Stability and Bifurcation Analysis in A SEIR Epidemic Model with Nonlinear Incidence Rates

Changjin Xu and Maoxin liao

***Abstract*—In this paper, a special SEIR epidemic model with nonlinear incidence rates is considered. By analyzing the associated characteristic transcendental equation, it is found that Hopf bifurcation occurs when these delays pass through a sequence of critical value. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Some numerical simulation for justifying the theoretical analysis are also presented. Finally, biological explanations and main conclusions are given.**

***Index Terms*—SEIR epidemic model; stability; Hopf bifurcation; periodic solution**

I. Introduction

In recent years, great attention has been paid to the dynamics properties (including stable, unstable, persistent and oscillatory behavior) of the epidemic models which have significant biological background. Many excellent and interesting results have been obtained [5-13]. It is well known that epidemic models are investigated on the transmission dynamics of infectious diseases in host population. In this paper, we assume that disease spreads in a single host population through direct contact of hosts and a host stays in a latent period before becoming infectious after the initial infection. An infectious host may die from disease or recover with acquired immunity to the disease at the infectious stage. The host population is partitioned into four classes: the susceptible, exposed (latent), infectious, and recovered with sizes denoted by $S, E, I,$ and R, respectively. The host total population $N = S + E + I + R$. Then, we consider the following differential equations:

$$\begin{cases} \dot{S}(t) = \mu - \mu S - \alpha I^p S^q, \\ \dot{E}(t) = \alpha I^p S^q - (\epsilon + \mu)E, \\ \dot{I}(t) = \epsilon E - (\gamma + \mu)I, \\ \dot{R}(t) = \gamma I - \mu R, \end{cases} \tag{1}$$

where $p, q, \alpha, \mu, \epsilon$ and γ are positive parameters. For the meaning of the parameters $p, q, \alpha, \mu, \epsilon$ and γ, one can see Sun et al.[5].

Considering the biological meaning of system (1), we can easily obtain that the feasible region for system (1) is R^4_+.

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C. Xu is with Guizhou Key Laboratory of Economics System Simulation, School of Mathematics and Statistics, Guizhou College of Finance and Economics, Guiyang, China. e-mail: xcj403@126.com.

M. Liao is with School of Mathematics and Physics, Nanhua University, Hengyang, 421001, P.R.China.

Adding the all the equations of (1), we get

$$\dot{S} + \dot{E} + \dot{I} + \dot{R} = -\mu(S + E + I + R - 1),$$

which has the following implication: the three-dimensional simplex

$$\Sigma = \{(S, E, I, R) \in R^4_+ : S + E + I + R = 1\}$$

is positively invariant. On the simplex Σ,

$$R(t) = 1 - S(t) - E(t) - I(t).$$

According the above discussion and under the assumption $p = 1$, Sun et al.[5] obtained the following three-dimensional system

$$\begin{cases} \dot{S}(t) = \mu - \mu S - \alpha I S^q, \\ \dot{E}(t) = \alpha I S^q - (\epsilon + \mu)E, \\ \dot{I}(t) = \epsilon E - (\gamma + \mu)I \end{cases} \tag{2}$$

and investigated the global stability of (2).

In order to reflect the dynamical behaviors of the models depending on the past information, it is more reasonable to incorporate time delays into the system. Based on this idea and under the assumption $p = q = 1$, in this paper, we consider the following delay differential equation:

$$\begin{cases} \dot{S}(t) = \mu - \mu S - \alpha I S, \\ \dot{E}(t) = \alpha I S - (\epsilon + \mu)E(t - \tau), \\ \dot{I}(t) = \epsilon E(t - \tau) - (\gamma + \mu)I. \end{cases} \tag{3}$$

The dynamics of system (3) with delays could be more complicated and interesting. To obtain a deep and clear understanding of dynamics of SEIR epidemic model with nonlinear incidence rates, in this paper, we study the stability, the local Hopf bifurcation for system (3).

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Biological explanations and some main conclusions are drawn in Section 5.

II. Stability of the positive equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations. One can see that if the following condition

$$\text{(H1)} \quad \alpha\epsilon\mu > \mu(\epsilon + \mu)(\gamma + \mu)$$

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holds, then Eq. (3) has an unique positive equilibrium $E_0(S^*, E^*, I^*)$, where

$$S^* = \frac{\mu}{\mu + \alpha I^*}, E^* = \frac{\gamma+\mu}{\epsilon} I^*, I^* = \frac{\alpha\epsilon\mu - \mu(\epsilon+\mu)(\gamma+\mu)}{\alpha(\epsilon+\mu)(\gamma+\mu)}. \tag{4}$$

Let $E = (S^*, E^*, I^*)$ be the arbitrary equilibrium point, and set $x(t) = S(t) - S^*, y(t) = E(t) - E^*, z(t) = I(t) - I^*$, then (4) becomes

$$\begin{cases} \dot{x}(t) = -(\mu + \alpha I^*)x(t) + \alpha S^* z(t) - \alpha x(t)z(t), \\ \dot{y}(t) = \alpha I^* x(t) + \alpha S^* z(t) - (\epsilon+\mu)y(t-\tau) + \alpha x(t)z(t), \\ \dot{z}(t) = -(\gamma+\mu)z(t) + \epsilon y(t-\tau). \end{cases} \tag{5}$$

