On the Blow-up Behavior of Solutions to Semi-linear Wave Models with Fractional Damping

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***Abstract*—The aim of this research study is to consider the Cauchy problem for the semi-linear wave equation**

$$u_{tt} - \Delta u + D_{0|t}^{\alpha+1} u + (-\Delta)^{\beta/2} u_t = h(x,t)\,|u|^p\,,$$

posed in $Q := (0+\infty)\times \mathbb{R}^N$, where $D^{\alpha+1}$, $0<\alpha<1$ denotes the time fractional derivative, $(-\Delta)^{\beta/2}$, $0<\beta<2$ is $\beta/2$ fractional power of $-\Delta$. In fact the researchers are interested in Fujita's exponent depending on all parameters α, β and the spatial dimension N, as well as in finding necessary conditions on the initial data which ensure nonexistence of local and global solutions.

***Index Terms*—Caputo derivative, fractional power derivative, critical exponent.**

I. Introduction

In this paper we will consider the Cauchy problem for the semi-linear wave equation with fractional damping

$$u_{tt} - \Delta u + D_{0|t}^{\alpha+1} u + (-\Delta)^{\beta/2} u_t = h(x,t)\,|u|^p \qquad (1)$$

posed in $Q = (0,+\infty)\times\mathbb{R}^N$, subject to the initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x).$$

Where $\Delta = \partial_1^2 + ... + \partial_N^2$ is the Laplacian in the space variable x, and $D^{\alpha+1}$ $(0<\alpha<1)$ is the time fractional derivative in the Caputo sense defined by

$$D_{0|t}^{\alpha+1} u(t) := \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{u_{tt}(\sigma)}{(t-\sigma)^{\alpha}} d\sigma.$$

(See [12],[13]) for more information on fractional time derivative. $(-\Delta)^{\beta/2}$ is $\beta/2$ fractional power of the Laplacian defined by

$$(-\Delta)^{\beta/2} v(x) = \mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(v)(\xi)\right)(x),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes the Fourier transform and its inverse, which stands for propagation media with impurities and u_0, u_1 are the given initial data. The time fractional derivative is found to be a very effective means to describe the anomalous attenuation behaviors(see [3,4,14,18],). Hanyga and Seredynska [14], studied the ordinary differential equation

$$D^2 u + \gamma D^{\eta+1} u + F(u) = 0,$$

where $D^{\eta+1}, 0 < \eta < 1$, represents the $(\eta+1)$-order fractional derivative in the sense of Caputo which models the anomalous attenuation, and γ is the collective thermoviscous coefficient. In [3], Chen and Holm studied the equation

$$\Delta v = \frac{1}{c_0^2} v_{tt} + \frac{2\alpha_0}{c_0^{1-\gamma}}(-\Delta)^{y/2} v_t,$$

which governs the propagation of sound through a viscous fluid, c_0 is the inviscid phase velocity, $2\alpha_0$ is the collective thermoviscous coefficient. with $x\in\mathbb{R}, t>0$, and ρ_0, λ are some constants that characterize the medium; $g(x,t)$ is a given function representing an external force. Recently Carvalho and Cholewa [2], dealt with the equation

$$u_{tt} = \Delta u + (-\Delta)^{\beta/2} u_t + |u|^p$$

The equations mentioned above are viewed to be a crucially important work undertaken by Greenberg, MacCamy and Mizel [5]. These researchers, consider the equation

$$\rho_0 u_{tt} = u_{xx} + \lambda u_{txx} + g(x,t)$$

In recent years, questions of global solution and blow-up of solution to semi-linear wave equations with damping term have been studied by many researchers(see [8],[9],[10],[11],[16]). As an illustration, the results found by Todorova and Yordanov[17] are mentioned below. Concerning the Cauchy problem for the semi-linear damped wave equation with the forcing term

$$u_{tt} - \Delta u + u_t = |u|^p\,, u(x,0) = u_0(x)\,, u_t(x,0) = u_1(x)\,,$$

where the authors solve the critical exponent problem. The main result is that the critical exponent $p_c(N)$ is exactly $1+2/N$. The number $p_c(N) = 1+2/N$ is the well known Fujita's critical exponent for the heat equation $u_t - \Delta u = |u|^p$ (see [17]). More precisely, they prove small data global existence when $p > p_c(N)$. If $1 < p < p_c(N)$, they prove blow up for all solutions with data positive on average. In [18] Zhang proves that the critical exponent belongs to the blow up region. This problem has been left open by Todorova and Yordanov [17].

The method used is based primarily on the articles of Mitidieri and Pohozahev [9], Pohozahev and Tesei [11], Pohozahev and Veron [12], Zhang [18]. It consists of a judicious choice of the test function in the weak formulation of the sought for solution of Eq. (1), in the second part of this paper, we establish necessary conditions on the initial datum assuring nonexistence of local and global solutions. Finally, the method used in Baras and Kersner [1], will be adapted.

