Common Fixed Points of Quasi-contractive Type Operators by a Generalized Iterative Process

Safeer Hussain Khan

***Abstract*— Common fixed points of two quasi-contractive type operators are approximated by using a three-steps iterative process in the setting of a normed space. This three-steps iterative process contains a number of existing iterative processes. Our main theorem generalizes and improves upon, among others, the corresponding results of Berinde, Khan, and those generalized therein.**

***Index Terms*— Three-steps iterative process, Quasi-contractive type operator, Common fixed point, Strong convergence.**

I. Introduction and Preliminaries

THROUGHOUT this paper, N denotes the set of all positive integers. Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be a mapping. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be appropriately chosen sequences in $[0,1]$.

We know that Picard iterative process or successive iterative method is defined by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tx_n, \quad n \in \mathrm{N} \end{cases} \tag{1}$$

The Mann iterative process [11] is defined by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n Tx_n, n \in \mathrm{N}. \end{cases} \tag{2}$$

The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n Ty_n, \\ y_n = (1-b_n)x_n + b_n Tx_n, \ n \in \mathrm{N} \end{cases} \tag{3}$$

is known as the Ishikawa iterative process [6].

We know that Ishikawa iterative process is a two-steps process. In 2000, Noor [12] introduced a three-steps iterative process to approximate solutions of variational inclusions in Hilbert spaces. Glowinski and Le Tallec [5] applied a three-steps iterative process for finding the approximate solution of the elastoviscoplasticity problem, eigen value problem and liquid crystal theory.

In 2002, Xu and Noor [16] introduced the following extension of the above Ishikawa iterative process:

$$\begin{cases} x_1 = x \in C, \\ z_n = (1-a_n)x_n + a_n Tx_n \\ y_n = (1-b_n)x_n + b_n Tz_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \ n \in \mathrm{N} \end{cases} \tag{4}$$

They used it to approximate fixed points in a uniformly convex Banach space.

Suantai [14] introduced the following iterative process:

$$\begin{cases} x_1 = x \in C, \\ z_n = (1-a_n)x_n + a_n Tx_n \\ y_n = b_n Tz_n + c_n Tx_n + (1-b_n-c_n)x_n \\ x_{n+1} = \alpha_n Ty_n + \beta_n Tz_n + (1-\alpha_n-\beta_n)x_n, \ n \in \mathrm{N} \end{cases} \tag{5}$$

He used it for weak and strong convergence of fixed points in a uniformly convex Banach space. It can be viewed as an extension of the iterative processes given by Noor [12], Glowinski and Le Tallec [5], Xu and Noor [16], Ishikawa [6] and Mann [11].

Obviously all these processes contain one mapping only. A two-mappings iterative process was considered by Das and Debata [4] as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n Sy_n \\ y_n = (1-b_n)\ x_n + b_n Tx_n, \ n \in \mathrm{N}. \end{cases} \tag{6}$$

This can be seen as a two-mappings version of Ishikawa process (3). Many authors have used this process both with and without error terms. See for example [10] and references cited therein. Approximating common fixed points has its own importance as it has a direct link with the

Manuscript received August 13, 2011
Safeer Hussain Khan is with the Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, State of Qatar. (e-mail: safeerhussain5@yahoo.com; safeer@qu.edu.qa).

minimization problem; see for example Takahashi [15]. Keeping this in mind, we extend (5) to two mappings case as follows.

$$\begin{cases} x_1 = x \in C, \\ z_n = (1-a_n)x_n + a_n S x_n \\ y_n = b_n S z_n + c_n T x_n + (1-b_n-c_n)x_n \\ x_{n+1} = \alpha_n T y_n + \beta_n S z_n + (1-\alpha_n-\beta_n)x_n, \ n \in \mathrm{N} \end{cases} \tag{7}$$

This process reduces to:

- (6) when $a_n = c_n = \beta_n = 0.$
- (5) when $S = T.$
- (4) when $c_n = \beta_n = 0, S = T.$
- (3) when either $S = I$ or $a_n = c_n = \beta_n = 0, S = T.$
- (2) when either $c_n = 0, S = I$ or
  $a_n = b_n = c_n = \beta_n = 0, S = T.$
- (1) when $c_n = 0, \alpha_n = 1, \quad S = I$ or
  $a_n = \alpha_n = 0, \beta_n = 1, S = T$ or
  $a_n = b_n = c_n = \beta_n = 0, \alpha_n = 1$

On the other hand Berinde [1] introduced a new class of quasi-contractive type operators and proved a strong convergence theorem for Ishikawa iterative process (3) to approximate fixed points in a normed space. To appreciate this class of operators, we have to go through some definitions in a metric space (X,d) .