The linearization of Eq. (5) at $(0, 0, 0)$ is

$$\begin{cases} \dot{x}(t) = -(\mu + \alpha I^*)x(t) + \alpha S^* z(t), \\ \dot{y}(t) = \alpha I^* x(t) + \alpha S^* z(t) - (\epsilon+\mu)y(t-\tau), \\ \dot{z}(t) = -(\gamma+\mu)z(t) + \epsilon y(t-\tau). \end{cases} \tag{6}$$

whose characteristic equation is

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + (q_1\lambda^2 + q_2\lambda + q_3)e^{-\lambda\tau} = 0, \tag{7}$$

where

$$\begin{aligned} p_1 &= 2\mu + \gamma + \alpha I^*, \\ p_2 &= (\mu + \alpha I^*)(\gamma + \mu), \\ q_1 &= \gamma + \mu, \\ q_2 &= (2\mu + \gamma + \alpha I^*)(\epsilon + \mu) - \epsilon\alpha S^*, \\ q_3 &= (\mu + \alpha I^*)(\epsilon+\mu)(\gamma+\mu) - \alpha^2\epsilon S^* I^* \\ &\quad -(\mu + \alpha I^*)\epsilon\alpha S^*. \end{aligned}$$

In order to investigate the distribution of roots of the transcendental equation 7), the following Lemma is useful.

Lemma 1 [2] *For the transcendental equation*

$$\begin{aligned} &P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \\ &\lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

For $\tau = 0$, (7) becomes

$$\lambda^3 + (p_1 + q_1)\lambda^2 + (p_2 + q_2)\lambda + q_3 = 0. \tag{8}$$

A set of necessary and sufficient conditions that all roots of (8) have a negative real part is given by the well-known Routh-Hurwitz criteria in the following form:

(H2) $(p_1 + q_1)(p_2 + q_2) - q_3 > 0, \; q_3 > 0.$

For $\omega > 0, i\omega$ is a root of (7) if and only if

$$-i\omega^3 - p_1\omega^2 + ip_2\omega + (-q_1\omega^2 + iq_2\omega + q_3)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} (q_3 - q_1\omega^2)\cos\omega\tau + q_2\omega\sin\omega\tau = p_1\omega^2, \\ q_2\omega\cos\omega\tau - (q_3 - q_1\omega^2)\sin\omega\tau = \omega^3 - p_2\omega. \end{cases} \tag{9}$$

which leads to

$$q_2^2\omega^2 + (q_3 - q_1\omega^2)^2 = p_1^2\omega^4 + (\omega^3 - p_2\omega)^2,$$

namely,

$$\omega^6 + (p_1^2 - 2p_2 + 2q_1q_3 - q_1^2)\omega^4 + (p_2^2 - q_2^2)\omega^2 - q_3^2 = 0. \tag{10}$$

Let $z = \omega^2$, then (10) become

$$z^3 + r_1z^2 + r_2z + r_3 = 0, \tag{11}$$

where

$$r_1 = p_1^2 - 2p_2 + 2q_1q_3 - q_1^2, \; r_2 = p_2^2 - q_2^2, \; r_3 = -q_3^2.$$

Denote

$$h(z) = z^3 + r_1z^2 + r_2z + r_3. \tag{12}$$

Since $\lim_{z\to+\infty} h(z) = +\infty$ and $r_3 < 0$, we can conclude that Eq. (11) has at least one positive root. Without loss of generality, we assume that (11) has three positive roots, defined by z_1, z_2, z_3, respectively. Then Eq. (10) has three positive roots

$$\omega_1 = \sqrt{z_1}, \; \omega_2 = \sqrt{z_2}, \; \omega_3 = \sqrt{z_3}.$$

By (9), we have

$$\cos\omega_k\tau = \frac{p_1\omega^2(q_3 - q_1\omega^2) + (\omega^3 - p_2\omega)q_2\omega}{(q_3 - q_1\omega^2)^2 + (q_2\omega)^2}.$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{ \arccos\left[\frac{p_1\omega^2(q_3 - q_1\omega^2) + (\omega^3 - p_2\omega)q_2\omega}{(q_3 - q_1\omega^2)^2 + (q_2\omega)^2}\right] + 2j\pi \right\}, \tag{13}$$

where $k = 1, 2, 3; j = 0, 1, \cdots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq. (7) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}, \; \omega_0 = \omega_{k_0}. \tag{14}$$

The above analysis leads to the following result:

Lemma 2 *If* $(H1)$ *and* $(H2)$ *hold, then all roots of (7) have a negative real part when* $\tau \in [0, \tau_0)$ *and (7) admits a pair of purely imaginary roots* $\pm\omega_k$ *when* $\tau = \tau_k^{(j)}$ $(k = 1, 2, 3; j = 0, 1, 2, \cdots)$.

