A Singly Diagonally Implicit Runge-Kutta-Nyström Method for Solving Oscillatory Problems

N. Senu, M. Suleiman, F. Ismail, and M. Othman

Abstract—**In this paper a singly diagonally implicit Runge-Kutta-Nyström (RKN) method is developed for second-order ordinary differential equations with periodical solutions. The method has algebraic order four and phase-lag order eight at a cost of four function evaluations per step. This new method is more accurate when compared with current methods of similar type for the numerical integration of second-order differential equations with periodic solutions, using constant step size.**

Keywords—**Runge-Kutta-Nyström methods; Diagonally implicit; Phase-lag; Oscillatory solutions.**

I. INTRODUCTION

THIS paper deals with numerical method for second-order ODEs, in which the derivative does not appear explicitly,

$$y'' = f(x, y), \ y(x_0) = y_0, \ y'(x_0) = y'_0, \qquad (1)$$

for which it is known in advance that their solution is oscillating. Such problems often arise in different areas of engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics and chemistry, and electronics. An s-stage Runge-Kutta-Nyström (RKN) method for the numerical integration of the IVP is given by

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{i=1}^{s} b_i k_i$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} b_i' k_i \qquad (2)$$

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where

$$k_i = f\left(x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{s} a_{ij} k_j \right) \qquad i = 1,..,s.$$

The RKN parameters a_{ij}, b_j, b_j' and c_j are assumed to be real and s is the number of stages of the method. Introduce the s-dimensional vectors c, b and b' and $s \times s$ matrix A, where $c = [c_1, c_2, \cdots, c_s]^T$, $b = [b_1, b_2, \cdots, b_s]^T$, $b' = [b_1', b_2', \cdots, b_s']^T$, $A = [a_{ij}]$ respectively. RKN methods can be divided into two broad classes: explicit ($a_{jk} = 0$, $k \geq j$) and implicit ($a_{jk} = 0$, $k > j$). The latter contains the class of diagonally implicit RKN (DIRKN) methods for which all the entries in the diagonal of A are equal. The RKN method above can be expressed in Butcher notation by the table of coefficients

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & b'^T \end{array}$$

Generally problems of the form (1) which have periodic solutions can be divided into two classes. The first class consists of problems for which the solution period is known a priori. The second class consists of problems for which the solution period is initially unknown. Several numerical methods of various types have been proposed for the integration of both classes of problems. See Stiefel and Bettis [3], van der Houwen and Sommeijer [12], Gautschi [16] and others.

When solving (1) numerically, attention has to be given to the algebraic order of the method used, since this is the main criterion for achieving high accuracy. Therefore, it is desirable to have a lower stage RKN method with maximal order. This will lessen the computational cost. If it is initially known that the solution of (1) is of periodic nature then it is essential to consider phase-lag (or dispersion) and amplification (or dissipation). These are actually two types of truncation errors. The first is the angle between the true and the approximated solution, while the second is the distance from a standard cyclic solution. In this paper we will derive a new diagonally implicit RKN method with four-stage fourth-order with dispersion of order eight.

A number of numerical methods for this class of problems of the explicit and implicit type have been extensively developed. For example, Okunbor and Skeel [1], van der Houwen and Sommeijer [12], Simos, Dimas and Sideridis, [15], and Senu, Suleiman and Ismail [18] have developed explicit RKN methods of algebraic order up to six for solving oscillatory problems. For implicit RKN methods, see for example van der Houwen and Sommeijer [13], Sharp, Fine and Burrage [14], Imoni, Otunta and Ramamohan [17] and Senu, Suleiman, Ismail, and Othman [21] and [23], and Al-Khasawneh, Ismail, Suleiman [22] and [24].

In this paper a dispersion relation is imposed and together with algebraic conditions to be solved explicitly. In the following section the construction of the new four-stage fourth-order diagonally implicit RKN method with dispersion of order eight is described. Its coefficients are given using the Butcher tableau notation. Finally, numerical tests on second order differential equation problems possessing an oscillatory solutions are performed.