A mapping $T : X \to X$ is called an a - contraction if

$$d(Tx,Ty) \le ad(x,y) \text{ for all } x,y \in X, \tag{8}$$

where $0 < a < 1.$

The map T is called Kannan mapping [7] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)] \tag{9}$$

for all $x, y \in X.$

A similar definition is due to Chatterjea [3]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)] \tag{10}$$

for all $x, y \in X.$

Combining the above three definitions, Zamfirescu [17] proved the following important result.

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a,b and c satisfying $0 < a < 1$, $b \in (0,\frac{1}{2})$, $c \in (0,\frac{1}{2})$ such that for each pair $x, y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx,Ty) \le ad(x,y)$ *for all* $x, y \in X$

(z_2) $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$ *for all* $x, y \in X$

(z_3) $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$ *for all* $x, y \in X$.

Then T has a unique fixed point p and the Picard iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \qquad n \in \mathrm{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

An operator T satisfying the contractive conditions $(z_1),(z_2)$ and (z_3) in the above theorem is called Zamfirescu operator. The class of Zamfirescu operators is one of the most studied classes of quasi contractive type operators. In this class, Mann and Ishikawa iterative processes are known to converge to a unique fixed point of T.

In 2005, Berinde [1] introduced a new class of quasi-contractive type operators on a normed space X satisfying

$$\|Tx - Ty\| \le \delta\|x - y\| + L\|Tx - x\| \tag{11}$$

for any $x, y \in X$, $0 < \delta < 1$ and $L \ge 0.$

Note that the contractive condition (8) makes T a continuous function on X while this is not the case with the contractive conditions (9)-(11).

Berinde [1] proved that the class of operators given by (11) is wider than the class of Zamfirescu operators and used the Ishikawa iterative process (3) to approximate fixed points of this class of operators in a normed space. Actually, his main theorem is the following:

Theorem 2. *Let C be a nonempty closed bounded convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (11). *Let $\{x_n\}$ be defined by the Ishikawa iterative process* (3). *If $F(T) \ne \phi$ and $\sum_{n=1}^{\infty} a_n = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Since our process contains two mappings, therefore we need to modify the definition (11) to the case of two mappings. One simple way is that we force both of our mappings to satisfy above kind of condition separately. That is, S and T satisfy $\|Sx - Sy\| \le \delta\|x - y\| + L\|Sx - x\|$ and $\|Tx - Ty\| \le \delta\|x - y\| + L\|Tx - x\|$ respectively. However, we rather follow [9] to have a more general extension as:

$$\begin{aligned} &\max(\|Sx - Sy\|, \|Tx - Ty\|) \\ &\le \delta\|x - y\| + L\max(\|Sx - x\|, \|Tx - x\|). \end{aligned} \tag{12}$$

This condition reduces to (11) as follows when either $S = T$ or one of the mappings is identity.

- The case $S = T$ is obvious.
- When one of the mappings, say S, is identity, then (12) reduces to

$$\max(\|x-y\|, \|Tx-Ty\|) \le \delta\|x-y\| + L\|Tx-x\|. \tag{13}$$

- If $\max(\|x-y\|, \|Tx-Ty\|) = \|Tx-Ty\|$, then clearly (13) reduces to (11).
- If $\max(\|x-y\|, \|Tx-Ty\|) = \|x-y\|$, then $\|Tx-Ty\| \le \|x-y\| \le \delta\|x-y\| + L\|Tx-x\|.$

Thus we conclude that (12) reduces to (11) when either $S = T$ or one of the mappings is identity.

Our goal in this paper is to prove a strong convergence theorem using iterative process (7) for two quasi-contractive type operators satisfying (12) to approximate common fixed points in normed spaces. As corollaries, we get corresponding results using (1)-(6). It will improve and unify a number of results including those in [1, 2, 8].

II. MAIN RESULTS

We now prove our main theorem as follows.

Theorem 3. *Let C be a nonempty closed convex subset of a normed space E. Let $S, T : C \to C$ be two operators satisfying* (12) *and $F(S) \cap F(T) \ne \phi$. Let $\{x_n\}$ be defined by the iterative process* (7). *If $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in (0,1]$ such that $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) = \infty$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Assume that $F(S) \cap F(T) \ne \phi$. Let $w \in F(S) \cap F(T)$. Then

$$\begin{aligned}\|x_{n+1} - w\| &= \|\alpha_n Ty_n + \beta_n Sz_n + (1-\alpha_n-\beta_n)x_n - w\| \\ &\le \alpha_n\|Ty_n - w\| + \beta_n\|Sz_n - w\| \\ &\quad +(1-\alpha_n-\beta_n)\|x_n - w\|. \end{aligned} \tag{14}$$