Let us first define the operator of fractional derivative in the Caputo sense $D_{0|t}^{\alpha}$, $0<\alpha<1$,

$$D_{0|t}^{\alpha} u(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} u_t(\tau) d\tau,\ \tau > 0,$$

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and in general

$$D_{0|t}^{\alpha}u(t):=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u^{(n)}(\tau)}{(t-\tau)^{-n+\alpha+1}}d\tau,$$

$$n=[\alpha]+1,\alpha>0$$

The fractional derivative in the Riemann-Liouville sense of the higher power α

$$D_{0|t}^{\alpha}u(t):=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{u(\tau)}{(t-\tau)^{-n+\alpha+1}}d\tau,$$

$$n=[\alpha]+1,\alpha>0$$

For $t>0$, we define the right-handed Riemann-Liouville fractional derivative by

$$D_{t|T}^{\alpha}=\frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_t^T\frac{u(\tau)}{(\tau-t)^{-n+\alpha+1}}d\tau,$$

$$n=[\alpha]+1,\alpha>0$$

This allows us to present formula integration by parts

$$\int_0^T f(t)(D_{0|t}^{\alpha}g)(t)dt=\int_0^T g(t)(D_{t|T}^{\alpha}f)(t),\ 0<\alpha<1,$$

(See[17], p.46)

A. Nonexistence result

The function h is assumed to be nonnegative and satisfying $h\left(xR,tR^{\frac{2}{\alpha+1}}\right)=R^{\sigma}h(x,t)$ for some positive constants σ and R large. Let us first clarify what is meant by a solution to problem (1). Q_T here will denote the set $Q_T:=(0,T)\times\mathbb{R}^N$.

Definition 1.1: Let $T>0$. For a given functions $u_0\in H^{\beta}\left(\mathbb{R}^N\right)$ and $u_1(x)\in L^1_{loc}\left(\mathbb{R}^N\right)$ with $0<\beta<2$, the function u defined on Q_T is said to be a weak solution to the problem (1) if $u\in L^p_{loc}(Q_T,hdxdt)$ and it satisfies

$$\begin{aligned}&\int_{Q_T}h(x,t)\varphi|u|^p\,dxdt+\int_{\mathbb{R}^N}\varphi(x,0)(-\Delta)^{\beta/2}u_0dx\\&+\textstyle\int_{\mathbb{R}^N}\varphi(x,0)u_1dx+\displaystyle\int_{\mathbb{R}^N}u_0D_{t|T}^{\alpha}\varphi(x,0)\,dx\\&+\int_{Q_T}u_1(x)\,D_{t|T}^{\alpha}\varphi(x,0)\,dx\\&=\int_{Q_T}u\varphi_{tt}dxdt-\int_{Q_T}u\Delta\varphi\ dxdt\\&-\int_{Q_T}u(-\Delta)^{\beta/2}\varphi_t\ dxdt-\int_{Q_T}uD_{t|T}^{\alpha+1}dxdt,\end{aligned}\qquad(2)$$

for any test function $\varphi\in C_{x,t}^{2,2}(\mathbb{R}^n\times[0,T])$, such as $\varphi\geq0$, $\varphi(x,T)=\varphi_t(x,0)=\varphi_t(x,T)=0$.

Now, we are able to announce our results.

Theorem 1.1: Let $p>1$ and

$$p\leq p_c=\min\left\{1+\frac{\rho+2}{\left(N+\frac{2}{\alpha+1}-2\right)},\frac{\rho+N+\frac{2}{\alpha+1}}{(N-\beta)}\right\}.$$

If the initial data satisfies $\int_{\mathbb{R}^N}(-\Delta)^{\beta/2}u_0>0$, and $\int_{\mathbb{R}^N}u_1>0,\int_{\mathbb{R}^N}u_0>0$ then, every weak solution of the problem (1) does not exist globally in time.

Proof. The proof is by contradiction. So we assume that the solution is global. Let Φ be a decreasing function $C_0^2(\mathbb{R}_+)$, $0\leq\Phi\leq1$ such as

$$\Phi(y):=\begin{cases}1 & if\quad 0\leq y\leq1\quad,\\0 & if\qquad y\geq2\quad.\end{cases}$$

Let us consider

$$\varphi(x,t):=\Phi^l\left(\frac{t^{2(\alpha+1)}}{R^4}\right)\Phi^l\left(\frac{|x|^2}{R^2}\right)=\varphi_1(t)\,\varphi_2(x),$$

where R, T are positive real numbers and the number l satisfies $l\geq2q$. Now, by using ε-Young inequality $ab\leq\varepsilon a^p+C(\varepsilon)b^q$ (where $1/p+1/q=1$) with $\varepsilon>0$, to the right hand side of (2), we find

$$\begin{aligned}&\int_{Q_T}\varphi h|u|^p\,dxdt+\int_{\mathbb{R}^N}\varphi(x,0)(-\Delta)^{\beta/2}u_0dx\\&+\int_{\mathbb{R}^N}\varphi(x,0)u_1dx+\int_{\mathbb{R}^N}u_0D_{t|T}^{\alpha}\varphi(0)\,dx+\\&\int_{Q_T}u_1D_{t|T}^{\alpha}\varphi dx\leq\\&\qquad C\int_{Q_T}(h\varphi)^{\frac{-q}{p}}\left(|\varphi_{tt}|^q+|\Delta\varphi|^q+\right.\\&\left.\left|D_{t|T}^{\alpha+1}\varphi\right|^q+\left|(-\Delta)^{\beta/2}\varphi_t\right|^q dxdt,\right.\end{aligned}\qquad(3)$$

for some positive constant C. The test function is chosen so that

$$\int_{Q_T}(h\varphi)^{\frac{-q}{p}}\left(|\varphi_{tt}|^q+|\Delta\varphi|^q+\left|D_{t|T}^{\alpha+1}\varphi\right|^q+\right.\left.\left|(-\Delta)^{\beta/2}\varphi_t\right|^q dxdt\right.$$

is finite.