II. ANALYSIS OF PHASE-LAG AND STABILITY

In this section we shall discuss the analysis of phase-lag for RKN method. The first analysis of phase-lag was carried out by Bursa and Nigro [10]. Then followed by Gladwell and Thomas [5] for the linear multistep method, and for explicit and implicit Runge-Kutta(-Nystrom) methods by van der Houwen and Sommeijer [12], [13]. The phase analysis can be divided in two parts; inhomogeneous and homogeneous components. Following van der Houwen and Sommeijer [12], inhomogeneous phase error is constant in time, meanwhile the homogeneous phase errors are accumulated as n increases. Thus, from that point of view we will confine our analysis to the phase-lag of homogeneous component only.

The phase-lag analysis of the method (2) is investigated using the homogeneous test equation

$$y'' = (i\lambda)^2 y(t). \tag{3}$$

Alternatively the method (2) can be written as

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i f(t_n + c_i h, Y_i)$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} b'_i f(t_n + c_i h, Y_i) \tag{4}$$

where

$$Y_i = y_n + c_i h y'_i + h^2 \sum_{j=1}^{s} a_{ij} f(t_n + c_i h, Y_j).$$

By applying the general method (2) to the test equation (3) we obtain the following recursive relation as shown by Papageorgiou, Famelis and Tsitouras [4]

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix} = D \begin{bmatrix} y_n \\ hy'_n \end{bmatrix}, \quad z = \lambda h ,$$

$$D(H) = \begin{pmatrix} 1 - Hb^T(I+HA)^{-1}e & 1 - Hb^T(I+HA)^{-1}c \\ -Hb'^T(I+HA)^{-1}e & 1 - Hb'^T(I+HA)^{-1}c \end{pmatrix} \tag{5}$$

where $H = z^2, e = (1\cdots1)^T, c = (c_1 \cdots c_m)^T$. Here $D(H)$ is the stability matix of the RKN method and its characteristic polynomial

$$\xi^2 - \text{tr}(D(z^2))\xi + \det(D(z^2)) = 0, \tag{6}$$

is the stability polynomial of the RKN method. Solving difference system (5), the computed solution is given by

$$y_n = 2 |c| \| \rho |^n \cos(\omega + n\phi). \tag{7}$$

The exact solution of (3) is given by

$$y(t_n) = 2 |\sigma| \cos(\chi + nz). \tag{8}$$

Eq. (7) and (8) led us to the following definition.

Definition 1. *(Phase-lag). Apply the RKN method (2) to (3). Then we define the phase-lag $\varphi(z) = z - \phi$. If $\varphi(z) = O(z^{q+1})$, then the RKN method is said to have phase-lag order q. Additionally, the quantity $\alpha(z) = 1 - |\rho|$ is called amplification error. If $\alpha(z) = O(z^{r+1})$, then the RKN method is said to have dissipation order r.*

Let us denote

$$R(z^2) = \text{trace}(D) \text{ and } S(z^2) = \det(D).$$

From Definition 1, it follows that

$$\varphi(z) = z - \cos^{-1}\left(\frac{R(z^2)}{2\sqrt{S(z^2)}} \right), \quad |\rho| = \sqrt{S(z^2)}.$$

Let us denote $R(z^2)$ and $S(z^2)$ in the following form

$$R(z^2) = \frac{2 + \alpha_1 z^2 + \cdots + \alpha_s z^{2s}}{(1 + \hat{\lambda} z^2)^s}, \tag{9}$$

$$S(z^2) = \frac{1 + \beta_1 z^2 + \cdots + \beta_s z^{2s}}{(1 + \hat{\lambda} z^2)^s}, \tag{10}$$

where $\hat{\lambda} = 2\lambda^2$ is diagonal element in the Butcher tableau. Here the necessary condition for the fourth-order accurate diagonally implicit RKN method (2) to have phase-lag order eight in terms of α_i and β_i is given by

$$\alpha_3 - \beta_3 = 32\lambda^6 - 24\lambda^4 + \frac{2\lambda^2}{3} - \frac{1}{360} \tag{11}$$

$$\frac{1}{2}\alpha_3 - \beta_4 + \alpha_4 = 16\lambda^8 - 10\lambda^4 + \frac{14\lambda^2}{45} - \frac{3}{2240} \qquad (12)$$

Notice that the fourth-order method is already dispersive order four and dissipative order five due to consistency of the method. Furthermore dispersive order is even and dissipative order is odd.

