V Ghodasara, Sweta Srivastav, V J Kaneria, "Cordial labeling for two cycle related graphs", *The Mathematics Student, J. of Indian Mathematical Society*. **76**, pp. 237-246, 2007.

[11] S K Vaidya, G V Ghodasara, Sweta Srivastav, V J Kaneria, "Some new cordial graphs", *Int. J. of scientific comp.* **2(1)**, pp. 81-92, 2008.

[12] S K Vaidya, Sweta Srivastav, G V Ghodasara, V J Kaneria, "Cordial labeling for cycle with one chord and its related graphs", *Indian J. of Math. and Math.Sci* **4(2)**, pp. 145-156, 2008.

[13] S K Vaidya, N A Dani, K K Kanani, P L Vihol, "Some wheel related 3-Equitable Graphs in the context of vertex duplication", *Advance Appl. in Discrete Math.* **4(1)**, pp. 71-85, 2009. (www.pphmj.com/journals/aadm.html)

[14] S K Vaidya, N A Dani, K K Kanani, P L Vihol, "Cordial and 3-Equitable labeling for some star related graphs", *Int. Math. Forum* **4(31)**, pp. 1543-1553, 2009. (www.m-hikari.com/imf.html)

[15] S K Vaidya, N A Dani, K K Kanani, P L Vihol, "Cordial and 3-Equitable labeling for some shell related graphs", *J. Sci. Res.* **1(3)**, pp. 438-449, 2009. (www.banglajol.info/index.php/JSR)

[16] S K Vaidya, K K Kanani, "Some new results on cordial labeling in the context of abritray supersubdivision of graph", *Applied Math. Sciences* **4(47)**, pp. 2323-2329, 2010. (www.m-hikari.com/ams/index.html)

[17] M. Z. Youssef, "A necessary condition on k-equitable labelings", *Util. Math.*, **64**, pp. 193-195, 2003.