Next we introduce the scaled variables ,$t=R^{\frac{2}{\alpha+1}}\tau$ and $x=Ry$, we define the set Ω and the function ψ by

$$\Omega:=\left\{(\tau,y)\in\mathbb{R}_+\times\mathbb{R}^N;\tau^{2(\alpha+1)}\leq2,|y|^2\leq2\right\}$$

and

$$\varphi(t,x)=\varphi\left(\tau R^{\frac{2}{\alpha+1}},Ry\right):=\psi(\tau,y),$$

respectively. Clearly, we have

$$dxdt=R^{N+\frac{2}{\alpha+1}}dyd\tau;\ D^{\alpha+1}_{t\left|TR^{\frac{2}{\alpha+1}}\right.}\varphi=R^{-(\alpha+1)}D_{\tau|T}^{\alpha+1}\psi;$$

$$(-\Delta)^{\beta/2}\varphi=R^{-\beta}(-\Delta)^{\beta/2}\psi.$$

Thus,

$$\begin{aligned}&\int_{Q_{TR^{2/(\alpha+1)}}}(h\varphi)^{\frac{-q}{p}}|\varphi_{tt}|^q\,dxdt=\\&R^{-\frac{4}{\alpha+1}q-\frac{q}{p}\rho+N+\frac{2}{\alpha+1}}\int_{\Omega}(h\psi)^{\frac{-q}{p}}|\psi_{\tau\tau}|^q\,dyd\tau,\\&\int_{Q_{TR^{2/(\alpha+1)}}}(h\varphi)^{\frac{-q}{p}}|\Delta\varphi|^q\,dxdt=\\&R^{-2q-\frac{q}{p}\rho+N+\frac{2}{\alpha+1}}\int_{\Omega}(h\psi)^{\frac{-q}{p}}|\Delta\psi|^q\,dyd\tau,\\&\int_{Q_{TR^{2/(\alpha+1)}}}(h\varphi)^{\frac{-q}{p}}\left|D_{t|TR^{2/(\alpha+1)}\alpha+1}\varphi\right|^q dxdt=\\&R^{-2q-\frac{q}{p}\rho+N+\frac{2}{\alpha+1}}\int_{\Omega}(h\psi)^{\frac{-q}{p}}\left|D_{\tau|T}^{\alpha+1}\psi\right|^q dyd\tau,\end{aligned}\qquad(4)$$

and

$$\int_{Q_{TR^{2/(\alpha+1)}}} (h\varphi)^{\frac{-q}{p}} \left|(-\Delta)^{\beta/2}\varphi_t\right|^q dxdt = R^{-\beta q-\frac{q}{p}\rho-\frac{2q}{\alpha+1}+N+\frac{2}{\alpha+1}} \int_{\Omega} (h\psi)^{\frac{-q}{p}} \left|(-\Delta)^{\beta/2}\psi_\tau\right|^q dyd\tau \quad (5)$$

These relations (3),(4) and (5) imply that

$$\int_{Q_{TR^{2/(\alpha+1)}}} h\varphi |u|^p dxdt + \int_{\mathbb{R}^N} u_1(x)dx + \int_{\mathbb{R}^N} (-\Delta)^{\beta/2} u_0(x)dx \le CR^{\max(k_1,k_2)} \times \int_\Omega (h\psi)^{\frac{-q}{p}} (|\psi_{\tau\tau}|^q + |\Delta\psi|^q + \left|D^{\alpha+1}_{\tau|T}\psi\right|^q + \left|(-\Delta)^{\beta/2}\psi_\tau\right|^q)dyd\tau. \quad (6)$$

Observe that our assumption $p \le p_c$ is equivalent to

$$k_1 = -2q - \frac{q}{p}\rho + N + \frac{2}{\alpha+1} \le 0,$$

$$k_2 = -\beta q - \frac{q}{p}\rho - \frac{2q}{\alpha+1} + N + \frac{2}{\alpha+1} \le 0$$

Suppose that $p < p_c$, then

$$\lim_{R\to+\infty} \int_{Q_{TR^{2/(\alpha+1)}}} h|u|^p = 0.$$

This implies that $u = 0$ a. e. on $\mathbb{R}^+ \times \mathbb{R}^N$,which is a contradiction.

In the critical case (i.e. $p = p_c$), we have by using again Hölder inequality to the first three terms in the right hand side of (2) and the remaining term by ε_1-Young inequality, it follows

$$\begin{aligned}&\int_{Q_T} h\varphi|u|^p dxdt + \int_{\mathbb{R}^N} u_1(x)dx + \int_{\mathbb{R}^N} (-\Delta)^{\beta/2} u_0(x)dx \\ &\le C\left(\int_{Q_T} h\varphi|u|^p dxdt\right)^{\frac1p} [\left(\int_{Q_T}(h\varphi)^{1-q}|\varphi_{tt}|^q dxdt\right)^{\frac1q} \\ &+ \left(\int_{Q_T} h\varphi|u|^p dxdt\right)^{\frac1p}\left(\int_{Q_T}(h\varphi)^{1-q}|\Delta\varphi|^q dxdt\right)^{\frac1q} + \\ &\left(\int_{Q_T} h\varphi|u|^p dxdt\right)^{\frac1p}\left(\int_{Q_T}(h\varphi)^{1-q}\left|(-\Delta)^{\beta/2}\varphi_t\right|^q dxdt\right)^{\frac1q} \\ &+\varepsilon_1\int_{Q_T}(h\varphi)|u|^p dxdt + \\ &C(\varepsilon_1)\int_{Q_T}(h\varphi)^{1-q}\left|D^{\alpha+1}_{t|T}\varphi\right|^q dxdt,\end{aligned} \quad (7)$$

for some positive constants C,and ε_1. Now modifying the above by introducing a new parameter $S > 1$ such as

$$\varphi(x,t) := \Phi^l\left(\frac{t^{2(\alpha+1)}}{(SR)^4}\right)\Phi^l\left(\frac{|x|^2}{R^2}\right)$$