We next discuss the stability properties of method for solving (1) by considering the scalar test problem (3) applied to the method (4), from which the expression given in (5) is obtain. Eliminating y'_n and y'_{n+1} in (5), we obtain a difference equation of the form

$$y_{n+2} - R(H)y_{n+1} + S(H)y_n = 0. \qquad (13)$$

The characteristic equation associated with equation (13) is given as in (6). Chawla and Sharma [11] have discussed the interval of periodicity and absolute stability of Nyström method. Since our concerned here is with the analysis of high order dispersive RKN method, we therefore drop the necessary condition of periodicity interval i.e $S(H) \equiv 1$. Hence the method derived will be with empty interval of periodicity. We now consider the interval of absolute stability of RKN method. We therefore need the characteristic equation (6) to have roots with modulus less than one so that approximate solution will converge to zero as n tends to infinity. For convenience, we note the following definition adopted for method (5).

Definition 2. An interval $(-H_a, 0)$ is called the interval of absolute stability of the method (5) if, for all $H \in (-H_a, 0)$, $|\xi_{1,2}| < 1$.

III. CONSTRUCTION OF THE METHOD

In the following we shall derive a four-stage fourth-order accurate diagonally implicit RKN method with dispersive order eight, by taking into account the dispersion relations in section II. The RKN parameters must satisfy the following algebraic conditions to find fourth-order accuracy as given in Hairer and Wanner [2], Butcher [6] and Dormand [9].

order 1
$$\sum b'_i = 1 \qquad (14)$$

order 2
$$\sum b_i = \frac{1}{2}, \quad \sum b'_i c_i = \frac{1}{2} \qquad (15)$$

order 3
$$\sum b_i c_i = \frac{1}{6}, \quad \frac{1}{2}\sum b'_i c_i^2 = \frac{1}{6} \qquad (16)$$

order 4
$$\frac{1}{2}\sum b_i c_i^2 = \frac{1}{24}, \quad \frac{1}{6}\sum b'_i c_i^3 = \frac{1}{24}, \quad \sum b'_i a_{ij} c_j = \frac{1}{24}. \qquad (17)$$

For most methods the c_i need to satisfy

$$\frac{1}{2}c_i^2 = \sum_{j=1}^s a_{ij} \quad (i = 1, ..., s). \qquad (18)$$

A four-stage method of algebraic order four ($p = 4$) with dispersive order eight ($q = 8$) and dissipative order five ($r = 5$) is now considered. The conditions (14)-(18) and dispersion relations (11)-(12) formed thirteen nonlinear equations with nineteen variables to be solved. Now, from algebraic conditions (14)-(18) and phase-lag relations of order six (11) and letting λ be a free parameter, then we solve it simultaneously. Therefore the following solution of one-parameter family is obtained

$$c_1 = 2\lambda, c_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}, c_3 = \frac{1}{2} + \frac{\sqrt{3}}{6}, c_4 = \frac{1}{2} - \frac{\sqrt{3}}{6}, a_{21} = \frac{1}{6} - \frac{\sqrt{3}}{12} - 2\lambda^2,$$

$$a_{32} = \frac{1}{6} + \frac{\sqrt{3}}{12} - 2\lambda^2, a_{43} = \frac{1}{6} - \frac{\sqrt{3}}{12} - 2\lambda^2, a_{31} = a_{41} = a_{42} = 0,$$

$$a_{11} = a_{22} = a_{33} = a_{44} = 2\lambda^2, b_1 = b_1' = b_2' = 0, b_3' = \frac{1}{4} - \frac{\sqrt{3}}{12}$$

$$b_2 = \frac{3(80\lambda^2 - 1)}{10(\sqrt{3} - 3 + 24\lambda^2\sqrt{3} + 24\lambda - 12\lambda\sqrt{3} - 288\lambda^3 + 72\lambda^2)},$$

$$b_4 = -\frac{1 - 60\lambda^2\sqrt{3} - 15\lambda + 5\lambda\sqrt{3} + 360\lambda^3 + 120\sqrt{3}\lambda^3}{5(\sqrt{3} - 3 + 24\lambda^2\sqrt{3} + 24\lambda - 12\lambda\sqrt{3} - 288\lambda^3 + 72\lambda^2)}.$$

