Three-fluid System Short-wave Instability and Gap-solitons

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Abstract—**The short-wave instability arising due to the resonace between two wave modes for a three-fluid system of a stratified and sheer flow is addressed. A weakly nonlinear analysis leads to a class of solitary waves, widely known as gap-solitons in other physical contexts. The essential ingredients are the existence of a spectral gap between two branches of the dispersion relation, and the development of a set of envelope equations to describe weakly nonlinear waves, whose carrier frequency and wavenumber belong to the centre of this gap. For the special case where the gap-soliton is a steady travelling wave of the full fluid system, we show that there is large class of such gap-solitons.**

Keywords: envelope soliraty waves, gap-solitons, nonlinear waves, stratified fluid flow

1 Introduction

In inviscid fluid flows instability generically arises due to a resonance between two wave modes. That is, as an appropriate parameter is varied, the phase speeds of the two waves coincide for some critical parameter value. A generic unfolding of this resonance yields either a stable "kissing" configuration, or "bubble" of instability, in the frequency-wavenumber space. Many illustrations of this concept are reported in [1] for shear flows, while a discussion on the physical processes involved can be found in [2].

The origin of the concept lies in the Hamiltonian structure of inviscid fluid flows. Indeed in a finite-dimensional Hamiltonian dynamical system, it is well-known that linearization about a steady state and a subsequent search for eigenfrequencies ω (i.e. a search for solutions proportional to $exp(-i\omega t)$ will generically lead to sets of quartets $\omega, -\omega^*; \omega^*, -\omega$. Here for a given eigenfrequency ω, $-\omega^*$ is also an eigenfrequency due to the real-valued nature of the system, while the pair ω^*, ω follow from the Hamiltonian structure. Instability occurs if $\mathrm{Im}\omega \neq 0$. Consider the situation as an appropriate parameter is varied. When stable, all the eigenfrequencies must lie on the real axis, with one eigenfrequency pair lying on the positive real axis, and the second pair being its mirror image on the negative real axis. For instability to occur as the external parameter is varied, the eigenfrequency pair on the positive real axis must come into coincidence, while the mirrorrimage pair on the negative real axis will do likewise. Further variation of the parameter which leads to instability will then cause the eigenfrequencies to split apart and move off the real axis, one member lying above the real axis, and the second member being its complex conjugate. The situation is sketched in figure 1.

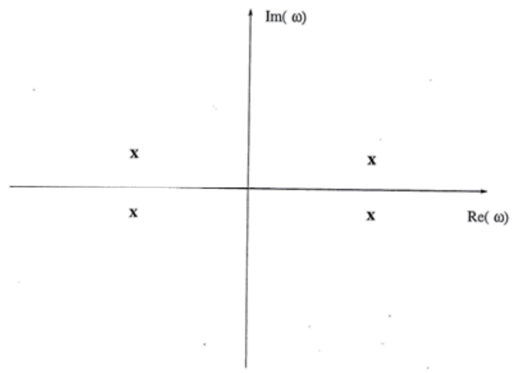

Figure 1: A sketch of the typical configuration of eigenfrequencies x in a Hamiltonian dynamical system.

The inference from this generic situation for fluid flows leads to the concept sketched in figure 2, where we plot the wave frequency ω as a function of the wavenumber k, while the unfolding parameter is δ. The further exploration of the implication for fluid flows depends on whether one is considering long waves, or short waves. Here we consider short waves, with analogous development for long waves described in [3] for a stratified shear flows, and for two-layer quasigeostrophic flows in [4, 5].

Exploring the generic weakly nonlinear unfolding of the basic resonance for short waves leads to the following system of evolution equations (see,[6]) for amplitudes A, B and group velocity c_g, for appropriate parameters

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$\gamma_1, \gamma_2, \nu, \mu, \tau$.