set $t = (SR)^{\frac{2}{\alpha+1}}\tau, x = Ry$, we rewrite (8) as follows

$$\begin{aligned}&\int_{\Sigma_1} h\varphi|u|^p \le CS^{-\frac{4}{\alpha+1}+\sigma\left(\frac1q-1\right)+\frac{2}{(\alpha+1)q}} \\ &\left(\int_{\Sigma_2} h\varphi u^p dxdt\right)^{\frac1p} \times \left(\int_{\Omega_2}(h\varphi)^{1-q}|\varphi_{\tau\tau}|^q dyd\tau\right)^{\frac1q} \\ &+S^{-\frac{2}{\alpha+1}+\sigma\left(\frac1q-1\right)+\frac{2}{(\alpha+1)q}}\left(\int_{\Sigma_2} h\varphi u^p dxdt\right)^{\frac1p} \\ &\left(\int_{\Omega_2}(h\varphi)^{1-q}\left|(-\Delta)^{\beta/2}\varphi_\tau\right|^q dyd\tau\right)^{\frac1q} + \\ &CS^{\frac{2}{(\alpha+1)q}}\left(\int_{\Sigma_3} h\varphi u^p dxdt\right)^{\frac1p}\left(\int_{\Omega_3}\varphi^{1-q}|\Delta\varphi|^q dyd\tau\right)^{\frac1q} \\ &C(\varepsilon)S^{-2q+\sigma(1-q)+\frac{2}{\alpha+1}}\int_{\Omega_1}(h\varphi)^{1-q}\left|D^{\alpha+1}_{\tau|T}\varphi\right|^q dyd\tau,\end{aligned} \quad (8)$$

here

$$\Sigma_1 = \left\{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+ : t^{2(\alpha+1)} \le 2(SR)^4,\ |x|^2 \le 2R^2\right\}$$

$$\Sigma_2 = \left\{(x,t) : (SR)^4 \le t^{2(\alpha+1)} \le 2(SR)^4,\ |x|^2 \le 2R^2\right\}$$

$$\Sigma_3 = \left\{(x,t) : t^{2(\alpha+1)} \le 2(SR)^4, R^2 \le |x|^2 \le 2R^2\right\},$$

and

$$\Omega_1 := \left\{(y,\tau) \in \mathbb{R}^N \times \mathbb{R}_+; \tau^{2(\alpha+1)} \le 2,\ |y|^2 \le 2\right\}$$

$$\Omega_2 := \left\{(y,\tau) \in \mathbb{R}^N \times \mathbb{R}_+; 1 \le \tau^{2(\alpha+1)} \le 2,\ |y|^2 \le 2\right\}$$

$$\Omega_3 := \left\{(y,\tau) \in \mathbb{R}^N \times \mathbb{R}_+; \tau^{2(\alpha+1)} \le 2, 1 \le |y|^2 \le 2\right\}.$$

As

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} h|u|^p dxdt < \infty$$

we have

$$\lim_{R\to+\infty}\int_{\Sigma_2} h|u|^p dxdt = \lim_{R\to+\infty}\int_{\Sigma_3} h|u|^p dxdt = 0.$$

Now let $R \to +\infty$

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} h|u|^p dxdt \le C(\varepsilon)S^{-2q+\sigma(1-q)+\frac{2}{\alpha+1}} \times \int_{\Omega_1}(h\varphi)^{1-q}\left|D^{\alpha+1}_{\tau|T}\varphi\right|^q dyd\tau, \quad (9)$$

now, by taking $\varepsilon_1 = 1/2$. Finally, let $S \to +\infty$ in (9), we infer that $\int_{\mathbb{R}^N\times\mathbb{R}^+} h|u|^p = 0$. We have $u = 0$, because $-2q + \sigma(1-q) + \frac{2}{\alpha+1} < 0$, which completes the proof.

When $\alpha \to 0$, $\beta \to 0$ and $h = 1$ the critical exponent is $p_c = 1 + \frac{2}{N}$ (see Todorova-Yordanov [16]).

II. Necessary conditions for local and global existence

In this section we will notice that the nonexistence of solutions depends on the behavior of the initial data in infinity

Theorem 2.1: Let $u_0,\ u_1 \in L^\infty(R^N), u_0 \ge 0,\ u_1 \ge 0, 0 < \alpha \le 1, 0 < \beta \le 2$ and $p > 1$. Let u be a local solution to (1) where $T < +\infty$, and $p < \frac{1}{1-\alpha}$. Then, there are positive constants K_1, K_2 such that

$$\lim_{|x|\to+\infty}\inf h^{q-1}u_0(x) \le K_1\left(T^{1+\alpha-2q} + T^{(1+\alpha)(1-q)}\right),$$

$$\lim_{|x|\to+\infty}\inf h^{q-1}u_1(x) \le K_2\left(T^{\alpha-2q} + T^{\alpha-(\alpha+1)q}\right).$$

Proof. We have by the definition of a weak solution

$$\int_{B_R}\varphi(x,0)u_1(x)dx + \int_{B_R}\varphi(x,0)(-\Delta)^{\beta/2}u_0(x)dx + \int_{B_R}u_0(x)D^\alpha_{t|T}\varphi(x,0)dx + \int_{B_R\times(0,T)}u_1 D^\alpha_{t|T}\varphi dxdt \le C\int_{B_R\times(0,T)}(h\varphi)^{\frac{-q}{p}} \left(|\varphi_{tt}|^q + |\Delta\varphi|^q + \left|D^{\alpha+1}_{t|T}\varphi\right|^q + \left|(-\Delta)^{\beta/2}\varphi_t\right|^q\right)dxdt. \quad (10)$$