From the above solution, we are going to derive a method with dispersion of order eight. The eight order dispersion relation (12) need to be satisfied and this can be written in terms of RKN parameters which corresponds to the above family of solution yields the following equation

$$(5806080\lambda^7 - 1451520\lambda^6 - 1451520\lambda^6\sqrt{3} + 241920\lambda^5\sqrt{3} - 967680\lambda^5 +$$
$$60480\lambda^4 + 181440\lambda^4\sqrt{3} - 80640\sqrt{3}\lambda^3 + 147168\lambda^3 + 44856\lambda^2 - 29736\lambda^2\sqrt{3} +$$
$$924\lambda\sqrt{3} - 1752\lambda - 585 + 349\sqrt{3})/[120960(12\lambda - \sqrt{3} + 3)] = 0.0,$$

and solving for λ yields

-0.2752157925,   -0.08524516029,   0.04719733276, 0.1682412065, 0.2490198846, 0.6846776634, and -0.1056624327.

Numerical results show that choosing $\lambda = -0.08524516029$ will give us more accurate scheme compared to the others and we mentioned here one fourth-order (p=4) with dispersive order eight (q=8) method. For $\lambda = -0.08524516029$, the following method will be produced. This method will be denoted by DIRKN4(4,8)NEW (see Table I) and its stability interval is approximately (-8.1877,0).

TABLE I THE DIRKN4(4,8)NEW METHOD

$$
\begin{array}{c|cccc}
c_1 & B \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{6}-\frac{\sqrt{3}}{12}-B & B \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & 0 & \frac{1}{6}+\frac{\sqrt{3}}{12}-B & B \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & 0 & 0 & \frac{1}{6}-\frac{\sqrt{3}}{12}-B & B \\
\hline
 & 0 & b_2 & \frac{1}{4}-\frac{\sqrt{3}}{12} & b_4 \\
 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

where c_1=-0.1704903206, b_2=0.2332957499, b_4=0.1610418175, and $B= 2\lambda^2$=0.01453347471.

This method has PLTE

$$\| \tau^{(5)} \|= 1.669237\times 10^{-3} \text{ and } \| \tau'^{(5)} \|_2 = 1.611272\times 10^{-3}.$$

Table II compares the properties of our method with the methods derived by van der Houwen and Sommeijer [20], Sharp, Fine and Burrage [14], Imoni, Otunta and Ramamohan [17] and Al-Khasawneh, Ismail and Suleiman [22].

TABLE II
SUMMARY OF THE CHARACTERISTIC OF THE FOURTH-ORDER DIRKN METHODS

Method	q	d	$\| \tau^{(p+1)} \|_2$	$\| \tau'^{(p+1)} \|_2$
DIRKN4(4,8)NEW	8	4.84×10^{-5}	1.67×10^{-3}	1.61×10^{-3}
DIRKN3(4,4)IMONI	4	-	3.75×10^{-2}	3.22×10^{-2}
DIRKN3(4,4)HS	4	1.43×10^{-4}	6.35×10^{-4}	1.59×10^{-4}
DIRKN3(4,6)SHARP	6	1.02×10^{-2}	1.85×10^{-3}	6.26×10^{-4}
DIRKN4(4,4)RAED	4	-1.80×10^{-2}	3.13×10^{-2}	1.71×10^{-2}

Notations : q – Dispersion order, d – Dissipation constant
$\| \tau^{(p+1)} \|_2$ – Error coefficient for y_n
$\| \tau'^{(p+1)} \|_2$ – Error coefficient for y_n'

IV. PROBLEM TESTED

In this section we use our method to solve homogeneous and inhomogeneous problems whose exact solution consists of a rapidly or/and a slowly oscillating function. For purposes of illustration, we will compare our results with those derived by using four methods; DIRKN three-stage fourth-order derived by van der Houwen and Sommeijer [20] and Imoni, Otunta, Ramamohan [17], three-stage fourth-order dispersive order six derived by Sharp, Fine and Burrage [14] and four-stage fourth-order derived by Al-Khasawneh, Ismail, Suleiman [22].

