Optimal Portfolio and Strategic Consumption Planning in a Life-Cycle of a Pension Plan Member in a Defined Contributory Pension Scheme

C. R. Nwozo and C. I. Nkeki , *Member, IAENG*

***Abstract*-We study the optimal portfolio and strategic life-cycle consumption process in a defined contributory pension plan. The pension plan members (PPMs) contribute flow of cash into the pension funds. These flow of cash are invested into a market structure that is characterized by a risk-free asset (cash account) and a risky asset (stock) by the pension fund administrator (PFA). The risk-free rate is assume to be deterministic. We find an explicit analytical solution to the non-linear partial differential equation (Hamilton Jacobi Bellman equation (HJB)) that arises from our problem. Also, we find that part of the portfolio value is proportional to the ratio of the present value of expected future contributions to the optimal portfolio value at time t. We further observe that the portfolio of the PPM will grow without bound, if the coefficient of the utility function is close to one, provided the expected growth rate of the risky asset is greater than the short term interest rate. We find an interesting result that shows that as the market evolve, part of the portfolio value should be transferred to the cash account overtime in order to offset unforeseen market shocks that may occur in future time. Also, we find that with the use of power utility function, the inflation risks that is associated with the PPM's contributions is hedged.**

***Index Terms*-Strategic consumption, optimal portfolio, power utility function, pension plan member, pension fund administration**

Mathematics Subject Classification: 91B28, 91B30, 93E20

I. INTRODUCTION

We consider optimal portfolio and consumption optimization problem in a defined contributory pension funds. We assume that an infinitely-lived individual with initial capital, works up to retirement period and consume continuously throughout his/her lifetime. The consumption terminates when the individual is dead. This paper is based on the already existing works. [11] provided an approximate solution and analytical results to the intertemporal consumption problem. [14] considered a tractable model of precautionary savings in continuous time and assumed that the uncertainty is about the timing of the income loss in addition to the assumption of non-stochastic asset return. [3] considered labour supply flexibility and portfolio choice of individual life cycle. They determined the objective of maximizing the expected discounted lifetime utility and assumed that the utility function has two argument (consumption and labour/leiture). [2] used the quadratic utility function that has the characterization of linear marginal utility. This utility function is not attractive in describing the behaviour of individual towards risk as it implies increasing absolute risk aversion. [3] concluded that labour income induces the individual to invest an additional amount of wealth to the risky asset. They shown that labour income and investment choices are related, while they failed to analyzed the optimal consumption process. [4] studied the optimal management of a defined contribution pension plan where the guarantee depends on the level of interest rates at the fixed retirement date. [5] considered the finding of the optimal dynamic asset allocation strategy for a DC pension plan, taking into account the stochastic features of the plan member's lifetime salary progression as well as the stochastic properties of the assets held in his accumulating pension fund. They emphasised that salary risk (the fluctuation in the plan member's earning in response to economic shocks) is not fully hedgeable using existing financial assets. They further emphasised that wage-indexed bonds could be used to hedge productivity and inflation shocks, but such bonds are not widely traded. They called the optimal dynamic asset allocation strategy *stochastic lifestyling***.** They compared it against various static and deterministic lifestyle strategies in order to calculate the costs of adopting suboptimal strategies. Their solution technique made use of the present value of future contribution premiums into the plan. This technique can be found in [4], [10], [1]. Deterministic lifestyling which is the gradual switch from equities to bonds according to present rules is a popular asset alloction strategy during the accumulation phase of DC pension plans and is designed to protect the pension fund from a catastrophic fall in the stock market just prior to retirement (see [5], [1], [6]). [9], [5] analysed extensively the occupational DC pension funds, where the contribution rate is a fixed percentage of salary.

The classical dynamic lifetime portfolio selection in a continuous time model is developed by [12], [13], who considered the market with a constant interest rate. Other than interest rate risk, salary risk, contribution risk, mortality rate and inflation risk are always considered in this framework. [9] used a dynamic programming approach to derived a formula for optimal investment allocation in a DC scheme and compared three risk measures to analyzed the final net replacement ratio achieved by members. They suggested that when the choice of investment strategy is determined, risk profiles of individual and different risk measures are both important factors which should be taken into

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C. R. Nwozo is with the Department of Mathematics, University of Ibadan, Ibadan, Oyo State, Nigeria. (email: crnwozo@yahoo.com).
C. I. Nkeki is with the Department of Mathematics, University of Benin, P. M. B. 1154, Benin City, Edo State, Nigeria. (email: nkekicharles2003@yahoo.com).

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consideration. In this paper, we aim at finding the optimal value function of investment portfolio and consumption strategies in a defined contributory pension funds using power utility function.

The structure of the remainder of the paper is as follows. Section 2 presents the formulation of the problem. In section 3, we presents the description of the model. In section 4, we give the optimal portfolio and consumption processes and expected utility of terminal wealth of the PPM using power utility function. In section 5, we presents the optimal expected value of the PPM's final wealth and the expected consumption process of the PPM life-cycle in pension plan. Finally, section 6 concludes the paper.