In the sequel, we assume that

(H3) $[(4p_2 + q_2) - 2p_1(p_1 + q_1)]^2 < 12p_2(p_2 + q_2).$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (7) near $\tau = \tau_k^{(j)}$, and $\alpha(\tau_k^{(j)}) = 0$, and $\omega(\tau_k^{(j)}) = \omega_0$. Due to functional differential equation theory, for every $\tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \cdots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^{(j)}| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand of (7) and taking derivative with respect to τ, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2p_1\lambda + p_2)e^{\lambda\tau}}{\lambda(q_1\lambda^2 + q_2\lambda + q_3)} + \frac{q_1\lambda + q_2}{\lambda(q_1\lambda^2 + q_2\lambda + q_3)} - \frac{\tau}{\lambda}.$$

Thus, Re $\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\lambda=i\omega_k} =$

$$\frac{(p_2 + q_2 - 3\omega_k^2)\omega_k[(q_3 - q_1\omega_k^2)\sin\omega_k\tau_k^{(j)} - q_2\omega_k\cos\omega_k\tau_k^{(j)}]}{M^2 + N^2}$$

$$+\frac{2(p_1 + q_1)\omega_k^2[(q_3 - q_1\omega_k^2)\cos\omega_k\tau_k^{(j)} + q_2\omega_k\sin\omega_k\tau_k^{(j)}]}{M^2 + N^2},$$

where

$$M = [\omega_0(q_3 - q_1\omega_k^2)\sin\omega_k\tau_k^{(j)} - q_2\omega_k^2\cos\omega_k\tau_k^{(j)}]^2$$
$$N = [\omega_0(q_3 - q_1\omega_k^2)\cos\omega_k\tau_k^{(j)} + q_2\omega_k^2\sin\omega_k\tau_k^{(j)}]^2.$$

Together with (9), it follows that Re $\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_k^{(j)}} =$

$$\frac{\omega_k^2\{3\omega_k^4 - [4p_2 + q_2 - 2p_1(p_1+q_1)]\omega_k^2 + p_2(p_2+q_2)\}}{M^2+N^2}$$

By the assumption (H3), so we have

$$\text{signRe}\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_k^{(j)}} = \text{signRe}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_k^{(j)}} > 0.$$

According to above analysis and the results of Kuang [3] and Hale[4], we have

Theorem 1 *If $(H1), (H2)$ and $(H3)$ hold, then the equilibrium E_0 of system (3) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau \geq \tau_0$, system (3) undergoes a Hopf bifurcation at the equilibrium E_0 when $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \cdots$.*

Proof The proof of the stability of the equilibrium E_0 can be obtained by Lemma 2. When $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \cdots$. (7) has a simple purely imaginary roots $\lambda = \pm\omega_k i$, and all roots $\lambda_j \neq \lambda, \overline{\lambda}$ satisfy $\lambda_j \neq im\omega_k$ for any integer m, since there is no other purely imaginary roots except for $\lambda = \pm i\omega_k$. Furthermore, $Re(\lambda'(\tau_k^{(j)})) > 0, k = 1, 2, 3; j = 0, 1, 2, \cdots$. Due to the Hopf bifurcation theorem [4], we complete the proof.

III. Direction and stability of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \cdots$. In this section, we shall derived the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E_0(S^*, E^*, I^*)$ at these critical value of τ, by using techniques from normal form and center manifold theory [1]. Throughout this section, we always assume that system (3) undergoes Hopf bifurcation at the positive equilibrium $E_0(S^*, E^*, I^*)$ for $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \cdots$, and then $\pm i\omega_0$ are corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E_0(S^*, E^*, I^*)$.

For convenience, let $\bar{x}(t) = x(\tau t), \bar{y}(t) = y(\tau t), \bar{z}(t) = z(\tau t)$ and $\tau = \tau_k^{(j)} + \mu$, where $\tau_k^{(j)}$ is defined by (2.10) and $\mu \in R$, drop the bar for the simplification of notations, then system (5) can be written as an FDE in $C = C([-1, 0]), R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{15}$$

where $u(t) = (x(t), y(t), z(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (x(t+\theta), y(t+\theta), z(t+\theta))^T \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by $L_\mu\phi =$

$$(\tau_k^{(j)} + \mu)\begin{pmatrix} -(\mu+\alpha)I^* & 0 & \alpha S^* \\ \alpha I^* & 0 & \alpha S^* \\ 0 & 0 & -(\gamma+\mu) \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix}$$

$$+(\tau_k^{(j)} + \mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\epsilon+\mu) & 0 \\ 0 & \epsilon & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \tag{16}$$

and

$$f(\mu, \phi) = (\tau_k^{(j)} + \mu)\begin{pmatrix} -\alpha\phi_1(0)\phi_3(-1) \\ \alpha\phi_1(0)\phi_3(-1) \\ 0 \end{pmatrix}, \tag{17}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$.