We consider $\Phi \in H^{\beta}(B_1)$ the first positive eigenfunction of the Dirichlet problem

$$\begin{cases} -\Delta\Phi(x) = k\Phi(x) & x \in B_1, \\ \Phi(x) = 0 & x \in \partial B_1, \end{cases}$$

and satisfies

$$(-\Delta)^{\beta/2}\Phi(x) = k'\Phi(x)$$

where $B_1 = \left\{x \in \mathbb{R}^N, 1 < |x| < 2\right\}$. Now, we define the following function

$$\varphi(x,t) = \Phi\left(\frac{|x|}{R}\right)\left(1 - \frac{t^2}{T^2}\right)^{2q}.$$

Clearly,

$$-\Delta\Phi\left(\frac{|x|}{R}\right) = kR^{-2}\Phi\left(\frac{|x|}{R}\right)$$

and

$$(-\Delta)^{\beta/2}\Phi\left(\frac{|x|}{R}\right) = k^{\beta/2}R^{-\beta}\Phi\left(\frac{x}{R}\right)$$

where φ satisfies the requirements

$$\varphi(x,T) = \varphi_t(x,T) = \varphi_t(x,0) = 0.$$

We remark that

$${}^{RL}D^{\alpha}_{t|T}\varphi = {}^{C}D^{\alpha}_{t|T}\varphi \quad and \quad {}^{RL}D^{\alpha-1}_{t|T}\varphi = {}^{C}D^{\alpha-1}_{t|T}\varphi.$$

Now, we estimate the right hand side in terms of T and R. If we use the change of variables $t = \tau T$, we obtain

$$\int_{B_R\times(0,T)} (h\varphi)^{\frac{-q}{p}} |\varphi_{tt}|^q\,dxdt \leq C_1T^{1-2q}\int_{B_R} h^{1-q}\Phi\left(\tfrac{x}{R}\right)dx, \quad (11)$$

$$\int_{B_R\times(0,T)} (h\varphi)^{\frac{-q}{p}} |\Delta\varphi|^q\,dxdt \leq C_2R^{-2q}T\int_{B_R} h^{1-q}\Phi\left(\frac{x}{R}\right)dx \quad (12)$$

and

$$\int_{B_R\times(0,T)} (h\varphi)^{\frac{-q}{p}} \left|(-\Delta)^{\beta/2}\varphi_t\right|^q dxdt \leq hfill\; C_3T^{1-q}R^{-\beta q}\int_{B_R}\Phi\left(\frac{x}{R}\right)dx, \quad (13)$$

for the term $\int_{B_R\times(0,T)} (h\varphi)^{\frac{-q}{p}} \left|D^{\alpha+1}_{t|T}\varphi\right|^q dxdt$, we need to compute $D^{\alpha+1}_{t|T}\varphi$. We have

$$D^{\alpha+1}_{t|T}\varphi = \frac{1}{\Gamma(1-\alpha)}\int_t^T (\sigma-t)^{-\alpha}\varphi''(\sigma)\,d\sigma,$$

by substituting the expression of φ'', we have

$$D^{\alpha+1}_{t|T}\varphi = \frac{-4q}{T^2\Gamma(1-\alpha)}\times \int_t^T\left[\left(1-\frac{\sigma^2}{T^2}\right)^{2q-1} - 2\frac{\sigma^2}{T^2}(2q-1)\left(1-\frac{\sigma^2}{T^2}\right)^{2q-2}\right]\times (\sigma-t)^{-\alpha}d\sigma$$

then

$$D^{\alpha+1}_{t|T}\varphi = \tfrac{-4qT^{-4q}}{\Gamma(1-\alpha)}\int_t^T \left(T^2-\sigma^2\right)^{2q-1}(\sigma-t)^{-\alpha}d\sigma + \tfrac{8q(2q-1)T^{-4q}}{\Gamma(1-\alpha)}\int_t^T \sigma^2\left(T^2-\sigma^2\right)^{2q-2}(\sigma-t)^{-\alpha}d\sigma \equiv I+J.$$

Using the Euler's change of variable

$$y = \frac{\sigma-t}{T-t} \Rightarrow \sigma - t = (T-t)\,y$$

we see that

$$1-y = \tfrac{T-\sigma}{T-t} \ and \ 1-y^2 = \tfrac{T^2-\sigma^2}{(T-t)^2} - 2t\tfrac{1-y}{T-t},$$
$$T^2-\sigma^2 = \left(1-y^2\right)(T-t)^2 + 2t(1-y)(T-t).$$

Therefore,

$$I = -\tfrac{4qT^{-4q}}{\Gamma(2-\alpha)}(T-t)^{-\alpha+2q}\times \int_0^1 (1-y)^{2q-1}\left((T-t)(1+y)+2t\right)^{2q-1}y^{-\alpha}dy$$

we notice that

$$(T-t)(1+y)+2t = (T+t) + y(T-t)$$

and as

$$y(T-t) < (T-t) \leq (T+t),\, for \quad y < 1,$$

then one can apply the Binomial formula for non integer power to the term$((T-t)(1+y)+2t)^{2q-1}$, we find

$$I = \tfrac{-4qT^{-4q}}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty} C_k^{2q-1}(T-t)^{-\alpha+2q+k}(T+t)^{2q-k-1}\times \int_0^1 (1-y)^{2q-1}y^{-\alpha+k}dy$$

where

$$C_k^{2q-1} = \frac{(2q-1)(2q-2)\times\ldots(2q-k)}{k\times(k-1)\times(k-2)\times\ldots\times 3\times 2\times 1}.$$