II. PROBLEM FORMULATION

We consider a continuous-time financial market where there are two investment instruments: a cash and a stock with price dynamics given, respectively, by

$$dB(t) = rB(t)dt, \qquad (1)$$

$$dS(t) = S(t)\left(\mu dt + \sigma^S dW(t)\right), \qquad (2)$$

$t \in \Re_+$. Here $r > 0, \mu > r, \sigma^S > 0$ are constants, μ is the predictable 1-dimensional process of excess appreciation rate in relation to the stock market, σ^S is a predictable process of volatility of the stock market. $W(t)$; $t \in \Re_+$ is a standard 1-dimensional Brownian motion on a filtered probability space $\left(\Omega, F, \{F_t\}_{t\in[0,T]}, P\right)$ with $W(0) = 0$ almost surely. We assume that the filtration $\{F_t\}_{t\in[0,T]}$ is generated by the Brownian motion and is right continuous, and that each F_t contains all the $P-$null sets of F. We denote by L_F^2 the set of square integrable $\{F_t\}_{t\in[0,T]}-$adapted processes,

$$L_F^2 = \left\{ \begin{array}{l} \Delta \middle| The\ process\ \Delta = \{\Delta(t)\}_{t\in[0,T]} \\ adapted\ process\ such\ that \\ \int_0^T E\left[\Delta^2(t)\right]dt < \infty \end{array} \right\}$$

and by $L_{F_T}^2$ the set of square integrable F_T-measurable random variables,

$$L_{F_T}^2 = \left\{ \Delta \middle| \begin{array}{l} \Delta\ is\ an\ F_T - measurable\ random \\ \text{var}iable\ such\ that\ E\left[\Delta^2\right] < \infty \end{array} \right\}.$$

Using *Ito's lemma* on Eq. (2), we obtain the following solutions

$$B(t) = \exp(rt),\ B(0) = 1; \qquad (3)$$

$$S(t) = S(0)\exp\left[\left(\mu - \frac{1}{2}\left(\sigma^S\right)^2\right)t + \sigma^S W(t)\right], \qquad (4)$$

We assume that the financial market is arbitrage-free, complete and continuously open between time 0 and T, i.e., there is only one process θ satisfying

$$\theta = (\sigma)^{-1}(\mu - r),\ \text{where}\ \sigma = \begin{pmatrix} \sigma^I & 0 \\ 0 & \sigma^S \end{pmatrix}$$

and σ^I is the volatility of salary of the PPM. The exponential process

$$Z(t) = \exp\left[-(\theta)' W(t) - \frac{1}{2}\|\theta\|^2 t\right], 0 \le t \le T \qquad (5)$$

is assumed to be a martingale, and the risk-neutral equivalent martingale measure, denoted by

$$\tilde{P}(A) = E\left[Z(T)1_A\right],\ A \in F(T). \qquad (6)$$

Definition 1: Let $\Delta(t)$ portfolio process and $C(t)$ the consumption process, then the pair (Δ, C) is said to be self-financing if the corresponding wealth process

$X^{\Delta,C}(t), t \in [0,T],$ satisfies

$$dX^{\Delta,C}(t) = \Delta(t)X^{\Delta,C}(t)\frac{dS(t)}{S(t)} +$$

$$\left(1 - \Delta(t)\right)X^{\Delta,C}(t)\frac{dB(t)}{B(t)} + H(t)dt - C(t)dt, \qquad (7)$$

where $1 - \Delta(t)$ represents the proportion of the portfolio invested in cash account at time t.

Note that if C=0, Eq.(7) becomes

$$dX^{\Delta,C}(t) = \Delta(t)X^{\Delta,C}(t)\frac{dS(t)}{S(t)} +$$

$$\left(1 - \Delta(t)\right)X^{\Delta,C}(t)\frac{dB(t)}{B(t)} + H(t)dt.$$

Substituting the assets returns in Eq.(1) and (2) into Eq.(7), we obtain the following

$$dX^{\Delta,C}(t) = \Delta(t)X^{\Delta,C}(t)\left\{\mu dt + \sigma^S dW(t)\right\} +$$

$$\left(1 - \Delta(t)\right)X^{\Delta,C}(t)rdt + H(t)dt - C(t)dt$$

$$= X^{\Delta,C}(t)rdt + \Delta(t)X^{\Delta,C}(t)(\mu - r)dt +$$

$$= \left(X^{\Delta,C}(t)r + \Delta(t)X^{\Delta,C}(t)(\mu - r) + H(t) - C(t)\right)dt$$

$$\sigma^S \Delta(t) X^{\Delta,C}(t) dW(t) + H(t)dt - C(t)dt$$

$$+ \sigma^S \Delta(t) X^{\Delta,C}(t) dW(t). \qquad (8)$$

This is our stochastic differential equation which represents the wealth process of the PPM at time t.

III. DESCRIPTION OF THE MODEL

In this section, we present the dynamics and the description of the model of a pension plan member in pension funds.

The dynamics of the PPM effective salary is given by

$$\frac{dH(t)}{H(t)} = \omega dt + \sigma^I dW(t), \qquad (9)$$

$$H(0) = h > 0,$$

where H(t) is the salary of the PPM at time t, ω is the expected growth rate of salary of PPM and σ^I is the volatility of salary which is driven by the source of uncertainty of inflation, $W(t)$. We assume that $\omega > 0$ and $\sigma^I > 0$ are constants. Using *Ito's Lemma* on Eq.(9), we have that the following is the solution to the stochastic differential equation:

$$H(t) = h \exp\left[\left(\omega - \frac{1}{2}\left(\sigma^I\right)^2\right)t + \sigma^I W(t)\right], 0 \le t \le T. \qquad (10)$$

Definition 2: The present value of the expected discounted future contributions process is defined as

$$Y(t) = E_t\left[\int_t^T \frac{\Lambda(s)}{\Lambda(t)} \beta H(s) ds\right], \qquad (11)$$

where $\beta > 0$ is the proportion of the salary contributed into the pension funds and E_t is the conditional expectation with respect to the Brownian filtration $\{F(t)\}_{t\ge 0}$

$$\Lambda(t) \equiv Z(t)\exp[-rt], 0 \le t \le T. \qquad (12)$$

is the stochastic discount factor which adjusts for nominal interest rate and market price of risk.