From the discussion in Section 2, we know that if $\mu = 0$, then system (15) undergoes a Hopf bifurcation at the positive equilibrium $E_0(S^*, E^*, I^*)$ and the associated characteristic equation of system (15) has a pair of simple imaginary roots $\pm\omega_0\tau_k^{(j)}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu), \theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad for \quad \phi \in C. \tag{18}$$

In fact, we can choose $\eta(\theta, \mu) =$

$$(\tau_k^{(j)} + \mu)\begin{pmatrix} -(\mu+\alpha I^* & 0 & 0 \\ \alpha I^* & -(\epsilon+\mu) & 0 \\ 0 & \epsilon & 0 \end{pmatrix}\delta(\theta)$$

$$-(\tau_k^{(j)} + \mu)\begin{pmatrix} -0 & 0 & -\alpha S^* \\ 0 & 0 & \alpha S^* \\ 0 & 0 & -(\gamma+\mu) \end{pmatrix}\delta(\theta+1), \tag{19}$$

where δ is the Dirac delta function.

For $\phi \in C([-1, 0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0 \end{cases} \tag{20}$$

and

$$R\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ f(\mu, \phi), & \theta = 0. \end{cases} \tag{21}$$

Then (15) is equivalent to the abstract differential equation

$$\dot{u_t} = A(\mu)u_t + R(\mu)u_t, \tag{22}$$

where $u_t(\theta) = u(t+\theta), \theta \in [-1, 0]$.

For $\psi \in C([0, 1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1, 0], R^3)$ and $\psi \in C([0, 1], (R^3)^*)$, define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_0\tau_k^{(j)}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_k^{(j)}$ and $-i\omega_0\tau_k^{(j)}$ respectively. By direct computation, we can obtain

$$q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\tau_k^{(j)}\theta}, q^*(s) = D(1, a_1^*, a_2^*)e^{i\omega_0\tau_k^{(j)}s},$$

$D=\frac{1}{B}$, where

$$\begin{aligned} a_1 &= \frac{(i\omega_0+\gamma+\mu)(i\omega_0+\mu+\alpha I^*)}{\alpha\epsilon S^* e^{i\omega_0\tau_k^{(j)}}}, \\ a_2 &= \frac{i\omega_0+\mu+\alpha I^*}{\alpha S^*}, a_1^*=\frac{-i\omega_0+\mu+\alpha I^*}{\alpha I^*}, \\ a_2^* &= \frac{\alpha S^*(-i\omega_0+2\alpha I^*+\mu)}{\alpha I^*(-i\omega_0+\gamma+\mu)}, \end{aligned}$$

$$B=1+\bar{a}_1 a_1^*+\bar{a}_2 a_2^*+a_1^*\bar{a}_1(\epsilon+\mu)e^{i\omega_0\tau_k^{(j)}}+\bar{a}_1 a_2^*\epsilon e^{i\omega_0\tau_k^{(j)}}.$$

Furthermore, $<q^*(s),q(\theta)>=1$ and $<q^*(s),\bar{q}(\theta)>=0$.

Next, we use the same notations as those in Hassard [1] and we first compute the coordinates to describe the center manifold C_0 at $\mu=0$. Let u_t be the solution of Eq. (15) when $\mu=0$.

Define

$$z(t)=<q^*,u_t>, W(t,\theta)=u_t(\theta)-2\mathrm{Re}\{z(t)q(\theta)\}. \quad (23)$$

on the center manifold C_0, and we have

$$W(t,\theta)=W(z(t),\bar{z}(t),\theta), \quad (24)$$

where

$$W(z(t),\bar{z}(t),\theta)=W_{20}\frac{z^2}{2}+W_{11}z\bar{z}+W_{02}\frac{\bar{z}^2}{2}+\cdots, \quad (25)$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t\in C_0$ of (15),

$$\begin{aligned} \dot{z}(t) &= i\omega_0\tau_k^{(j)}z+\bar{q}^*(\theta)f(0,W(z,\bar{z},\theta)+2\mathrm{Re}\{zq(\theta)\} \\ &= i\omega_0\tau_k^{(j)}z+\bar{q}^*(0)f_0. \end{aligned}$$

That is

$$\dot{z}(t)=i\omega_0\tau_k^{(j)}z+g(z,\bar{z}),$$

where

$$g(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots.$$

Hence, we have $g(z,\bar{z})=\bar{q}^*(0)f_0(z,\bar{z})=f(0,u_t)=$

$$\begin{aligned} &\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)a_2 z^2+2\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)Re\{a_2\}z\bar{z} \\ &+\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)\bar{a}_2^2\bar{z}^2+\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*) \\ &\times\left[\frac{1}{2}W_{20}^{(1)}(0)\bar{a}_2+\frac{1}{2}W_{20}^{(3)}(0)+W_{11}^{(1)}(0)a_2+W_{11}^{(3)}(0)\right] \\ &\times z^2\bar{z}+\text{h.o.t.} \end{aligned}$$

Then we obtain

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)a_2, \\ g_{11} &= 2\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)\mathrm{Re}\{a_2\}, \\ g_{02} &= 2\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)\bar{a}_2^2, \\ g_{21} &= 2\bar{D}\tau_k^{(j)}\alpha(1+\bar{a}_1^*)\left[\frac{1}{2}W_{20}^{(1)}(0)\bar{a}_2+\frac{1}{2}W_{20}^{(3)}(0)\right. \\ &\quad \left.+W_{11}^{(1)}(0)a_2+W_{11}^{(3)}(0)\right]. \end{aligned}$$