Problem 1(*Homogenous*)

$$\frac{d^2 y(t)}{dt^2} = -100 y(t), \quad y(0) = 1, \quad y'(0) = -2$$

Exact solution $y(t) = -\frac{1}{5}\sin(10t) + \cos(10t)$

Problem 2 (*Inhomogenous*)

$$\frac{d^2 y(t)}{dt^2} = -y(t) + t, \quad y(0) = 1, \quad y'(0) = 2$$

Exact solution $y(t) = \sin(t) + \cos(t) + t$
Source : Allen and Wing [19]

Problem 3(*Inhomogeneous system*)

$$\frac{d^2 y_1(t)}{dt^2} = -v^2 y_1(x) + v^2 f(t) + f''(t),$$
$$y_1(0) = a + f(0), \quad y_1'(0) = f'(0),$$
$$\frac{d^2 y_2(t)}{dt^2} = -v^2 y_2(t) + v^2 f(t) + f''(t),$$
$$y_2(0) = f(0), \quad y_2'(0) = va + f'(0),$$

Exact solution is
$y_1(t) = a\cos(vt) + f(t), \quad y_2(t) = a\sin(vt) + f(t), \quad f(t)$ is chosen to be $e^{-0.05t}$ and parameters v and a are 20 and 0.1 respectively.
Source : Lambert and Watson [7]

Problem 4 (*An almost Periodic Orbit problem*)

$$\frac{d^2 y_1(t)}{dt^2} + y_1(t) = 0.001\cos(t), \quad y_1(0) = 1, \quad y_1'(0) = 0$$

$$\frac{d^2 y_2(t)}{dt^2} + y_2(t) = 0.001\sin(t), \quad y_2(0) = 0, \quad y_2'(0) = 0.9995$$

Exact solution $y_1(t) = \cos(t) + 0.0005t\sin(t)$,
$y_2(t) = \sin(t) - 0.0005t\cos(t)$
Source : Stiefel and Bettis [3]

The following notations are used in Tables III-VI and Figs. 1-4:

- **DIRKN4(4,8)NEW** : A four-stage fourth-order dispersive order eight method with 'small' dissipation constant derived in this paper.
- **DIRKN3(4,4)IMONI** : A three-stage fourth-order derived by Imoni, Otunta and Ramamohan [17].
- **DIRKN3(4,4)HS** : A three-stage fourth-order dispersive order four derived by van der Houwen and Sommeijer [20].
- **DIRKN3(4,6)SHARP** : A three-stage fourth-order dispersive order six as in Sharp, Fine and Burrage [14].

- ▪ **DIRKN4(4,4)RAED** : A four-stage fourth-order drived by Al-Khasawneh, Ismail, Suleiman [22].

V. NUMERICAL RESULTS

The results for the four problems above are tabulated in Tables III-VI. One measure of the accuracy of a method is to examine the Emax(T), the maximum error which is defined by

$$\text{Emax}(T) = \max \| y(t_n) - y_n \|,$$

where $t_n = t_0 + nh, \quad n = 1, 2, ? , \dfrac{T - t_0}{h}$.

Tables III-VI show the absolute maximum error for DIRKN4(4,8)NEW, DIRKN3(4,4)IMONI, DIRKN3(4,4)HS, DIRKN3(4,6)SHARP and DIRKN4(4,4)RAED methods when solving Problems 1-4 with three different step values. From numerical results in Table III-VI, we observed that the new method is more accurate compared with DIRKN3(4,4)IMONI, DIRKN3(4,4)HS and DIRKN4(4,4)RAED methods which do not relate to the dispersion order of the method. Also the new method is more accurate compared with DIRKN3(4,6)SHARP method because the new method has dispersion order eight which is the highest and also the dissipation constant for our method is small (see Table II). Notice that all the methods are of the same algebraic order.