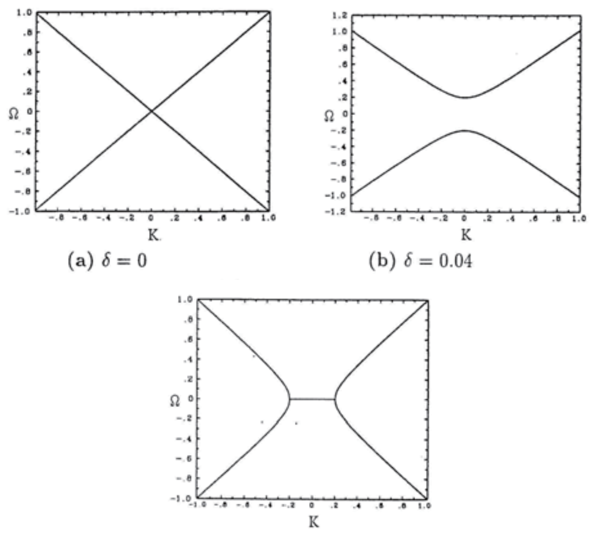
$$i(A_T + c_g A_X) + \gamma_1 B + \gamma_1\tau(|A|^2 + \nu|B|^2)A + \gamma_1\mu B^2 A * + \ldots, \qquad (1)$$

$$i(B_T - c_g B_X) + \gamma_2 A + \gamma_2\tau(|B|^2 + \nu|A|^2)B + \gamma_2\mu A^2 B * + \ldots, \qquad (2)$$

In general, the system (1)−(2) may also contain mean flow terms (see [6]). Note that the assumed symmetry in the nonlinear terms in system (1)−(2) ensures that there is a Hamiltonian structure of the form

$$i\gamma_2 A_T = -\frac{\delta H}{\delta A^*}, \quad i\gamma_1 B_T = -\frac{\delta H}{\delta B^*} \qquad (3)$$

where the Hamiltonian H is a conserved quantity and can be found in [6].



(a) $\delta = 0$ (b) $\delta = 0.04$ (c) $\delta = -0.04$

Figure 2: A schematic sketch of the dispersion relation for mode resonance, where is the difference between frequency and resonant frequency, K the difference between wavenumber and resonant wavenumber and is the unfolding parameter. (a) uncoupled case, (b) stable case, (c) unstable case.

In the stable case ($\gamma_1\gamma_2 > 0$) the system (1)−(2) can support gap solitons (see [7]), that is, envelope solitary waves, whose speeds and frequencies are such that they lie in the gap in the linear spectrum (see figure 2b). In the unstable case ($\gamma_1\gamma_2 < 0$) two basic scenarios could be expected, depending on the relative coefficients of the nonlinear terms (see [8]). In one case, solutions develop a singularity in finite time, while in the other case, the solutions evolve into successively finer temporal and spatial structures, due to modulational instability.

In section II, we consider the specific case of a three-layered stratified shear flow that leads to a set of coupled evolution equations confirming the theoretical results regarding short-wave instability above. In section III we discuss traveling waves solutions in the form of gap-solitons based on the obtained evolution equations of section III.

2 Formulation and weakly nonlinear analysis

We consider a fluid composed of three layers, each of constant density ρ_i and with a basic constant horizontal velocity U_i $(i = 1, 2, 3)$, as shown in figure 3. The two interfaces are described by $z = \eta(x, t)$ and $z = H_2 + \zeta(x, t)$. We assume that the fluid is inviscid and incompressible and that the flow in each layer is irrotational and two-dimensional, with velocity potentials ϕ_i $(i = 1, 2, 3)$.

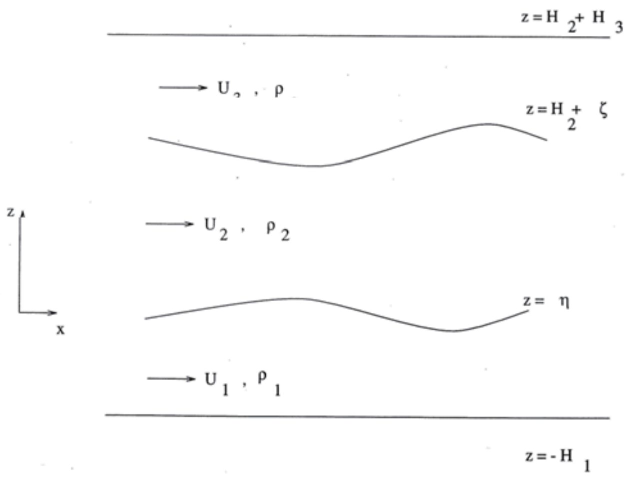


Figure 3: A sketch of the coordinate system for a three-layered fluid.