Definition 3: The present value of the expected discounted future consumption process is defined by

$$\Phi(t) = E_t\left[\int_t^\infty \frac{\Lambda(u)}{\Lambda(t)} C(u) du\right], \ t \in [T, \infty). \qquad (13)$$

We take limits of integration in Eq.(13) from $t = T$ to $t = \infty$ because, we assume that consumption starts when the PPM retired and consume his/ her investment (benefits) till he/she is dead.

Theorem 1: Let $Y(t)$ be the present value of expected future contributions into the pension funds, then

$$Y(t) = \frac{1}{\omega - r - \theta\sigma} \times$$

$$\left(\exp[(\omega - r - \theta\sigma)(T - t)] - 1\right) H(t) \qquad (14)$$

Proof: By definition 2, we have that

$$Y(t) = E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} \beta H(u) du\right],$$

$$= \beta H(t) E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} \frac{H(u)}{H(t)} du\right], \qquad (15)$$

Since the processes $\Lambda(.)$ and $H(.)$ are geometric Brownian motions, it follows that $\frac{\Lambda(u)}{\Lambda(t)}\frac{H(u)}{H(t)}$ is independent of $F(t)$ *for* $u \ge t$. Hence,

$$Y(t) = \beta H(t) g(t,T). \qquad (16)$$

There, the deterministic function $g(t,T)$, is defined by

$$g(t,T) \equiv E\int_0^{T-t} \Lambda(u) \frac{H(u)}{H(0)} du \qquad (17)$$

$$\int_0^{T-t} E\left[\Lambda(u)\frac{H(u)}{H(0)}\right] du = \frac{1}{\varphi}\left(\exp[\varphi(T-t)] - 1\right), \quad (18)$$

where $\varphi = \omega - r - \theta\sigma$.

Hence,

$$Y(t) = \frac{\beta H(t)}{\varphi}\left(\exp[\varphi(T-t)] - 1\right).$$

(19)

Find the differential of bothsides of Eq.(19), we obtain

$$dY(t) = Y(t)\left[(r + \sigma\theta)dt + \sigma^I dW(t)\right] - H(t)dt \quad (20)$$

Definition 4: The value of wealth of the PPM is define as

$$V(t) = X^{\Delta,C}(t) + Y(t). \qquad (21)$$

Theorem 2: The change in wealth of the PPM is given by the dynamics:

$$dV(t) = \begin{pmatrix} \Delta(t) X^{\Delta,C}(t)(\mu - r) + rX^{\Delta,C}(t) + \\ Y(t)(r + \sigma\theta) - C(t) \end{pmatrix} dt +$$

$$\left(\Delta(t)X^{\Delta,C}(t)\sigma^S+Y(t)\sigma^I\right)dW(t) \quad (22)$$

Proof: $dV(t)=dX^{\Delta,C}(t)+dY(t)$ (23)

$$=\Delta(t)X^{\Delta,C}(t)\left(\mu dt+\sigma^S dW(t)\right)+$$
$$rX^{\Delta,C}(t)(1-\Delta(t))dt+Y(t)\times$$
$$\left((r+\sigma\theta)dt-C(t)dt+\sigma^I dW(t)\right)$$
$$=rX^{\Delta,C}(t)dt+\Delta(t)X^{\Delta,C}(t)(\mu-r)dt+$$
$$Y(t)(r+\sigma\theta)dt-C(t)dt+$$
$$Y(t)\sigma^I dW(t)+\Delta(t)X^{\Delta,C}(t)\sigma^S dW(t)$$
$$=\begin{pmatrix}\Delta(t)X^{\Delta,C}(t)(\mu-r)+rX^{\Delta,C}(t)+\\ Y(t)(r+\sigma\theta)-C(t)\end{pmatrix}dt+$$
$$\left(\Delta(t)X^{\Delta,C}(t)\sigma^S+Y(t)\sigma^I\right)dW(t) \quad (24)$$

IV. OPTIMAL PORTFOLIO AND CONSUMPTION PROCESS OF A PPM

In this section, we derived the optimal portfolio and consumption process of a PPM using HJB equation. We describe the portfolio and consumption problems faced by the PFA in pension plan. Hence, we assume that the PFA chooses power utility function in maximizing the expected utility of the terminal wealth and consumption process. The choice of the power utility is motivated by the fact that pension scheme are in general large investment conpanies whose strategic plans are with respect to the size of funds they are managing. Again, pension funds are regulated in such a way that its values can not be negative. Therefore, the risk aversion of the PFA is described by a linear combination of two power utility functions. The first term is with respect to the wealth process while the second is with respect the consumption process:

$$U(V)=\frac{(X+Y)^\gamma}{\gamma}-\frac{C(t)^\gamma}{\gamma},\gamma\in(-\infty,1)\setminus\{0\}. \quad (25)$$

We now give the optimal portfolio and consumption process for pension funds planners at time t. Let

$$dV(t)=\begin{pmatrix}X^{\Delta,C}(t)r+\Delta(t)X^{\Delta,C}(t)(\mu-r)+\\ Y(t)(r+\sigma\theta)-C(t\end{pmatrix}dt+$$
$$\left(\sigma^S\Delta(t)X^{\Delta,C}(t)+Y(t)\sigma^I\right)dW(t), \quad (26)$$

be the change in wealth process at time t. Let

$$U(V,t)=\sup_{\substack{\{\Delta\}\in\Pi_X\\ C\in\Pi_C}} E\left[U(V(T))\mid X^{\Delta,C}(t)=X,Y(t)=Y\right] \quad (27)$$

be the value function, where Π_C is a set of admissible consumption strategy and Π_X set of admissible portfolio strategy that are F_V – progressively measureable, that satisfy the integrability conditions

$$E\left[\int_0^T C(s)^2 ds\right]<\infty,\ E\left[\int_0^T \Delta(s)^2 ds\right]<\infty,$$

Then,

$$\Delta^*(t)=\frac{-U_X(V,t)(\mu-r)-\sigma^S\sigma^I YU_{XY}(V,t)}{X(\sigma^S)^2U_{XX}(V,t)}, \quad (28)$$

$$C^*=I\left(U_X(V,t)\exp[\rho t]\right), \quad (29)$$

where, $I=\left(\frac{dU(C)}{dC}\right)^{-1}$.