Figs. 1-4 show the decimal logarithm of the maximum global error for the solution (MAXE) versus the number of function evaluations. From Figs. 1-4, it can be seen that for the same number of function evaluations the new DIRKN4(4,8)NEW method is more accurate when compared to the current same type and the same order of DIRKN methods, DIRKN4(4,4)RAED, DIRKN3(4,4)HS, DIRKN3(4,6)SHARP and DIRKN3(4,4)IMONI.

TABLE III
COMPARING OUR RESULTS WITH THE METHODS IN THE LITERATURE FOR PROBLEM 1

h	Method	T=100	T=1000	T=4000
0.0025	**DIRKN4(4,8)NEW**	1.4194(-9)	1.0464(-7)	7.7266(-7)
	DIRKN3(4,4)IMONI	1.5646(-2)	1.4622(-1)	4.7069(-1)
	DIRKN3(4,4)HS	1.2561(-7)	1.3689(-6)	5.8314(-6)
	DIRKN3(4,6)SHARP	3.0150(-7)	3.0229(-6)	1.2120(-5)
	DIRKN4(4,4)RAED	9.2774(-6)	9.2904(-5)	3.7129(-4)
0.005	**DIRKN4(4,8)NEW**	1.5124(-9)	1.5126(-8)	5.3047(-7)
	DIRKN3(4,4)IMONI	2.0121(-2)	1.8480(-1)	5.6322(-1)
	DIRKN3(4,4)HS	6.6977(-7)	6.6966(-6)	2.7338(-5)
	DIRKN3(4,6)SHARP	2.5569(-6)	2.5624(-5)	1.0255(-4)
	DIRKN4(4,4)RAED	1.4811(-4)	1.4849(-3)	5.9392(-3)
0.01	**DIRKN4(4,8)NEW**	4.5984(-8)	4.1025(-7)	1.875664(-6)
	DIRKN3(4,4)IMONI	5.9680(-2)	4.6223(-1)	9.286052(-1)
	DIRKN3(4,4)HS	3.2305(-5)	3.2361(-4)	1.295597(-3)
	DIRKN3(4,6) SHARP	3.1342(-4)	3.1448(-3)	1.263707(-2)
	DIRKN4(4,4)RAED	2.3699(-3)	2.3786(-2)	9.536865(-2)

TABLE IV
COMPARING OUR RESULTS WITH THE METHODS IN THE LITERATURE FOR PROBLEM 2

h	Method	T=100	T=1000	T=4000
0.065	**DIRKN4(4,8)NEW**	2.4682(-8)	2.7634(-8)	3.8789(-8)
	DIRKN3(4,4)IMONI	5.2936(-3)	5.3213(-2)	2.0104(-1)
	DIRKN3(4,4)HS	6.8021(-7)	6.8361(-6)	2.7394(-5)
	DIRKN3(4,6) SHARP	4.0017(-6)	4.1061(-5)	1.6419(-4)
	DIRKN4(4,4)RAED	5.8594(-5)	5.8706(-4)	2.3509(-3)
0.125	**DIRKN4(4,8)NEW**	3.4752(-7)	5.1872(-7)	1.0986(-6)
	DIRKN3(4,4)IMONI	1.0214(-2)	1.01103(-1)	3.6319(-1)
	DIRKN3(4,4)HS	1.0871(-5)	1.0930(-4)	4.3835(-4)
	DIRKN3(4,6)SHARP	1.3006(-4)	1.3398(-3)	5.3657(-3)
	DIRKN4(4,4)RAED	8.0270(-4)	8.0329(-3)	3.2192(-2)
0.25	**DIRKN4(4,8)NEW**	5.8948(-6)	1.2772(-5)	3.5775(-5)
	DIRKN3(4,4)IMONI	1.9124(-2)	1.8683(-1)	6.2958(-1)
	DIRKN3(4,4)HS	1.7332(-4)	1.7444(-3)	7.0007(-3)
	DIRKN3(4,6) SHARP	4.4802(-3)	4.6441(-2)	1.9520(-1)
	DIRKN4(4,4)RAED	1.2897(-2)	1.2969(-1)	5.3226(-1)