Using the Beta formula

$$\int_0^1 (1-\tau)^{u-1}\tau^{v-1}d\tau = B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, u,v>0,$$

we obtain

$$I = \frac{-4qT^{-4q}}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty} C_k^{2q-1}(T-t)^{-\alpha+2q+k}(T+t)^{2q-k-1} \times\frac{\Gamma(2q)\Gamma(1-\alpha+k)}{\Gamma(2q+1-\alpha+k)}.$$

similarly for J

$$J = \tfrac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\int_0^1 (t+(T-t)\,y)^2 (1-y)^{2q-2}\left((T+t)+y(T-t)\right)^{2q-2} (T-t)^{-\alpha+2q}y^{-\alpha}dy$$
$$= \tfrac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty} C_k^{2q-2}\int_0^1 (t+(T-t)\,y)^2 (T+t)^{2q-2-k}(T-t)^{k-\alpha-1+2q} \times(1-y)^{2q-2}y^{k-\alpha}dy,$$

then we have

$$J = \tfrac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)} \sum_{k=0}^{+\infty} C_k^{2q-2}\Big[t^2(T+t)^{2q-2-k}(T-t)^{k-\alpha-1+2q} \int_0^1 (1-y)^{2q-2}y^{k-\alpha}dy + 2t(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q} \times\int_0^1 (1-y)^{2q-2}y^{k+1-\alpha}dy + (T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+1} \times\int_0^1 (1-y)^{2q-2}y^{k+2-\alpha}dy\Big]$$

which is

$$J=\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\Big[t^2\left(T+t\right)^{2q-2-k}$$
$$\left(T-t\right)^{k-\alpha+2q-1}\frac{\Gamma(2q-1)\Gamma(1-\alpha+k)}{\Gamma(2q-\alpha+k)}$$
$$+2t\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q}\frac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}$$
$$+\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q+1}\frac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\Big].$$

Hence

$$D_{t|T}^{\alpha+1}\left(1-\frac{t^2}{T^2}\right)^{2q}=\frac{-4qT^{-4q}}{\Gamma(1-\alpha)}\Big[\sum_{k=0}^{\infty}C_k^{2q-1}$$
$$\left(T-t\right)^{-\alpha+2q+k}$$
$$\times\left(T+t\right)^{2q-k-1}\frac{\Gamma(2q)\Gamma(1-\alpha+k)}{\Gamma(2q+1-\alpha+k)}\Big]+$$
$$\frac{8q(2q-1)T^{-4q}}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\Big[t^2\left(T+t\right)^{2q-2-k}$$
$$\left(T-t\right)^{k-\alpha+2q-1}$$
$$\frac{\Gamma(2q-1)\Gamma(-\alpha+k)}{\Gamma(2q-1-\alpha+k)}$$
$$+2t\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q}\frac{\Gamma(2q-1)\Gamma(1-\alpha+k)}{\Gamma(2q+1-\alpha+k)}$$
$$+\left(T+t\right)^{2q-2-k}\left(T-t\right)^{k-\alpha+2q+1}\frac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\Big].$$

if we set $t=\tau T$ we find

$$D_{t|T}^{\alpha+1}\left(1-\frac{t^2}{T^2}\right)^{2q}=\frac{-4qT^{-\alpha-1}}{\Gamma(2-\alpha)}\Big[\sum_{k=0}^{\infty}C_k^{2q-1}$$
$$\left(1-\tau\right)^{1-\alpha+2q+k}$$
$$\times\left(1+\tau\right)^{2q-k-1}\frac{\Gamma(2q)\Gamma(2-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\Big]+$$
$$\frac{8q(2q-1)}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\Big[\tau^2\left(1+\tau\right)^{2q-2-k}\left(1-\tau\right)^{k-\alpha+2q}$$
$$\frac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}$$
$$+2\tau\left(1+\tau\right)^{2q-2-k}\left(1-\tau\right)^{k-\alpha+2q+1}\frac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}$$
$$+\left(1+\tau\right)^{2q-2-k}\left(1-\tau\right)^{k-\alpha+2q+2}\frac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\Big].$$

Therefore

$$D_{t|T}^{\alpha+1}\left(1-\frac{t^2}{T^2}\right)^{2q}\le C_4T^{-\alpha-1}\tag{14}$$

for some positive constant C_4 depend of q and α.

Using (12) to compute

$$\int_{B_R\times(0,T)}\left(h\varphi\right)^{1-q}D_{t|T}^{\alpha+1}\varphi^q=$$
$$\int_{B_R\times(0,T)}\Phi\left(\frac{x}{R}\right)\left(1-\frac{t^2}{T^2}\right)^{2q(1-q)}D_{t|T}^{\alpha}\varphi^{\frac{p}{p-1}},$$

we have the estimate

$$\int_{B_R\times(0,T)}\left(h\varphi\right)^{-\frac{q}{p}}D_{t|T}^{\alpha+1}\varphi^q\le$$
$$C_4T^{1-(\alpha+1)q}\int_0^1\left(1-\tau\right)^{2q(1-q)}\tau^{2q(1-q)}d\tau\int_{\mathbb{R}^N}\Phi\left(\frac{x}{R}\right)$$

Then we have

$$\int_{B_R\times(0,T)}\left(h\varphi\right)^{-\frac{q}{p}}D_{t|T}^{\alpha+1}\varphi^q\le$$
$$C_5T^{1-(\alpha+1)q}\int_{B_R}\Phi\left(\frac{x}{R}\right)dx,\tag{15}$$

where $C_2=\frac{C_1\Gamma\left(2q\left(1-\frac{p}{p-1}\right)+1\right)^2}{\Gamma(2-\alpha)\Gamma\left(4q\left(1-\frac{p}{p-1}\right)+2\right)}$