Given

$$U(V,t)=\sup_{\substack{\{\Delta\}\in\Pi_X\\ C\in\Pi_C}} E\left[U(V(T))\middle| X^{\Delta,C}(t)=X,Y(t)=Y\right],$$

Eq.(26) becomes:

$$U(V,t)\geq E\left[U(V(t+h),t+h)\middle| X^{\Delta,C}(t)=X,Y(t)=Y\right]$$
$$\geq U(V,t)+$$
$$E\left[\int_t^{t+h}\begin{pmatrix}U_t(V(u),u)+\\ Y(r+\sigma\theta)U_Y\\ U_X(V(u),u)\times\\ \begin{pmatrix}rX+\Delta(u)\times\\ X(\mu-r)-C(u)\end{pmatrix}\\ +U(C(t))\exp(-\rho t)\end{pmatrix}du\middle| X^{\Delta,C}(u)=X,Y(u)=Y\right.$$
$$+\frac{1}{2}E\left[\int_t^{t+h}\begin{pmatrix}(\sigma^S)^2\Delta(u)^2X^2U_{XX}(V(u),u)+\\ 2\sigma^S\sigma^I\Delta(u)XYU_{XY}(V,t)+\\ Y^2(\sigma^I)^2U_{YY}(V(u),u)\end{pmatrix}du \middle| X^{\Delta,C}(u)=X,Y(u)=Y\right] \quad (30)$$

where, $\rho>0$ is the preference rate of consumption. By subtracting $U(V,t)$ from bothsides of Eq.(30) and then divide bothsides by h and allow h to tends to zero, we obtain:

$$0\geq U_t(V,t)+U_X(V,t)\left(rX+\Delta(t)X(\mu-r)\right)+$$
$$Y(r+\sigma\theta)U_Y+\frac{1}{2}U_{XX}(V,t)\Delta(t)^2X^2(\sigma^S)^2+$$

$$\sigma^S \sigma^I \Delta(u) XYU_{XY}(V,t) + \frac{1}{2} U_{YY}(V(u),u) Y^2 (\sigma^I)^2$$

$$- C(t) U_X(V,t) + U(C(t)) \exp(-\rho t)$$

This yields the HJB equation for the value function

$$U_t(V,t) +$$

$$\max_{\substack{\Delta \in \Pi_X \\ C \in \Pi_C}} \left\{ \begin{array}{l} U_X(V,t)(rX + \Delta(t) X(\mu - r)) \\ + \frac{1}{2} U_{XX}(V,t) \Delta(t)^2 X^2 (\sigma^S)^2 - \\ C(t) U_X(V,t) + U(C(t)) \exp(-\rho t) \\ + \sigma^S \sigma^I \Delta(u) XYU_{XY}(V,t) \end{array} \right\} +$$

$$rXU_X(X,t) + Y(r + \sigma\theta) U_Y +$$

$$\frac{1}{2} U_{YY}(V(t),t) Y^2 (\sigma^I)^2 = 0 \qquad (31)$$

where Δ and $C \in \Pi_C$ are sets of admissible portfolio and consumption strategies, respectively.

Assuming U is concave and

$U((V,t) \in C^{1,2}(\Re \times [0,T])$, then Eq.(31) has a unique smooth solution and the maximum in Eq.(31) is well-defined. Hence, we have the following:

$$\frac{\partial H}{\partial \Delta} = U_X(V,t) X(\mu - r) + \sigma^S \sigma^I XYU_{XY}(V,t)$$

$$+ U_{XX}(V,t) \left[\Delta^*(t) X^2 (\sigma^S)^2 \right] = 0 \qquad (32)$$

$$\frac{\partial H}{\partial C} = -U_X(V,t) + \frac{\partial U(C(t))}{\partial C} C^*(t) \exp(-\rho t) = 0 \qquad (33)$$

where, H is the Hamiltonian of Eq.(31).

From Eq.(32), we obtain:

$$\Delta^*(t) = \frac{-(\mu - r) U_X(V,t) - \sigma^S \sigma^I Y(t) U_{XY}(V,t)}{X(t)(\sigma^S)^2 U_{XX}(V,t)}, \qquad (34)$$

From Eq.(33), we obtain:

$$C^*(t) = I(U_X(V,t) \exp[\rho t]), \qquad (35)$$

Theorem 3: Suppose that Eq.(34) and Eq.(35) hold, then the following Eq.(31) implies Eq.(36).