Now for $x = w$ and $y = y_n$, (12) gives $\max(\|Sy_n - w\|, \|Ty_n - w\|) \le \delta\|x-y\|$ and so

$$\|Ty_n - w\| \le \delta\|y_n - w\|. \tag{15}$$

Similarly, with $x = w$ and $y = z_n$, we get

$$\|Sz_n - w\| \le \delta\|z_n - w\|. \tag{16}$$

Also, the choice $x = w$ and $y = x_n$ provides

$$\|Sx_n - w\| \le \delta\|x_n - w\|. \tag{17}$$

But

$$\begin{aligned}\|z_n - w\| &\le a_n\|Sx_n - w\| + (1-a_n)\|x_n - w\| \\ &\le a_n\delta\|x_n - w\| + (1-a_n)\|x_n - w\| \\ &= (1 - a_n(1-\delta))\|x_n - w\|. \end{aligned} \tag{18}$$

Thus

$$\begin{aligned}\|y_n - w\| &= \left\| \begin{matrix} b_n Sz_n + c_n Tx_n \\ +(1-b_n-c_n)x_n - w \end{matrix} \right\| \\ &\le b_n\|Sz_n - w\| + c_n\|Tx_n - w\| \\ &\quad +(1-b_n-c_n)\|x_n - w\| \\ &\le b_n\delta\|z_n - w\| + c_n\delta\|x_n - w\| \\ &\quad +(1-b_n-c_n)\|x_n - w\| \\ &\le b_n\delta(1-a_n(1-\delta))\|x_n - w\| \\ &\quad +c_n\delta\|x_n - w\| + (1-b_n-c_n)\|x_n - w\| \\ &= \begin{bmatrix} b_n\delta(1-a_n(1-\delta)) + c_n\delta \\ +(1-b_n-c_n) \end{bmatrix}\|x_n - w\|. \end{aligned} \tag{19}$$

Then using (14) through (19), we obtain

$$\begin{aligned}\|x_{n+1} - w\| &\le \alpha_n\|Ty_n - w\| + \beta_n\|Sz_n - w\| \\ &\quad +(1-\alpha_n-\beta_n)\|x_n - w\| \\ &\le \alpha_n\delta\|y_n - w\| + \beta_n\delta\|z_n - w\| \\ &\quad +(1-\alpha_n-\beta_n)\|x_n - w\| \\ &\le \left\{ \begin{matrix} \alpha_n\delta\begin{bmatrix} b_n\delta(1-a_n(1-\delta)) \\ +c_n\delta + (1-b_n-c_n) \end{bmatrix} \\ +\beta_n\delta(1-a_n(1-\delta)) \\ +(1-\alpha_n-\beta_n) \end{matrix} \right\}\|x_n - w\|. \end{aligned}$$

Rearranging the terms, we get

$$\begin{aligned}\|x_{n+1} - w\| &\le \left\{ \begin{matrix} 1-(1-\delta)\alpha_n\begin{bmatrix} 1+a_nb_n \\ +(b_n+c_n)\delta \end{bmatrix} \\ -(1-\delta)\beta_n[1+a_n\delta] \end{matrix} \right\}\|x_n - w\| \\ &\le [1-(1-\delta)\alpha_n - (1-\delta)\beta_n]\|x_n - w\| \\ &= [1-(1-\delta)(\alpha_n+\beta_n)]\|x_n - w\| \end{aligned}$$

for all $n \in \mathbb{N}$.

By induction,

$$\|x_{n+1}-w\| \le \prod_{k=1}^{n}\left[1-(1-\delta)(\alpha_k+\beta_k)\right]\|x_1-w\|$$
$$=\|x_1-w\|\exp\left(\sum_{k=1}^{n}-(1-\delta)(\alpha_k+\beta_k)\right)$$
$$=\|x_1-w\|\exp\left(-(1-\delta)\sum_{k=1}^{n}(\alpha_k+\beta_k)\right)$$

for all $n \in \mathrm{N}$.

Since $0<\delta<1, \alpha_n+\beta_n \in (0,1]$ and $\sum_{n=1}^{\infty}(\alpha_n+\beta_n)=\infty$, we get

$$\limsup_{n\to\infty}\|x_n-w\| \le \limsup_{n\to\infty}\|x_1-w\|$$
$$\times \exp\left(-(1-\delta)\sum_{k=1}^{\infty}(\alpha_k+\beta_k)\right)$$
$$\le 0.$$

Hence $\lim_{n\to\infty}\|x_n-w\|=0$. Consequently $x_n \to w \in F(S)\cap F(T)$. This completes the proof.

We now have the following corollaries.

Corollary 1. ([8], Theorem 3) *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (11)*. Let $\{x_n\}$ be defined by the iterative process* (5)*. If $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, $b_n+c_n \in [0,1]$ and $\alpha_n+\beta_n \in (0,1]$ such that $\sum_{n=1}^{\infty}(\alpha_n+\beta_n)=\infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof Choose $S=T$ in the above theorem.