The governing equations are then given by

$$\phi_{ixx} + \phi_{izz} = 0\,, \quad (i = 1, 2, 3) \qquad (4)$$

subject to the boundary conditions

$$\phi_{1z} = 0\,, \quad (z = -H_1) \qquad (5)$$

$$\phi_{3z} = 0\,, \quad (z = H_2 + H_3) \qquad (6)$$

where $w_i = \phi_{iz}$ are the vertical perturbation velocities in each layer.

At the two interfaces the kinematic conditions are given respectively by

$$\eta_t + (U_i + u_i)\eta_x = w_i\,, \quad (i = 1, 2) \quad \text{at } z = \eta\,, \qquad (7)$$

$$\zeta_t + (U_i + u_i)\zeta_x = w_i\,, \quad (i = 2, 3) \quad \text{at } z = H_2 + \zeta\,, \qquad (8)$$

while the dynamic conditions are given respectively by

$$\rho_1(\phi_{1t} + U_1\phi_{1x} + \frac{1}{2}|\nabla\phi_1|^2 + g\eta) - \rho_2(\phi_{2t} + U_2\phi_{2x} + \frac{1}{2}|\nabla\phi_2|^2 + g\eta) = \frac{\sigma\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\,, \qquad (9)$$

$$\rho_2(\phi_{2t} + U_2\phi_{2x} + \frac{1}{2}|\nabla\phi_2|^2 + g\zeta) - \rho_3(\phi_{3t} + U_3\phi_{3x} + \frac{1}{2}|\nabla\phi_3|^2 + g\zeta) = \frac{\sigma''\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}}\,, \qquad (10)$$

where $u_i = \phi_{ix}$ are the horizontal perturbation velocities in each layer and σ, σ' are the surface tension coefficients at each interface.

2.1 Linear analysis

Considering the linearized version of system (4)−(10) above and seeking solutions of the form

$$\eta = Ae^{i\theta} + c.c.\,, \quad \zeta = Be^{i\theta} + c.c.\,, \tag{11}$$

where $\theta = kx - \omega t$ and $c.c.$ denotes the complex conjugate, it can be established (see [6]) that

$$D_1 A + EB = 0\,, \qquad EA + D_3 B = 0\,, \tag{12}$$

$$D_1(\omega, k) =$$
$$g(\rho_1 - \rho_2) + \sigma k^3 - \{\frac{\rho_1(\omega - kU_1)^2}{kT_1} + \frac{\rho_2(\omega - kU_2)^2}{kT_2}\}\,,$$
$$D_3(\omega, k) =$$
$$g(\rho_2 - \rho_3) + \sigma' k^3 - \{\frac{\rho_2(\omega - kU_2)^2}{kT_2} + \frac{\rho_3(\omega - kU_3)^2}{kT_3}\}\,,$$
$$E(\omega, k) = \frac{\rho_2(\omega - kU_2)^2}{kS_2}\,,$$

where $T_i = \tanh kH_i$ and $S_i = \sinh kH_1$ for $i = 1, 2, 3$. The dispersion relation is the given by

$$D_1 D_3 - E^2 = 0\,. \tag{13}$$

Instability occurs for the zeros of (13) with $\mathrm{Im}\omega \neq 0$, with a two-mode resonance instability occuring for D_1, D_2 and E independently equal to 0. This is the case when

$$\omega = kU_2\,, \tag{14}$$

$$g(\rho_1 - \rho_2) + \sigma k^2 = \frac{\rho_1 k(U_2 - U_1)^2}{T_1}\,, \tag{15}$$

$$g(\rho_2 - \rho_3) + \sigma' k^2 = \frac{\rho_3 k(U_2 - U_3)^2}{T_3}\,. \tag{16}$$