$$U_t(V,t) - \frac{1}{2} \frac{U_X^2(V,t)}{U_{XX}(V,t)} \theta^2 - \frac{1}{2} \frac{(\sigma^I)^2 Y^2 U_{XY}^2(V,t)}{U_{XX}(V,t)}$$

$$- \frac{(\mu - r) \sigma^I YU_X(V,t) U_{XY}(V,t)}{(\sigma^S) U_{XX}(V,t)} +$$

$$rXU_X(V,t) - I(U_X(V,t) \exp(\rho t)) U_X(V,t) +$$

$$Y(r + \sigma\theta) U_Y + \frac{1}{2} U_{YY}(V(t),t) Y^2 (\sigma^I)^2 +$$

$$\exp(-\rho t) U(I(U_X(V,t) \exp(\rho t))) = 0, \qquad (36)$$

Proof: Substituting Eq.(34), and (35) into Eq.(31), we have

$$U_t(V,t) + \frac{1}{2} \frac{U_X^2(V,t)}{U_{XX}(V,t)} \theta^2 +$$

$$\frac{(\mu - r) \sigma^I YU_X(V,t) U_{XY}(V,t)}{(\sigma^S) U_{XX}(V,t)} +$$

$$\frac{1}{2} \frac{(\sigma^I)^2 Y^2 U_{XY}^2(V,t)}{U_{XX}(V,t)} - \frac{U_X^2(V,t)}{U_{XX}(V,t)} \theta^2 \times$$

$$\frac{(\mu - r) \sigma^I YU_X(V,t) U_{XY}(V,t)}{(\sigma^S) U_{XX}(V,t)} -$$

$$\frac{(\mu - r) \sigma^I YU_X(V,t) U_{XY}(V,t)}{(\sigma^S) U_{XX}(V,t)} -$$

$$\frac{(\sigma^I)^2 Y^2 U_{XY}^2(V,t)}{U_{XX}(V,t)} + rXU_X(V,t) -$$

$$I(U_X(V,t) \exp(\rho t)) U_X(V,t) +$$

$$Y(r + \sigma\theta) U_Y + \frac{1}{2} U_{YY}(V(t),t) Y^2 (\sigma^I)^2$$

$$+ \exp(-\rho t) U(I(U_X(V,t) \exp(\rho t))) = 0.$$

$$\Rightarrow U_t(V,t) - \frac{1}{2} \frac{U_X^2(V,t)}{U_{XX}(V,t)} \theta^2 -$$

$$\frac{1}{2} \frac{(\sigma^I)^2 Y^2 U_{XY}^2(V,t)}{U_{XX}(V,t)} -$$

$$\frac{(\mu - r) \sigma^I YU_X(V,t) U_{XY}(V,t)}{(\sigma^S) U_{XX}(V,t)} +$$

$$rXU_X(V,t) - I(U_X(V,t) \exp(\rho t)) U_X(V,t)$$

$$+Y(r+\sigma\theta)U_Y+\frac{1}{2}U_{YY}(V(t),t)Y^2\left(\sigma^I\right)^2+$$

$$\exp(-\rho t)U\left(I\left(U_X(V,t)\exp(\rho t)\right)\right)=0. \qquad (37)$$

Theorem 4: Let $U\left(X^{\Delta},Y,t\right)=\frac{[(X+Y)A(t)B(t)]^\gamma}{\gamma}$,

$\gamma<1,\gamma\neq 0$ be the solution of the HJB equation (37), and let $U(C(t))=\frac{[C(t)A(t)B(t)]^\gamma}{\gamma}$,

then,

$$U(V,t)=\frac{[(X+Y)A(t)B(t)]^\gamma}{\gamma}-\frac{[C(t)A(t)B(t)]^\gamma}{\gamma}$$

is given by

$$\begin{cases} U(V,t)=\left(\frac{(X+Y)^\gamma}{\gamma}-\frac{[C(t)]^\gamma}{\gamma}\right)\times \\ \exp\left[\left(r\gamma-\frac{\gamma\theta^2}{(\gamma-1)^2}\right)(T-t)\right]\times \\ \left\{K\exp\left[\left(\frac{-\rho+\gamma r}{\gamma-1}-\frac{\gamma\theta^2}{2(\gamma-1)^2}\right)(T-t)\right]+\lambda\right\} \\ U(V,T)=\left(\frac{(X+Y)^\gamma}{\gamma}-\frac{C^\gamma}{\gamma}\right)\{K+\lambda\} \end{cases}$$

where,

$$K=\frac{\left(\frac{\left(1-3\gamma+12\gamma^2-\gamma^3\right)}{1-\gamma}\right)\exp\left[\frac{\gamma\theta^2T}{(\gamma-1)^2}\right]}{2(1-\gamma)^\gamma\eta}$$

and $\lambda=\frac{\left(2\gamma-\left(1+\gamma^2\right)\right)\exp\left[\frac{\gamma\theta^2T}{(\gamma-1)^2}\right]}{2(1-\gamma)^\gamma}$

Proof:

$$U_t=V^\gamma(A(t)B(t))^{\gamma-1}(A'(t)B(t)+A(t)B'(t)) \qquad (38)$$

$$U_X=V^{\gamma-1}(A(t)B(t))^\gamma \qquad (39)$$

$$U_{XX}=(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma \qquad (40)$$

$$U_Y=V^{\gamma-1}(A(t)B(t))^\gamma \qquad (41)$$

$$U_{YY}=(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma \qquad (42)$$

$$U_{XY}=(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma \qquad (43)$$

Substituting Eq.(38)-Eq.(43) into Eq.(37), we obtain the following:

$$V^\gamma(A(t)B(t))^{\gamma-1}(A'(t)B(t)+A(t)B'(t))$$

$$-\frac{V^\gamma(A(t)B(t))^\gamma\theta^2}{2(\gamma-1)}-$$

$$\frac{V^{\gamma-2}Y^2(\gamma-1)(A(t)B(t))^\gamma}{2}-$$

$$\frac{\sigma^IV^{\gamma-1}Y(A(t)B(t))^\gamma(\mu-r)}{\sigma^S}-$$

$$I\left(V^{\gamma-1}(A(t)B(t))^\gamma\exp[\rho t]\right)V^{\gamma-1}(A(t)B(t))^\gamma$$