Corollary 2. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (11)*. Let $\{x_n\}$ be defined by the iterative process* (4)*. If $F(T)\neq\phi$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Choose $c_n=\beta_n=0$ and $S=T$.

Theorem 3 (as well as Corollaries 1 and 2) now immediately gives Theorem 1 of [1] as the following:

Corollary 3. ([1], Theorem 1) *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (11)*. Let $\{x_n\}$ be defined by the iterative process* (3)*. If $F(T)\neq\phi$ and $\sum_{n=1}^{\infty}a_n=\infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

The above corollary also generalizes Theorem 2 of [2]. Similar results using (1) and (2) can also be obtained as corollaries.

Noting that class of operators given by (12) is wider than the class of Zamfirescu operators, we can get the results for such operators using iterative processes (1) through (7). We just mention the following special cases which constitute immediate generalizations of the indicated results.

Corollary 4. ([1], Theorem 2) *Let E be a Banach space and C a non-empty closed convex subset of E. Let $T : C \to C$ be a Zamfirescu operator. Let $\{x_n\}$ be defined by the Ishikawa iterative process* (3) *with $\sum_{n=1}^{\infty}a_n=\infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The operator T has a unique fixed point by Theorem 1 and hence the result follows from Theorem 3 by putting $a_n=c_n=\beta_n=0$ and $S=T$.

Corollary 5. *Let E be a Banach space and C a non-empty closed convex subset of E. Let $T : C \to C$ be a Zamfirescu operator. Define $\{x_n\}$ by the Mann iterative process* (2) *with $\sum_{n=1}^{\infty}a_n=\infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Set $a_n=b_n=c_n=\beta_n=0$ in Theorem 3.

Remarks.(1) *The Chatterjea's and the Kannan's contractive conditions* (9) *and* (10) *are both included in the class of Zamfirescu operators and so their convergence theorems for the iterative processes* (1) *through* (7) *are obtained in Theorem* 3.

(2) *Theorem* 4 *of Rhoades* [13] proved *in the context of Mann iterative process on a uniformly convex Banach space has been extended not only to more general processes* (3) *through* (7) *but also to more general spaces, namely normed spaces.*

REFERENCES

[1] V. Berinde, A convergence theorem for some mean value fixed point iterations procedures, Dem.Math., 38(1)2005, 177-184.

[2] V. Berinde, On the convergence of Ishikawa iteration in the class of quasi contractive operators, Acta.Math.Univ.Comenianae, **LXXIII** (1) 2004, 119-126.

[3] S.K.Chatterjea, Fixed point theorems, C.R. Acad.Bulgare Sci., **25** (1972), 727-730.

[4] G. Das and J. P. Debata, Fixed points of Quasi-nonexpansive mappings, Indian J. Pure. Appl. Math., 17 (1986), 1263--1269.

[5] R. Glowinski, P. Le Tallec, Augmented Lagrangian and operator-splitting methods in nonlinear mechanics, SIAM, Philadelphia,1989.

[6] S.Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math.Soc., **44** (1974), 147-150.

[7] R. Kannan, Some results on fixed points, Bull.Calcutta Math. Soc., **10**(1968), 71-76.

[8] S.H. Khan, Fixed points of quasi-contractive type operators in normed spaces by a three-step iteration process, Proceedings of the World Congress on Engineering 2011 Vol I, WCE 2011, July 6 - 8, 2011, London, U.K, pp 144-147.

[9] S.H.Khan, Approximating Common fixed points by an iterative process involving two steps and three mappings, J.Concr. Appl. Math., 8(3) 2010, 407-415.

[10] S.H.Khan, Common fixed points of two quasi contractive operators in normed spaces by iteration, Int. Journal of Math. Analysis, Vol. 3, 2009, no. 3, 145 - 151

[11] W.R. Mann, Mean value methods in iterations, Proc. Amer. Math. Soc., **4** (1953), 506-510.

[12] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., **251** (2000), 217-229.
[13] B.E. Rhoades, Fixed point iteration using infinite matrices, Trans. Amer. Math .Soc., **196** (1974), 161-176.
[14] Suantai, Weak and strong convergence criteria of Noor iteration for asymptotically nonexpansive mappings, J.Math. Anal. Appl. 311(2)2005, 506-517.
[15] W. Takahashi, Iterative methods for approximation of fixed points and their applications, J.Oper.Res.Soc. Jpn., **43(1)** (2000), 87 -108.
[16] B. Xu, M.A.Noor, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl., **224** (1998), 91-101.
[17] T. Zamfirescu, Fix point theorems in metric spaces, Arch. Math. (Basel), 23(1972), 292-298.