TABLE V
COMPARING OUR RESULTS WITH THE METHODS IN THE LITERATURE FOR PROBLEM 3

h	Method	T=100	T=1000	T=4000
0.0025	**DIRKN4(4,8)NEW**	8.6798(-10)	2.0910(-8)	1.5309(-7)
	DIRKN3(4,4)IMONI	5.9756(-3)	4.6003(-2)	9.1502(-2)
	DIRKN3(4,4)HS	3.9675(-7)	3.9897(-6)	1.6028(-5)
	DIRKN3(4,6)SHARP	1.8995(-6)	1.9004(-5)	7.6048(-5)
	DIRKN4(4,4)RAED	2.9121(-5)	2.9123(-4)	1.1650(-3)
0.005	**DIRKN4(4,8)NEW**	2.1642(-8)	9.4393(-8)	4.1014(-7)
	DIRKN3(4,4)IMONI	1.1371(-2)	7.0105(-2)	9.9201(-2)
	DIRKN3(4,4)HS	6.3468(-6)	6.3496(-5)	2.5414(-4)
	DIRKN3(4,6)SHARP	6.1529(-5)	6.1776(-4)	2.4938(-3)
	DIRKN4(4,4)RAED	4.6623(-4)	4.6689(-3)	1.8754(-2)
0.01	**DIRKN4(4,8)NEW**	5.1541(-7)	3.4561(-6)	1.3384(-5)
	DIRKN3(4,4)IMONI	1.9988(-2)	9.0402(-2)	1.0002(-1)
	DIRKN3(4,4)HS	1.0142(-4)	1.0156(-3)	4.0589(-3)
	DIRKN3(4,6) SHARP	2.0819(-3)	2.2852(-2)	1.2662(-1)
	DIRKN4(4,4)RAED	7.5063(-3)	7.7409(-2)	2.5731(-1)

TABLE VI
COMPARING OUR RESULTS WITH THE METHODS IN THE LITERATURE FOR PROBLEM 4

h	Method	T=100	T=1000	T=4000
0.065	**DIRKN4(4,8)NEW**	2.4953(-8)	2.7438(-8)	4.6060(-8)
	DIRKN3(4,4)IMONI	3.9398(-3)	4.0219(-2)	2.1108(-1)
	DIRKN3(4,4)HS	5.6025(-7)	5.8308(-6)	3.2036(-5)
	DIRKN3(4,6) SHARP	3.4938(-6)	3.6382(-5)	1.9990(-4)
	DIRKN4(4,4)RAED	4.1138(-5)	4.2777(-4)	2.3486(-3)
0.125	**DIRKN4(4,8)NEW**	3.5178(-7)	4.8299(-7)	1.2310(-6)
	DIRKN3(4,4)IMONI	7.3190(-3)	7.3627(-2)	3.7216(-1)
	DIRKN3(4,4)HS	7.6595(-6)	7.9664(-5)	4.3794(-4)
	DIRKN3(4,6)SHARP	9.3794(-5)	9.7492(-4)	5.3673(-3)
	DIRKN4(4,4)RAED	5.6329(-4)	5.8856(-3)	3.2103(-2)
0.25	**DIRKN4(4,8)NEW**	5.8234(-6)	1.0223(-5)	3.6041(-5)
	DIRKN3(4,4)IMONI	1.3995(-2)	1.3655(-1)	6.7067(-1)
	DIRKN3(4,4)HS	1.2220(-4)	1.2725(-3)	7.0099(-3)
	DIRKN3(4,6) SHARP	3.2209(-3)	3.3895(-2)	1.9396(-1)
	DIRKN4(4,4)RAED	9.0479(-3)	9.4159(-2)	5.1301(-1)