For unknown $W_{20}^{(1)}(0),W_{20}^{(3)}(0),W_{11}^{(1)}(0),W_{11}^{(3)}(0)$ in g_{21}, we still need to compute them.
Form (22), (23), we have

$$\begin{aligned} W' &= \begin{cases} AW-2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1\le\theta<0, \\ AW-2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}+\bar{f}, & \theta=0 \end{cases} \\ &= AW+H(z,\bar{z},\theta), \end{aligned} \quad (26)$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots. \quad (27)$$

Comparing the coefficients, we obtain

$$(AW-2i\tau_k^{(j)}\omega_0)W_{20}=-H_{20}(\theta), \quad (28)$$

$$AW_{11}(\theta)=-H_{11}(\theta). \quad (29)$$

We know that for $\theta\in[-1,0)$,

$$\begin{aligned} H(z,\bar{z},\theta) &= -\bar{q}^*(0)f_0 q(\theta)-q^*(0)\bar{f}_0\bar{q}(\theta) \\ &= -g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta). \end{aligned} \quad (30)$$

Comparing the coefficients of (30) with (27) gives that

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta). \quad (31)$$

$$H_{11}(\theta)=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta). \quad (32)$$

From (3.14),(3.17) and the definition of A , we get

$$\dot{W}_{20}(\theta)=2i\omega_0\tau_k^{(j)}W_{20}(\theta)+g_{20}q(\theta)+\bar{g_{02}}\bar{q}(\theta). \quad (33)$$

Noting that $q(\theta)=q(0)e^{i\omega_0\tau_k^{(j)}\theta}$, we have

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0\tau_k^{(j)}}q(0)e^{i\omega_0\tau_k^{(j)}\theta}+\frac{i\bar{g}_{02}}{3\omega_0\tau_k^{(j)}}\bar{q}(0)e^{-i\omega_0\tau_k^{(j)}\theta} \\ &\quad +E_1 e^{2i\omega_0\tau_k^{(j)}\theta}, \end{aligned} \quad (34)$$

where $E_1=(E_1^{(1)},E_1^{(2)},E_1^{(3)})\in R^3$ is a constant vector.
Similarly, from (29), (32) and the definition of A, we have

$$\dot{W}_{11}(\theta)=g_{11}q(\theta)+\bar{g_{11}}\bar{q}(\theta), \quad (35)$$

$$W_{11}(\theta)=-\frac{ig_{11}}{\omega_0\tau_k^{(j)}}q(0)e^{i\omega_0\tau_k^{(j)}\theta}+\frac{i\bar{g}_{11}}{\omega_0\tau_k^{(j)}}\bar{q}(0)e^{-i\omega_0\tau_k^{(j)}\theta}+E_2. \quad (36)$$

where $E_2=(E_2^{(1)},E_2^{(2)},E_2^{(3)})\in R^3$ is a constant vector

In what follows, we shall seek appropriate E_1,E_2 in (34), (36), respectively. It follows from the definition of A and (31), (32) that

$$\int_{-1}^{0}d\eta(\theta)W_{20}(\theta)=2i\omega_0\tau_k^{(j)}W_{20}(0)-H_{20}(0) \quad (37)$$

and

$$\int_{-1}^{0}d\eta(\theta)W_{11}(\theta)=-H_{11}(0), \quad (38)$$

where $\eta(\theta)=\eta(0,\theta)$.
From (28), we have

$$H_{20}(0)=-g_{20}q(0)-\bar{g_{02}}\bar{q}(0)+2\tau_k^{(j)}\begin{pmatrix}-\alpha a_2\\ \alpha\bar{a}_1^* a_2\\ 0\end{pmatrix}, \quad (39)$$

$$H_{11}(0)=-g_{11}q(0)-\bar{g_{11}}(0)\bar{q}(0)+2\tau_k^{(j)}\begin{pmatrix}-\alpha\mathrm{Re}\{a_2\}\\ \alpha\bar{a}_1^*\mathrm{Re}\{a_2\}\\ 0\end{pmatrix}. \quad (40)$$

Noting that

$$\left(i\omega_0\tau_k^{(j)}I-\int_{-1}^{0}e^{i\omega_0\tau_k^{(j)}\theta}d\eta(\theta)\right)q(0)=0,$$
$$\left(-i\omega_0\tau_k^{(j)}I-\int_{-1}^{0}e^{-i\omega_0\tau_k^{(j)}\theta}d\eta(\theta)\right)\bar{q}(0)=0$$

and substituting (34) and (39) into (37), we have

$$(2i\omega_0\tau_k^{(j)}I-\int_{-1}^{0}e^{2i\omega_0\tau_k^{(j)}\theta}d\eta(\theta))E_1=2\tau_k^{(j)}\begin{pmatrix}-\alpha a_2\\ \alpha\bar{a}_1^*a_2\\ 0\end{pmatrix}.$$