The condition (14) states that the phase speed $c = \omega/k = U_2$, while the conditions (15) and (16) imply that the phase speed of a wave on the lower (upper) interface, considered independently of the upper (lower) interface, is also just U_2. Without loss of generality we can set $U_2 = 0$ and choose $U_3 > 0$. Then the resonance conditions (14)−(16) reduce to $\omega = 0$ and

$$U_1 = \mp\{\frac{T_1}{k}\frac{g(\rho_1 - \rho_2)}{\rho_1} + \frac{\sigma k^2}{\rho_1}\}^{1/2}\,, \tag{17}$$

$$U_3 = \{\frac{T_3}{k}\frac{g(\rho_2 - \rho_3)}{\rho_3} + \frac{\sigma' k^2}{\rho_3}\}^{1/2}\,. \tag{18}$$

The choice of sign in (17) corresponds to a shear flow ($U_1 < 0$) or a jet flow ($U_1 > 0$) respectively. For a given wavenumber k, the conditions (17−(18) define the basic velocities for which a resonance can occur. Alternatively, elimination of k defines a functional relationship between U_1, U_3 for which a resonance of this type can occur.

We next let A, B in (11) depend on slow variables $X = \alpha^2 x$ and $T = \alpha^2 t$ and expand the dispersion relation (13) around resonant ω, k. We then readily obtain (at resonance)

$$D_{1\omega} = \frac{2\rho_1 U_1}{T_1}\,, \qquad D_{3\omega} = \frac{2\rho_3 U_3}{T_3}\,, \tag{19}$$

$$V_1 = \frac{1}{2}U_1\{1 - \frac{kH_1T_1}{S_1^2}\} - \frac{\sigma k T_1}{\rho_1 U_1}\,, \tag{20}$$

$$V_3 = \frac{1}{2}U_3\{1 - \frac{kH_3T_3}{S_3^2}\} - \frac{\sigma' k T_3}{\rho_3 U_3}\,, \tag{21}$$

where $D_{1,3\omega}$ are the derivatives of $D_{1,3}$ w.r.t. ω, and V_1, V_3 are the corresponding group velocities. In general $V_1 \neq V_3$, an assumption that fails to hold only for very special parameter values. Note that the resonance generates an instability when $D_{1\omega}D_{3\omega} < 0$, which corresponds to the shear-flow case. This instability can, perhaps rather loosely, be called a "Holmboe" instability. In contrast, it is also true that the flow configuration described in figure 3 can support a Kelvin-Helmholtz instability. For instance, $D_1 = D_{1\omega} = D_{1k} = E = 0$ simoultaneously and $D_3 \neq 0$ describes a situation in which a Kelvin-Helmholtz instability occurs on the lower interface, which is only weakly coupled to the upper interface. Note that unlike the "Holmboe" instability, this situation can only be realized by letting the limit $H_2 \to 0$ be the condition leading to $E = 0$.

2.2 Nonlinear analysis

Next we move to the weakly nonlinear analysis by replacing the linear solution (11) with

$$\eta = \alpha A(X,T)e^{i\theta} + \alpha^2 A_2(X,T)e^{2i\theta} + c.c. + \alpha^2\bar{\eta}(X,T) + O(\alpha^3)\,, \tag{22}$$

$$\zeta = \alpha B(X,T)e^{i\theta} + \alpha^2 B_2(X,T)e^{2i\theta} + c.c. + \alpha^2\bar{\zeta}(X,T) + O(\alpha^3)\,. \tag{23}$$