$$+Y(r+\sigma\theta)V^{(\gamma-1)}(A(t)B(t))^\gamma+$$

$$\frac{1}{2}Y^2\left(\sigma^I\right)^2(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma+$$

$$U\left[I\left(V^{\gamma-1}(A(t)B(t))^\gamma\exp[\rho t]\right)\right]\exp[-\rho t]+$$

$$rX.V^{\gamma-1}(A(t)B(t))^\gamma=0$$

$$V^\gamma(A(t)B(t))^{\gamma-1}(A'(t)B(t)+A(t)B'(t))-$$

$$\frac{V^\gamma(A(t)B(t))^\gamma\theta^2}{2(\gamma-1)}-$$

$$\frac{\sigma^IV^{\gamma-1}Y(A(t)B(t))^\gamma(\mu-r)}{\sigma^S}-$$

$$\left(V^{\gamma-1}(A(t)B(t))^\gamma\exp[\rho t]\right)^{\frac{1}{\gamma-1}}V^{\gamma-1}(A(t)B(t))^\gamma$$

$$+Y(r+\sigma\theta)V^{\gamma-1}(A(t)B(t))^\gamma+$$

$$(A(t)B(t))^{\frac{\gamma^2}{\gamma-1}}\exp\left[\frac{\rho t}{\gamma-1}\right]V^\gamma+$$

$$\frac{1}{\gamma}\left[\left(V^\gamma(A(t)B(t))^{\frac{\gamma^2}{\gamma-1}}\exp\left[\frac{\rho t}{\gamma-1}\right]\right)\right]\exp[-\rho t]$$

$$+rX.V^{\gamma-1}(A(t)B(t))^\gamma=0$$

$$V^\gamma(A(t)B(t))^{\gamma-1}A'(t)B(t)+$$

$$r\left(X+Y\right)V^{\gamma-1}\left(A(t)B(t)\right)^{\gamma}=0 \qquad (44)$$

$$V^{\gamma}\left(A(t)B(t)\right)^{\gamma-1}A(t)B'(t)-\frac{V^{\gamma}\left(A(t)B(t)\right)^{\gamma}\theta^{2}}{2(\gamma-1)}+$$

$$\frac{1-\gamma}{\gamma}\left[\left(V^{\gamma}\left(A(t)B(t)\right)^{\frac{\gamma^2}{\gamma-1}}\exp\left[\frac{\rho t}{\gamma-1}\right]\right)\right]=0. \qquad (45)$$

From Eq.(44), we have that

$$A'(t)+rA(t)=0 \qquad (46)$$

Solving Eq.(45), we obtain

$$\begin{cases} A(t)=\exp\left[r(T-t)\right] \\ A(T)=1 \end{cases} \qquad (47)$$

Substituting (47) into Eq.(45), we have

$$B'(t)=\frac{B(t)\theta^{2}}{2(\gamma-1)}-\frac{1-\gamma}{\gamma}\times$$

$$\left[\left(B(t)^{\frac{2\gamma-1}{\gamma-1}}\exp\left[\frac{\rho t+r\gamma(T-t)}{\gamma-1}\right]\right)\right] \qquad (48)$$

Using Mathematica 6.0 on Eq.(48), we obtain the following:

$$B(t)=\frac{1}{4\eta(1-\gamma)}\left(\begin{array}{l}\exp\left[\frac{\gamma\theta^{2}T}{(\gamma-1)^{2}}\right]\times \\ \left\{\begin{array}{l}\left(\frac{2\left(1-3\gamma+12\gamma^{2}-\gamma^{3}\right)}{1-\gamma}\right)\times \\ \exp\left[\frac{\left(\eta-2\gamma\theta^{2}\right)(T-t)}{2(\gamma-1)^{2}}\right]+ \\ \eta\left(4\gamma-2\left(1+\gamma^{2}\right)\right)\times \\ \exp\left[\frac{-\gamma\theta^{2}(T-t)}{(\gamma-1)^{2}}\right]\end{array}\right\}\end{array}\right)^{\frac{1}{\gamma}} \qquad (49)$$

where $\eta=2\rho-\gamma\theta^{2}-2\gamma\rho-2\gamma r+2\gamma^{2}r$

From Eq.(49), setting

$$K=\frac{\left(\frac{\left(1-3\gamma+12\gamma^{2}-\gamma^{3}\right)}{1-\gamma}\right)\exp\left[\frac{\gamma\theta^{2}T}{(\gamma-1)^{2}}\right]}{2(1-\gamma)^{\gamma}\eta}$$

and $\lambda=\dfrac{\left(2\gamma-\left(1+\gamma^{2}\right)\right)\exp\left[\frac{\gamma\theta^{2}T}{(\gamma-1)^{2}}\right]}{2(1-\gamma)^{\gamma}}$

we therefore obtain the following:

$$\begin{cases} B(t)=\left(\begin{array}{l}K\exp\left[\frac{\left(\eta-2\gamma\theta^{2}\right)(T-t)}{2(\gamma-1)^{2}}\right]+ \\ \lambda\exp\left[\frac{-\gamma\theta^{2}(T-t)}{(\gamma-1)^{2}}\right]\end{array}\right)^{\frac{1}{\gamma}} \\ B(T)=0 \end{cases} \qquad (50)$$