Notation : 1.2345(-4) means 1.2345×10^{-4}
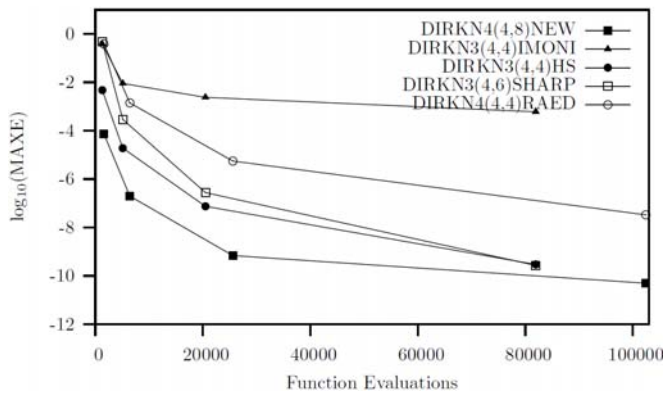
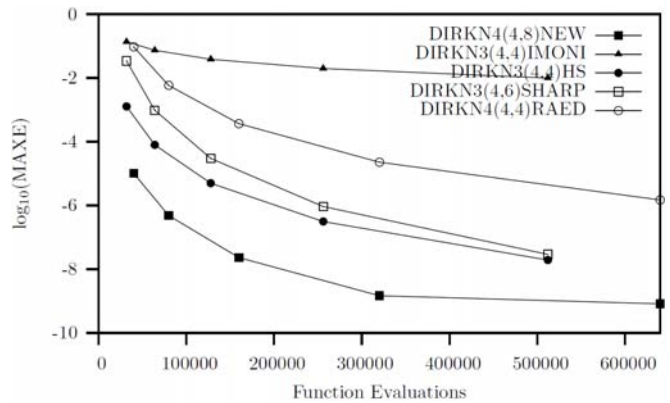Fig. 1 Efficiency curves for Problem 1 for $h = 1/4^i$, $i = 2,...,5$ and $0 \le t \le 10$

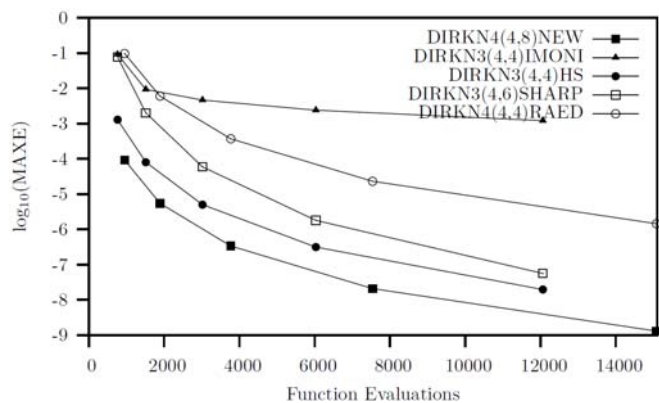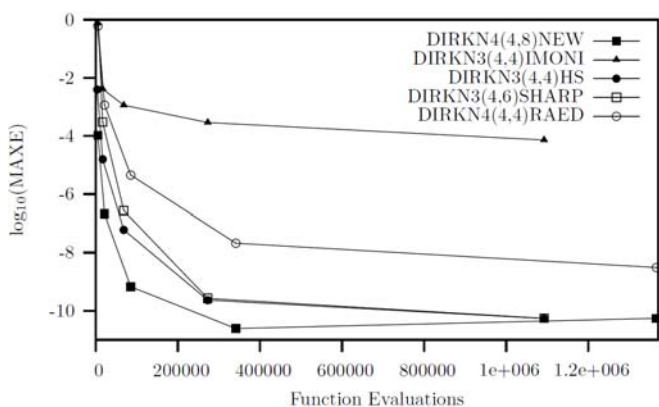Fig. 4 Efficiency curves for Problem 4 for $h = 1/2^i$, $i = 2,...,6$ and $0 \le t \le 1000$

VI. CONCLUSION

In this paper we have derived diagonally implicit four-stage fourth-order and dispersive order eight with 'small' dissipation constant. We have also performed various numerical tests. From the results tabulated in Table III-VI, and from the Figs. 1-4 we conclude that the new method is more accurate for integrating second-order equations possessing an oscillatory solution when compared to the current DIRKN methods derived by van der Houwen and Sommeijer [20], Sharp, Fine and Burrage [14], Imoni, Otunta and Ramamohan [17] and Al-Khasawneh, Ismail and Suleiman [22].

Fig. 2 Efficiency curves for Problem 2 for $h = 1/2^i$, $i = 1,...,5$ and $0 \le t \le 15\pi$

Fig. 3 Efficiency curves for Problem 3 for $h = 0.6/4^i$, $i = 2,...,6$ and $0 \le t \le 20$

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