The analysis at leading order $O(\alpha)$ recovers the results of the linear case above. The second harmonic terms A_2, B_2 and the mean-flow terms $\bar{\eta}, \bar{\zeta}$ are then determined by substituting the expansions (22)−(23) into the fully nonlinear set of equations, and examining the terms of $O(\alpha^2)$. The analysis for the second harmonic terms results in obtaining

$$D_1(2\omega, 2k)\eta_2 = -\frac{1}{2}k^2A^2\rho_1U_1^2\frac{1}{T_1^2} + \frac{4}{T_1T_{12}} - 1\,, \tag{24}$$

$$D_3(2\omega, 2k)\zeta_2 = -\frac{1}{2}k^2B^2\rho_3U_3^2\frac{1}{T_3^2} + \frac{4}{T_1T_{32}} - 1\,. \tag{25}$$

where $T_{12}, T_{32} = \tanh 2kH_1, \tanh 2kH_3$, respectively. The expressions above are simply the linear dispersion operators for each interface considered separately, but evaluated at $(2\omega, 2k)$ instead of (ω, k). Assuming that there is no second harmonic resonance, then these are not 0, and hence they uniquely define η_2, ζ_2.

The equations governing the mean flow expressions $\bar{\eta}, \bar{\zeta}$ are most readily obtained by averaging the fully nonlinear system of equations (4)−(10) over the phase θ. Upon introduction of mean Eulerian horizontal velocity fields $\alpha^2 \bar{u}_i{}^2$ $(i = 1, 2, 3)$, we obtain the following system of mean-flow equations by averaging the kinematic boundary conditions at each interface, with the variable C being related to the mean vertical velocity in the middle layer

$$\bar{\eta}_T + U_1 \bar{\eta}_X + H_1 \bar{u}_{1X} = \frac{2kU_1}{T_1} |A|^2_X \,, \tag{26}$$

$$\bar{\eta}_T - C = 0 \,, \tag{27}$$

$$\bar{\zeta}_T + H_2 \bar{u}_{2X} - C = 0 \,, \tag{28}$$

$$\bar{\zeta}_T + U_3 \bar{\zeta}_X + H_3 \bar{u}_{3X} = -\frac{2kU_3}{T_3} |B|^2_X \,, \tag{29}$$

The system is closed by averaging the pressure boundary conditions at each interface to get

$$\rho_1(\bar{u}_{1T} + U_1 \bar{u}_{1X}) - \rho_2 \bar{u}_{2T} + g(\rho_1 - \rho_2)\bar{\eta}_X = -\rho_1 U_1^2 k^2 \frac{1 - T_1^2}{T_1^2} |A|^2_X, \tag{30}$$

$$\rho_2 \bar{u}_{2T} - \rho_3(\bar{u}_{3T} + U_3 \bar{u}_{3X}) + g(\rho_2 - \rho_3)\bar{\zeta}_X = \rho_3 U_3^2 k^2 \frac{1 - T_3^2}{T_3^2} |B|^2_X \,. \tag{31}$$

Finally, proceeding to the third order $O(\alpha^3)$ in the expansion (see [6] for all details), we obtain a coupled system of evolution equations, given by

$$iD_{1\omega}(A_T + V_1 A_X) + \gamma B + \mu_1 |A|^2 A + \nu_1 A \bar{\eta} - kD_{1\omega} A \bar{u}_1 = 0 \,, \tag{32}$$

$$iD_{3\omega}(B_T + V_3 B_X) + \gamma A + \mu_3 |B|^2 B + \nu_3 B \bar{\zeta} - kD_{3\omega} B \bar{u}_3 = 0 \,, \tag{33}$$

Here, the parameter $\gamma > 0$ is a tuning parameter for the resonance, formally defined by $E = \alpha^2 \gamma$, so that $\omega = \alpha\Omega$, where $\gamma = \rho_2 \Omega^2 / kS_2$. The remaining coefficients can be found in [9]. Equations (32)−(33) are the counterpart of the schematic equations (1)−(2) presented in section I, but coupled to the set of mean flow equations (26)−(31).