By the same method we compute $D_{t|T}^{\alpha}\left(1-\frac{t^2}{T^2}\right)^{2q}$ to find

$$D_{t|T}^{\alpha}\left(1-\frac{t^2}{T^2}\right)^{2q}=\frac{4qT^{-4q}}{\Gamma(2-\alpha)}\Big[\left(T-t\right)^{2q-\alpha+1}$$
$$\sum_{k=0}^{\infty}C_{2q-1}^{k}\left(T+t\right)^{2q-1-k}\left(T-t\right)^k$$
$$\times\int_0^1y^{k+3-\alpha}\left(1-y\right)^{2q-1}dy+\left(T-t\right)^{2q-\alpha}\tag{16}$$
$$t\sum_{k=0}^{\infty}C_{2q-1}^{k}\left(T+t\right)^{2q-1-k}\left(T-t\right)^k$$
$$\int_0^1y^{k+2-\alpha}\left(1-y\right)^{2q-1}dydt$$

In particular we have

$$D_{t|T}^{\alpha}\varphi\left(0\right)=\frac{4qT^{-\alpha}}{\Gamma\left(1-\alpha\right)}\sum_{k=0}^{\infty}C_{2q-1}^{k}\frac{\Gamma\left(k+2-\alpha\right)\Gamma\left(2q\right)}{\Gamma\left(k+2-\alpha+2q\right)}.\tag{17}$$

Then, we have

$$\int_{B_R}u_0D_{t|T}^{\alpha}\xi\left(0\right)=C_6T^{-\alpha}\int_{B_R}u_0\Phi\left(\frac{x}{R}\right)dx.\tag{18}$$

On the other hand (14) implies that

$$\int_{B_R\times(0,T)}u_1D_{t|T}^{\alpha}\varphi=$$
$$\frac{4qT^{-4q}}{\Gamma(2-\alpha)}\int_{B_R}u_1\left(x\right)\Phi\left(\frac{x}{R}\right)\int_0^T[\left(T-t\right)^{2q-\alpha+1}$$
$$\times\sum_{k=0}^{\infty}C_{2q-1}^{k}\left(T+t\right)^{2q-1-k}\left(T-t\right)^k$$
$$\int_0^1y^{k+1-\alpha}\left(1-y\right)^{2q-1}dy$$
$$+\left(T-t\right)^{2q-\alpha}t\sum_{k=0}^{\infty}C_{2q-1}^{k}\left(T+t\right)^{2q-1-k}\left(T-t\right)^k$$
$$\int_0^1y^{k-\alpha}\left(1-y\right)^{2q-1}dy]dt$$

hence

$$\int_{B_R\times(0,T)}u_1D_{t|T}^{\alpha}\varphi=\frac{C_3T^{-\alpha+1}}{\Gamma(1-\alpha)}\int_{B_R}u_1\left(x\right)\Phi\left(\frac{x}{R}\right).\tag{19}$$

Gathering all the estimates (12)-(19) together with (2), we find

$$C_1T^{-\alpha}\int_{B_R}u_0\Phi\left(\frac{x}{R}\right)dx+C_2T^{-\alpha+1}\int_{B_R}u_1\left(x\right)\Phi\left(\frac{x}{R}\right)\le$$
$$[C_3T^{1-2q}+C_4R^{-2q}T+C_5T^{1-(\alpha+1)q}+C_6T^{1-q}R^{-\beta q}]\times$$
$$\int_{B_R}h^{1-q}\Phi\left(\frac{x}{R}\right)dx,\tag{20}$$

for some positive constants $C_i, i=1,...,5$. On the other hand we have

$$\int_{B_R}u_0\Phi\left(\frac{x}{R}\right)\ge\int_{\frac{R}{2}<|x|<R}u_0\Phi\left(\frac{x}{R}\right)\ge$$
$$\inf_{|x|>\frac{R}{2}}h^{q-1}u_0\left(x\right)\int_{B_R}h^{1-q}\Phi\left(\frac{x}{R}\right)dx$$
$$\int_{B_R}u_1\Phi\left(\frac{x}{R}\right)\ge\int_{\frac{R}{2}<|x|<R}u_1\Phi\left(\frac{x}{R}\right)\ge\tag{21}$$
$$\inf_{|x|>\frac{R}{2}}h^{q-1}u_1\left(x\right)\int_{B_R}h^{1-q}\Phi\left(\frac{x}{R}\right)dx,$$

Combining (18) and (19), we find

$$C_1T^{-\alpha}\inf_{|x|>R}h^{q-1}u_0\left(x\right)\int_{B_R}h^{1-q}\Phi\left(\frac{x}{R}\right)dx+$$
$$C_2T^{-\alpha+1}\int_{B_R}u_1\left(x\right)\Phi\left(\frac{x}{R}\right)\le$$
$$\left(C_3T^{1-2q}+C_4R^{-2q}T+C_5T^{1-(\alpha+1)q}+C_6T^{1-q}R^{-\beta q}\right)$$
$$\times\int_{B_R}h\Phi\left(\frac{x}{R}\right)dx,\tag{22}$$

dividing both sides of (21) by $\int_{B_R}h^{1-q}\Phi\left(\frac{x}{R}\right)dx>0$, we infer that

$$\left(C_1T^{-\alpha}\inf_{|x|>R}h^{q-1}u_0\left(x\right)+T^{-\alpha+1}C_2\inf_{|x|>R}h^{q-1}u_1\left(x\right)\right)$$
$$\le C_3T^{1-2q}+C_4R^{-2q}T+C_5T^{1-(\alpha+1)q}+$$
$$C_6T^{1-q}R^{-\beta q}.\tag{23}$$