That is

$$\begin{pmatrix}l_1 & 0 & -\alpha S^*\\ -\alpha I^* & l_2 & 0\\ 0 & -\epsilon e^{-2i\omega_0\tau_k^{(j)}} & l_3\end{pmatrix}E_1=2\begin{pmatrix}-\alpha a_2\\ \alpha\bar{a}_1^*a_2\\ 0\end{pmatrix},$$

where $l_1=2i\omega_0+\mu+\alpha I^*, l_2=2i\omega_0+(\epsilon+\mu)e^{-2i\omega_0\tau_k^{(j)}}$, $l_3=2i\omega_0+\gamma+\mu$. It follows that

$$E_1^{(1)}=\frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)}=\frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)}=\frac{\Delta_{13}}{\Delta_1},\tag{41}$$

where

$$\begin{aligned}\Delta_1&=\det\begin{pmatrix}l_1 & 0 & -\alpha S^*\\ -\alpha I^* & l_2 & 0\\ 0 & -\epsilon e^{-2i\omega_0\tau_k^{(j)}} & l_3\end{pmatrix},\\ \Delta_{11}&=2\det\begin{pmatrix}-\alpha a_2 & 0 & -\alpha S^*\\ \alpha a_1^*a_2 & l_2 & 0\\ 0 & -\epsilon e^{-2i\omega_0\tau_k^{(j)}} & l_3\end{pmatrix},\\ \Delta_{12}&=2\det\begin{pmatrix}l_1 & -\alpha a_2 & -\alpha S^*\\ -\alpha I^* & \alpha a_1^*a_2 & 0\\ 0 & 0 & l_3\end{pmatrix},\\ \Delta_{13}&=2\det\begin{pmatrix}l_1 & 0 & -\alpha a_2\\ -\alpha I^* & l_2 & \alpha a_1^*a_2\\ 0 & -\epsilon e^{-2i\omega_0\tau_k^{(j)}} & 0\end{pmatrix}.\end{aligned}$$

Similarly, substituting (35) and (40) into (38), we have

$$\int_{-1}^{0}d\eta(\theta))E_2=2\tau_k^{(j)}\begin{pmatrix}-\alpha\mathrm{Re}\{a_2\}\\ \alpha\bar{a}_1^*\mathrm{Re}\{a_2\}\\ 0\end{pmatrix}.$$

That is

$$\begin{pmatrix}\mu+\alpha I^* & 0 & -\alpha S^*\\ -\alpha I^* & \epsilon+\mu & -\alpha S^*\\ 0 & -\epsilon & \gamma+\mu\end{pmatrix}E_2=2\begin{pmatrix}-\alpha\mathrm{Re}\{a_2\}\\ \alpha\bar{a}_1^*\mathrm{Re}\{a_2\}\\ 0\end{pmatrix}.$$

It follows that

$$E_2^{(1)}=\frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)}=\frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)}=\frac{\Delta_{23}}{\Delta_2},\tag{42}$$

where

$$\begin{aligned}\Delta_2&=\det\begin{pmatrix}\mu+\alpha I^* & 0 & -\alpha S^*\\ -\alpha I^* & -(\epsilon+\mu) & -\alpha S^*\\ 0 & -\epsilon & -(\gamma+\mu)\end{pmatrix},\\ \Delta_{21}&=2\det\begin{pmatrix}-\alpha\mathrm{Re}\{a_2\} & 0 & -\alpha S^*\\ \alpha\bar{a}_1^*\mathrm{Re}\{a_2\} & \epsilon+\mu & -\alpha S^*\\ 0 & -\epsilon & \gamma+\mu\end{pmatrix},\\ \Delta_{22}&=2\det\begin{pmatrix}\mu+\alpha I^* & -\alpha\mathrm{Re}a_2 & -\alpha S^*\\ -\alpha I^* & -\alpha\bar{a}_1^*\mathrm{Re}a^2 & -\alpha S^*\\ 0 & 0 & -(\gamma+\mu)\end{pmatrix},\\ \Delta_{23}&=2\det\begin{pmatrix}\mu+\alpha I^* & 0 & -\alpha\mathrm{Re}a_2\\ -\alpha I^* & -(\epsilon+\mu) & -\alpha\bar{a}_1^*\mathrm{Re}a^2\\ 0 & -\epsilon & 0\end{pmatrix}.\end{aligned}$$

From (34),(36),(41),(42), we can calculate g_{21} and derive the following values:

$$\begin{aligned}c_1(0)&=\frac{i}{2\omega_0\tau_k^{(j)}}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right)+\frac{g_{21}}{2},\\ \mu_2&=-\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k^{(j)})\}},\\ \beta_2&=2\mathrm{Re}(c_1(0)),\\ T_2&=-\frac{\mathrm{Im}\{c_1(0)\}+\mu_2\mathrm{Im}\{\lambda'(\tau_k^{(j)})\}}{\omega_0\tau_k^{(j)}}.\end{aligned}$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (15) at $\tau=\tau_k^{(j)},(k=1,2,3;j=0,2,3,\cdots)$ on the center manifold. From the discussion above, we have the following result:

Theorem 2 *The periodic solution is supercritical (subcritical) if $\mu_2>0$ ($\mu_2<0$); the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2<0$ ($\beta_2>0$); the periodic of the bifurcating periodic solutions increase (decrease) if $T_2>0$ ($T_2<0$).*