A preliminary qualitative analysis shows that one can obtain steady traveling wave solutions of this equation set, analogous to the gap-soliton solutions found in [7] for a model problem. This is presented in the next section. Moreover, in the linearly unstable case (see [8]) a system of the form (1)−(2) can support collapsing solutions if the coefficients of the out-of-phase (such as μ) nonlinear terms are large enough compared to the coefficients of the in-phase nonlinear terms. Otherwise, modulational instability of plane wave solutions leading to spatio-temporal chaos would be the dominant scenario. Because the present system (26)−(33) apparently has no nonlinear out-of-phase terms, we would expect that there are no collapsing solutions, but that the solutions might exhibit spatio-temporal chaos. Finally it can be shown that the system (26)−(33) is Hamiltonian (see [6] for an explicit demonstration).

3 Gap solitons

We then seek traveling wave solutions of (32)−(33) of the form

$$A(X, T) = R(\xi) e^{i\phi(\xi)} e^{-i\Omega T} \,, \tag{34}$$

$$B(X, T) = S(\xi) e^{i\psi(\xi)} e^{-i\Omega T}, \tag{35}$$

$$\bar{m}(X, T) = \bar{m}(\xi), \tag{36}$$

where $\xi = X - VT$, R, S are real-valued amplitudes and ϕ, ψ are real-valued phases, while $\bar{m}$ denotes the mean-flow variables. We shall assume further that $R, S \to 0$ as $\xi \to \pm\infty$, so that these will be solitary waves, that is gap-solitons.

Substitution of (34)−(36) into the mean flow equations (26)−(31) yields a system of ordinary differential equations whose solution is

$$\bar{\eta} = \eta_A |A|^2 + \eta_B |B|^2 \,, \tag{37}$$

$$\bar{\zeta} = \zeta_A |A|^2 + \zeta_B |B|^2 \,, \tag{38}$$

$$\bar{u}_1 = u_{1A} |A|^2 + u_{1B} |B|^2, \tag{39}$$

$$\bar{u}_3 = u_{3A} |A|^2 + u_{3B} |B|^2, \tag{40}$$

where the coefficients can be found in [9]. Then substitution of (34)−36) into (32−(33) yields

$$iD_{1\omega}(V_1 - V)R' - D_{1\omega}(V_1 - V)\phi' R + (\mu_1 + \nu_1 \eta_A - kD_{1\omega} u_{1A})|R|^2 R + (-\nu_1 \eta_B - kD_{1\omega} u_{1B})|S|^2 R + \Omega D_{1\omega} R = -\gamma S e^{i(\psi - \phi)} \,, \tag{41}$$

$$iD_{3\omega}(V_3 - V)S' - D_{3\omega}(V_3 - V)\psi' S + (\mu_3 + \nu_3 \zeta_B - kD_{3\omega} u_{3B})|S|^2 S + (-\nu_3 \zeta_A - kD_{3\omega} u_{3A})|R|^2 S + \Omega D_{3\omega} S = -\gamma R e^{i(\phi - \psi)} \,. \tag{42}$$

An analysis of the imaginary and the real parts of system (41)−(42), yields the following relation between amplitudes R and S,

$$D_{1\omega}(V_1 - V)R^2 = -D_{3\omega}(V_3 - V)S^2 \,, \tag{43}$$

where for a solution to exist we must satisfy

$$D_{1\omega}(V_1 - V)D_{3\omega}(V - V_3) > 0 \,. \tag{44}$$

Eventually (for a complete analysis, see [9]), we obtain a solitary wave solution of the form

$$WR^2 = -(\tilde{\gamma} - \tilde{\Omega}) \frac{2\mathrm{sech}^2(K\xi)}{1 + \beta^2 \tanh^2(K\xi)} \,, \quad \text{for} \quad \tilde{W} < 0 \,, \tag{45}$$

$$WR^2 = (\tilde{\gamma} - \tilde{\Omega}) \frac{2\mathrm{sech}^2(K\xi)}{\tanh^2(K\xi) + \beta^2} \,, \quad \text{for} \quad \tilde{W} > 0 \,, \tag{46}$$

where,

$$\tilde{\gamma} = \pm \frac{2\gamma}{\sqrt{D_{1\omega}(V_1 - V) D_{3\omega}(V - V_3)}} ,$$