Hence,

$$U(V,t)=\left(\frac{\left(X(t)+Y(t)\right)^{\gamma}}{\gamma}-\frac{\left[C(t)\right]^{\gamma}}{\gamma}\right)\times$$

$$\left\{\left(\begin{array}{l}K\exp\left[\frac{\left(\eta-2\gamma\theta^{2}\right)(T-t)}{2(\gamma-1)^{2}}\right]+ \\ \lambda\exp\left[\frac{-\gamma\theta^{2}(T-t)}{(\gamma-1)^{2}}\right]\end{array}\right)^{\frac{1}{\gamma}}\exp\left[r(T-t)\right]\right\}^{\gamma}$$

$$=\left(\frac{\left(X(t)+Y(t)\right)^{\gamma}}{\gamma}-\frac{\left[C(t)\right]^{\gamma}}{\gamma}\right)\times$$

$$\left(\begin{array}{l}K\exp\left[\left(\frac{\left(\eta-2\gamma\theta^{2}\right)}{2(\gamma-1)^{2}}+\gamma r\right)(T-t)\right]+ \\ \lambda\exp\left[\left(\gamma r-\frac{\gamma\theta^{2}}{(\gamma-1)^{2}}\right)(T-t)\right]\end{array}\right)$$

$$=\left(\frac{\left(X(t)+Y(t)\right)^{\gamma}}{\gamma}-\frac{\left[C(t)\right]^{\gamma}}{\gamma}\right)\times$$

$$\left(\begin{array}{l}K\exp\left[\left(\begin{array}{l}\frac{\left(2\rho(1-\gamma)-2\gamma r(1-\gamma)-3\gamma\theta^{2}\right)}{2(\gamma-1)^{2}} \\ +\gamma r\end{array}\right)(T-t)\right] \\ +\lambda\exp\left[\left(\gamma r-\frac{\gamma\theta^{2}}{(\gamma-1)^{2}}\right)(T-t)\right]\end{array}\right)$$

$$=\left(\frac{\left(X(t)+Y(t)\right)^{\gamma}}{\gamma}-\frac{\left[C(t)\right]^{\gamma}}{\gamma}\right)\times$$

$$\begin{pmatrix} K\exp\left[\left(\frac{(-\rho+\gamma r)}{(\gamma-1)}-\frac{3\gamma\theta^2}{2(\gamma-1)^2}+\gamma r\right)(T-t)\right] \\ +\lambda\exp\left[\left(\gamma r-\frac{\gamma\theta^2}{(\gamma-1)^2}\right)(T-t)\right] \end{pmatrix}$$

$$=\left(\frac{(X(t)+Y(t))^\gamma}{\gamma}-\frac{[C(t)]^\gamma}{\gamma}\right)\times$$

$$\exp\left[\gamma\left(r-\frac{\theta^2}{(\gamma-1)^2}\right)(T-t)\right]\times.$$

$$\left\{\left(K\exp\left[\frac{1}{(\gamma-1)}\left(-\rho+\gamma r+\frac{\gamma\theta^2}{2(1-\gamma)}\right)(T-t)\right]+\lambda\right)\right\} \quad (51)$$

We observe that the assumption of concavity of U turns out to be true, as

$$U_{XX}=U_{YY}=U_{XY}=(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma<0,$$

since, $\gamma<1$.

Theorem 5: The optimal protfolio and consumption process of the PPM are respectively be given by

$$\Delta(t)=\frac{(\mu-r)(X(t)+Y(t))+\sigma^S\sigma^I Y(t)(\gamma-1)}{X(t)(\sigma^S)^2(1-\gamma)},$$

$$C^*(t)=V\psi\exp\left[\frac{1}{(\gamma-1)^2}\left(-\rho+r+\frac{\gamma\theta^2}{(1-\gamma)}\right)(T-t)\right]\times$$

$$\left(K\exp\left[\frac{1}{(\gamma-1)}\left(-\rho+r\gamma+\frac{\gamma\theta^2}{2(1-\gamma)}\right)(T-t)\right]+\lambda\right)^{\frac{1}{\gamma-1}}$$

where $\psi=\exp\left[\frac{\rho T}{(\gamma-1)^2}\right]$.

Proof: By substituting Eq.(39) - Eq.(43) into Eq.(33), we have the following

$$\Delta^*(t)=\frac{-(\mu-r)V^{\gamma-1}(A(t)B(t))^\gamma}{X(\sigma^S)^2(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma}-$$

$$\frac{\sigma^S\sigma^I Y(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma}{X(\sigma^S)^2(\gamma-1)V^{\gamma-2}(A(t)B(t))^\gamma},$$

$$=\frac{(\mu-r)(X(t)+Y(t))+\sigma^S\sigma^I Y(\gamma-1)}{X(t)(\sigma^S)^2(1-\gamma)},$$

$$=\frac{(\mu-r)(X(t)+Y(t))+\sigma^S\sigma^I Y(t)(\gamma-1)}{X(t)(\sigma^S)^2(1-\gamma)},$$

$$=\frac{(\mu-r)}{(1-\gamma)(\sigma^S)^2}+\frac{(\mu-r)Y(t)}{X(t)(1-\gamma)(\sigma^S)^2}-\frac{\sigma^I Y(t)}{X(t)(\sigma^S)}. \quad (52)$$

Therefore, the proportion of wealth in cash account is obtain as:

$$1-\Delta^*(t)=1+\frac{\sigma^I Y(t)}{X(t)(\sigma^S)}-\frac{X(t)+Y(t)}{X(t)(1-\gamma)}\frac{(\mu-r)}{(\sigma^S)^2}.$$