Remark 1 *A τT-periodic solution of (15) is a T-periodic solution of (5).*

IV. NUMERICAL EXAMPLES

In this section, we present some numerical results of system (3) to verify the analytical predictions obtained in the previous section. From section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

$$\begin{cases}\dot{S}(t)=0.2-0.2S-2IS,\\ \dot{E}(t)=2IS-0.5E(t-\tau),\\ \dot{I}(t)=0.3E(t-\tau)-0.4I,\end{cases}\tag{43}$$

which has a positive equilibrium $E_0(S^*,E^*,I^*)\approx(\frac{1}{3},\frac{4}{15},\frac{1}{5})$ and satisfies the conditions indicated in Theorem 1. When $\tau=0$, the positive equilibrium $E_0=(\frac{1}{3},\frac{4}{15},\frac{1}{5})$ is asymptotically stable. Take $j=0$ for example, by some complicated computation by means of Matlab 7.0, we get $\omega_0\approx0.4112,\tau_0\approx5.67,\lambda'(\tau_0)\approx0.4209-5.1315i$. Thus we can calculate the following values:

$$c_1(0)\approx-0.7802-4.0452i,\mu_2\approx0.4122,\beta_2\approx-2.3552,$$

$T_2\approx6.3235$. Furthermore, it follows that $\mu_2>0$ and $\beta_2<0$. Thus, the positive equilibrium $E_0=(\frac{1}{3},\frac{4}{15},\frac{1}{5})$ is stable when $\tau<\tau_0$ as is illustrated by the computer simulations (see Figs.1-7). When τ passes through the critical value τ_0, the positive equilibrium $E_0=(\frac{1}{3},\frac{4}{15},\frac{1}{5})$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the positive equilibrium $E_0=(\frac{1}{3},\frac{4}{15},\frac{1}{5})$. Since $\mu_2>0$ and $\beta_2<0$, the direction of the Hopf bifurcation is $\tau>\tau_0$, and these bifurcating periodic solutions from $E_0=(\frac{1}{3},\frac{4}{15},\frac{1}{5})$ at τ_0 are stable, which are depicted in Figs.8-14.

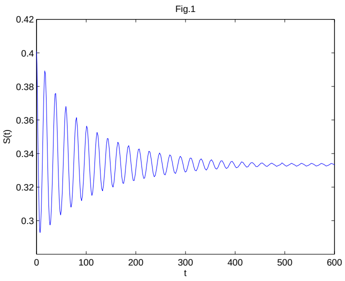

Fig. 1. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

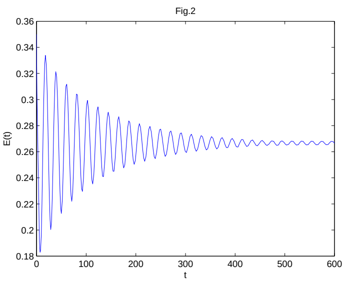

Fig. 2. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

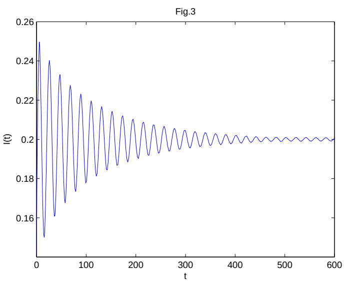

Fig. 3. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

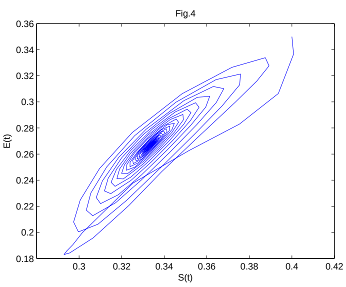

Fig. 4. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

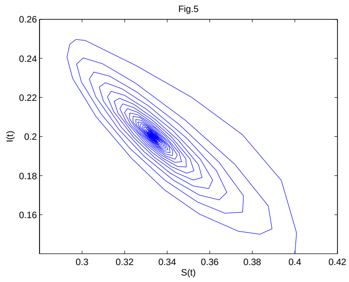

Fig. 5. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

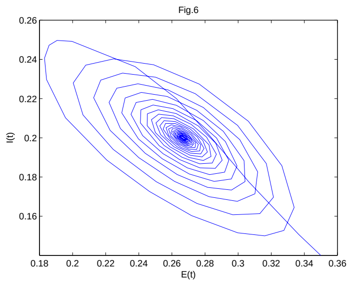

Fig. 6. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

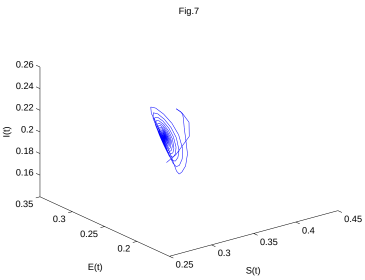

Fig. 7. Behavior and phase portrait of system (43) with $\tau = 5.5 < \tau_0 \approx 5.67$. The positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$ is asymptotically stable. The initial value is (0.4,0.35,0.14).