$$\tilde{\Omega} = \Omega \frac{V_3 - V_1}{(V_1 - V)(V - V_3)} .$$

$$\beta^2 = \frac{\tilde{\gamma} - \tilde{\Omega}}{\tilde{\gamma} + \tilde{\Omega}} \quad \text{and} \quad K = -\frac{1}{2}\beta(\tilde{\gamma} + \tilde{\Omega}) .$$

and $\pm$ corresponds to sign$\{D_{1\omega}(V_1 - V)\}$. Note that W changes sign with sign$\{D_{1\omega}(V_1 - V)\}$ and so it is useful to write $\tilde{W} = \pm W$ accordingly.

In each case there are two solutions for R, identical except for the sign, so that one may be interpreted as an elevation wave (say $R > 0$) and the other is then a depression wave ($R < 0$). With R known we can determine S, where the sign of S is required to be the same as that of R. It turns out that in the "+"-case, $\tilde{W} < 0$, $\Phi = 0$ at the wave crest where $\xi = 0$, and so the two interfaces are in phase. In the opposite "−"-case, $\tilde{W} > 0$, $\Phi = \pm\pi$ at the wave crest, and the two interfaces are now out of phase.

In general, this solution contains three free parameters, V, Ω, γ and although these are constrained by (44) for V and by the requirement that $\tilde{\gamma}^2 > \tilde{\Omega}^2$, there still remains a very large parameter space, in addition to the system parameters (the basic fluid densities, flow velocities and layer depths).

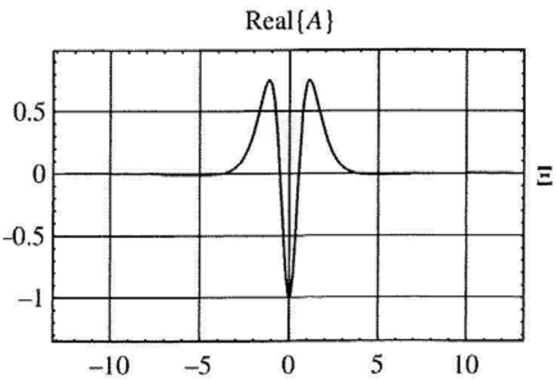


Figure 4: Steady gap-soliton for normalized and dimensionless amplitude and ξ (Ξ), dimensionless $\sigma = 1.50$ and $\sigma' = 0.40$, for $k = 1m^{-1}$, $H_2 = 50m$, $H_1 = H_3 = 500m$, on the lower interface.

4 Conclusion

The analysis presented here has yielded a solitary wave solution that, despite its large parameter space, can lead to a reduced class of steady solutions corresponding to $V = 0$, $\Omega = 0$. This has been done in [9], where one can obtain gap-solitons on both interfaces either in-phase or out-of-phase. In figures 4 and 5 we demonstrate a case of out-of-phase waves. Although the existence of gap-solitons is well-known in other branches of phsyics, such as nonlinear optics (see [10], for instance), this was the first time they have been found in a fluid flow.

The natural next step toward exploring the short-wave instability discussed here would be the rigorous study of the fully nonlinear system (4)−(10) through a dynamical systems approach, similar to the one used in [11] for a coupled Korteweg-de Vries system.

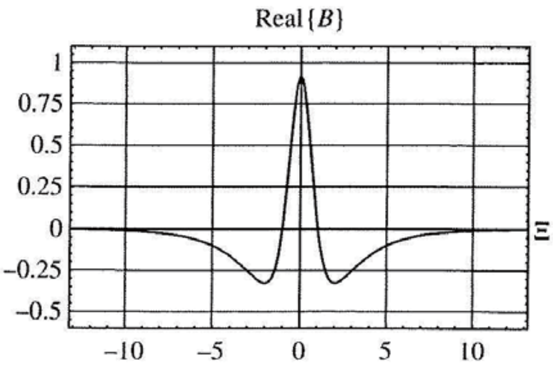


Figure 5: As for figure 4, on the upper interface.

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