Finally, passing to the limit $R \to +\infty$, we obtain

$$\begin{array}{l} C_1 T^{-\alpha} \lim_{|x| \to +\infty} \inf h^{q-1} u_0(x) + \\ C_2 T^{-\alpha+1} \lim_{|x| \to +\infty} \inf h^{q-1} u_1(x) \\ \leq C_3 T^{1-2q} + C_5 T^{1-(\alpha+1)q}, \end{array} \tag{24}$$

it follows from (24) that

$$\begin{array}{l} \lim_{|x| \to +\infty} \inf h^{q-1} u_0(x) \leq \\ C_6 \left(T^{1+\alpha-2q} + T^{(1+\alpha)(1-q)}\right), \end{array}$$

and

$$\begin{array}{l} \lim_{|x| \to +\infty} \inf h^{q-1} u_1(x) \leq \\ C_7 \left(T^{\alpha-2q} + T^{\alpha-(\alpha+1)q}\right) \end{array}$$

Theorem 2.2: Suppose the problem (1) has a nontrivial global weak solution. Then, there are positive constants K_1, and K_2such that

$$\lim_{|x| \to +\infty} \inf \left(|x|^{2q-2(1+\alpha)} h^{q-1} u_0(x)\right) \leq K_1,$$

and

$$\lim_{|x| \to +\infty} \inf \left(|x|^{2(q-\alpha)} h^{q-1} u_1(x)\right) \leq K_2,$$

provided that $q > \alpha + 1$.

Proof. From the relation (18), we infer that

$$\begin{array}{l} C_1 \inf_{|x|>R} u_0(x) \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx \leq (C_3 T^{\alpha+1-2q} + \\ C_4 R^{-2q} T^{1+\alpha} + C_5 T^{(1+\alpha)(1-q)} + \\ C_6 T^{1+\alpha-q} R^{-\beta q}) \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx, \\ \leq (C_4 R^{-2q} T^{1+\alpha} + \left(C_8 + C_6 R^{-\beta q}\right) T^{1+\alpha-q}) \times \\ \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx, \end{array} \tag{25}$$

we notice that $\left(C_8 + C_6 R^{-\beta q}\right)$ is bounded, then by taking $T = R^2$, we find

$$\begin{array}{l} C_1 \inf_{|x|>R} h^{q-1} u_0(x) \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx \leq \\ \int_{B_R} u_0(x) \Phi\left(\frac{x}{R}\right) dy \leq \\ M R^{-2q+2(1+\alpha)} \int_{B_1} h^{1-q}(Ry) \Phi(y) dy \end{array}$$

Using the hypothesis on the support of Φ, we get

$$\begin{array}{l} C_1 \inf_{|x|>R} \left(u_0(x) |x|^{2q-2(1+\alpha)} h^{q-1}\right) \\ \int_{B_1} h^{1-q}(Ry) |y|^{-2q+2(1+\alpha)} \Phi(y) dx \leq \\ M R^{2q-2(1+\alpha)} \int_{B_1} |x|^{-2q+2(1+\alpha)} h^{1-q} \Phi\left(\frac{x}{R}\right) dx, \end{array} \tag{26}$$

Next, taking the sup with respect to R of the both sides of (24) and dividing by

$$\int_{B_R} |x|^{-2q+2(1+\alpha)} h^{1-q} \Phi\left(\frac{x}{R}\right) dx > 0$$

we have

$$\lim_{|x| \to +\infty} \inf \left(u_0(x) h^{q-1} |x|^{2q-2(1+\alpha)}\right) \leq K_1$$

and

$$\begin{array}{l} C_2 \int_{B_R} u_1(x) \Phi\left(\frac{x}{R}\right) \leq (C_3 T^{\alpha-2q} + C_4 R^{-2q} T^{\alpha} + \\ C_5 T^{\alpha-(\alpha+1)q} + C_6 T^{\alpha-q} R^{-\beta q}) \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx, \end{array} \tag{27}$$

then

$$\begin{array}{l} C_2 \int_{B_R} u_1(x) \Phi\left(\frac{x}{R}\right) \leq ((c_3 + c_5 + C_6 R^{-\beta q}) T^{\alpha-q} \\ \quad + C_4 R^{-2q} T^{\alpha}) \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx. \end{array} \tag{28}$$

Now, taking as before $T = R^2$, we realize

$$\begin{array}{c} C_2 \int_{B_R} u_1(x) \Phi\left(\frac{x}{R}\right) \leq \\ (c_3 + c_5 + C_4 + C_6 R^{-\beta q}) R^{2(\alpha-q)} \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx, \end{array}$$

Thus

$$\int_{B_R} u_1(x) \Phi\left(\frac{x}{R}\right) \leq C R^{2(\alpha-q)} \int_{B_R} h^{1-q} \Phi\left(\frac{x}{R}\right) dx,$$

arguing as above

$$\begin{array}{c} \left(\inf_{|x|>R} |x|^{2(q-\alpha)} h^{q-1} u_1(x)\right) \int_{B_R} |x|^{2(\alpha-q)} h^{1-q} \Phi\left(\frac{x}{R}\right) \leq \\ C R^{2(\alpha-q)} \int_{B_R} h^{1-q} |x|^{2(\alpha-q)} \Phi\left(\frac{x}{R}\right) dx, \end{array}$$

The conclusion follows by dividing by $\int_{B_R} h^{1-q} |x|^{2(\alpha-q)} \Phi\left(\frac{x}{R}\right)$, and passing to the limit when $R \to +\infty$.

III. Conclusion

I wish that the present manuscript will open wide avenues for further research in the field of Fractional derivative and other domains.

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