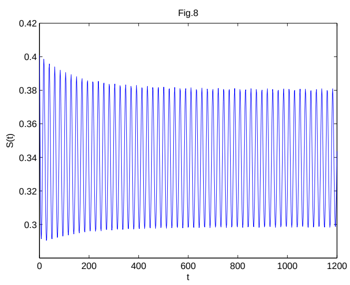

Fig. 8. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

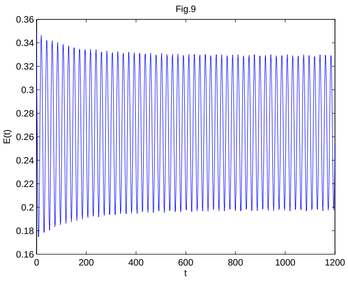

Fig. 9. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

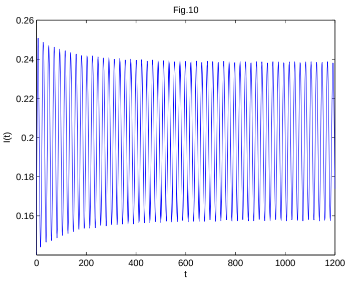

Fig. 10. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

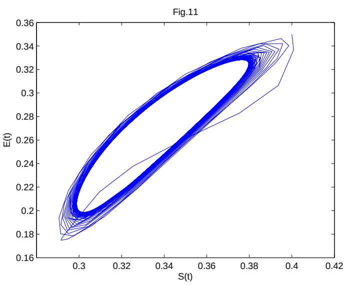

Fig. 11. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

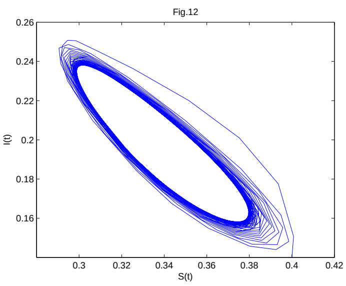

Fig. 12. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

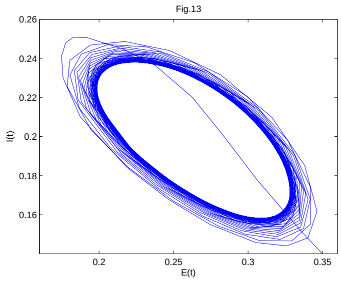

Fig. 13. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

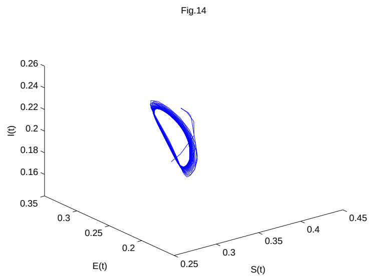

Fig. 14. Behavior and phase portrait of system (43) with $\tau = 5.8 > \tau_0 \approx 5.67$. Hopf bifurcation occurs from the positive equilibrium $E_0 = (\frac{1}{3}, \frac{4}{15}, \frac{1}{5})$. The initial value is (0.4,0.35,0.14).

V. Biological explanations and Conclusions

1 Biological explanations

From the analysis in Section 2, we know that if the conditions (H1), (H2) and (H3) hold, then the positive equilibrium $E_0(S^*, E^*, I^*)$ of system (3) is asymptotically stable when $\tau \in [0, \tau_0)$, and unstable when $\tau > \tau_0$. This shows that, in this case, the susceptible, exposed (latent), infectious host populations will tend to stabilization, that is, the susceptible host populations will tend to S^*, the exposed (latent) host populations will tend to E^* and the infectious host populations will tend to I^*, and this fact is not influenced by the delay $\tau \in [0, \tau_0)$. When τ crosses through the critical value τ_0, the positive equilibrium $E_0(S^*, E^*, I^*)$ of system (3) loses stability and a Hopf bifurcation occurs. If the periodic solution bifurcating from the Hopf bifurcation is stable, then this shows that the susceptible, exposed (latent), infectious host populations may coexist and keep in an oscillatory mode. From discussion in Section 2, we know that the positive equilibrium $E_0(S^*, E^*, I^*)$ is always unstable when $\tau > \tau_0$. Therefore, if the above bifurcating periodic solution is unstable, then it is at least semi-stable (stable inside and unstable outside) and hence the susceptible, exposed (latent), infectious host populations may keep in an oscillatory mode near the positive equilibrium $E_0(S^*, E^*, I^*)$.

2 Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E_0(S^*, E^*, I^*)$ and local Hopf bifurcation in a special SEIR epidemic model with nonlinear incidence rates. we have showed that if the conditions (H1), (H2) and (H3) hold, the positive equilibrium $E_0(S^*, E^*, I^*)$ of system (3) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. We have also showed that, if the conditions (H1), (H2) and (H3) hold, as the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(S^*, E^*, I^*)$, i.e., a family of periodic orbits bifurcates from the the positive equilibrium $E_0(S^*, E^*, I^*)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. A numerical example verifying our theoretical results is also correct.

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Changjin Xu is a lecturer in Guizhou Key Laboratory of Economics System Simulation, School of Mathematics and Statistics, Guizhou College of Finance and Economics, Guiyang, China. He received his M.S. from Kunming University of Science and Technology, Kunming, in 2004 and Ph. D. from Central South University, Changsha, China in 2010. His research interests focus on the stability and bifurcation theory of delayed differential and equation..