From Eq.(35), we have that

$C^*(t)=I(U_X(V,t)\exp[\rho t])$. Substituting Eq.(29) into Eq.(35), we obatin

$$C^*(t)=I\left(V^{\gamma-1}(A(t)B(t))^\gamma\exp[\rho t]\right)$$

$$=\left(V^{\gamma-1}(A(t)B(t))^\gamma\exp[\rho t]\right)^{\frac{1}{\gamma-1}}$$

$$=V\left\{\left(\begin{matrix}\exp[r(T-t)]\times \\ \begin{pmatrix} K\exp\left[\frac{(\eta-2\gamma\theta^2)(T-t)}{2(\gamma-1)^2}\right]+ \\ \lambda\exp\left[\frac{-\gamma\theta^2(T-t)}{(\gamma-1)^2}\right]\end{pmatrix}\end{matrix}\right)\exp[\rho t]\right\}^{\frac{1}{\gamma-1}}$$

$$=V\exp\left[\frac{r(T-t)+\rho t}{\gamma-1}\right]\times$$

$$\begin{pmatrix} K\exp\left[\begin{matrix}\frac{(2\rho-\gamma\theta^2-2\gamma\rho-2\gamma r)(T-t)}{2(\gamma-1)^2} \\ +\frac{(2\gamma^2 r-2\gamma\theta^2)(T-t)}{2(\gamma-1)^2}\end{matrix}\right] \\ +\lambda\exp\left[\frac{-\gamma\theta^2(T-t)}{(\gamma-1)^2}\right]\end{pmatrix}^{\frac{1}{\gamma-1}}$$

$$=V\begin{pmatrix} K\exp\left[\begin{matrix}\frac{(2\rho(1-\gamma)-2\gamma r(1-\gamma)-3\gamma\theta^2)(T-t)}{2(\gamma-1)^2} \\ +\frac{r(T-t)+\rho t}{\gamma-1}\end{matrix}\right]+ \\ \lambda\exp\left[\frac{-\gamma\theta^2(T-t)}{(\gamma-1)^2}+\frac{r(T-t)+\rho t}{\gamma-1}\right]\end{pmatrix}^{\frac{1}{\gamma-1}}$$

$$= V\left(\begin{array}{l} K\exp\left[\begin{array}{l}\dfrac{(-\rho+\gamma r+r)(T-t)}{(\gamma-1)}- \\ \dfrac{3\gamma\theta^2(T-t)}{2(\gamma-1)^2}+\dfrac{\rho t}{\gamma-1}\end{array}\right]+ \\ \lambda\exp\left[\dfrac{-\gamma\theta^2(T-t)}{(\gamma-1)^2}+\dfrac{r(T-t)+\rho t}{\gamma-1}\right]\end{array}\right)^{\frac{1}{\gamma-1}}$$

$$= V\psi\exp\left[\frac{1}{(\gamma-1)^2}\left(-\rho+r+\frac{\gamma\theta^2}{(1-\gamma)}\right)(T-t)\right]\times$$

$$\left(K\exp\left[\frac{1}{(\gamma-1)}\left(-\rho+r\gamma+\frac{\gamma\theta^2}{2(1-\gamma)}\right)(T-t)\right]+\lambda\right)^{\frac{1}{\gamma-1}}$$

Now, for $0<\gamma<1$, we have intuitively that the growth rate (GR) of the expected optimal consumption is given by

$$GR = \frac{1}{(\gamma-1)^2}\left(r-\rho+\frac{\gamma\theta^2}{(1-\gamma)}\right). \quad (53)$$

This is referred to as the Euler equation for the intertemporal maximization of consumption under uncertainty. The positive term N captures the uncertainty of the financial martet. When the financial market is risky, it will induce investors to shift consumption over time. From (52), the first two terms are the variational form of the classical portfolio strategy while the last term is an inter-temporal hedging strategy that offset any shock to the salary of the PPM.

V. OPTIMAL EXPECTED VALUE OF WEALTH AND EXPECTED CONSUMPTION PROCESS OF THE PPM

In this section, we present the optimal expected value of wealth of the PPM in pension plan at time t. From (26), we have that

$$dE(V(t)) = E\left[\left(\begin{array}{l}\Delta(t)X^{\Delta,C}(t)(\mu-r)+rX^{\Delta,C}(t)+ \\ Y(t)(r+\sigma\theta)-C(t)\end{array}\right)\right]dt$$

.Now, substituting (52) and using (19), we obtain the following

$$dE(V^*(t)) =$$

$$E\left[\left(\begin{array}{l}\dfrac{(\mu-r)^2}{(\sigma^S)^2(1-\gamma)}V^*(t)+ \\ \dfrac{\sigma^I\beta h}{\sigma^S\varphi}\exp(\omega t)(\exp(\varphi(T-t))-1)+ \\ rV^*(t)+\dfrac{\beta h\sigma\theta}{\varphi}\exp(\omega t)\times \\ (\exp(\varphi(T-t))-1)-C^*(t)\end{array}\right)\right]dt$$

$$= \left(\begin{array}{l}\left(\dfrac{\theta^2}{(1-\gamma)}+r\right)E(V^*(t))+\left(\dfrac{\sigma^I\beta h}{\sigma^S\varphi}+\dfrac{\beta h\sigma\theta}{\varphi}\right)\times \\ \exp(\omega t)(\exp(\varphi(T-t))-1)-C^*(t)\end{array}\right)dt \quad (54)$$

Solving the ordinary differential equation (54), we obtain the following

$$E(V^*(t)) = v_0\exp(\delta t)+$$

$$\frac{\alpha\exp(\varphi T)(\exp(\delta t)-\exp(-(\varphi-\omega)t))}{\varphi+\delta-\omega}+$$

$$\frac{\alpha(\exp(\delta t)-\exp(\omega t))}{\omega-\delta}-$$

$$\int_0^t E(C^*(s))\exp(\delta(t-s))ds, \quad (55)$$

where, $\delta=\dfrac{\theta^2}{1-\gamma}+r$, $\alpha=\dfrac{\beta h}{\varphi}\left(\dfrac{\sigma^I}{\sigma^S}+\sigma\theta\right)$

and $v_0 = x_0+\dfrac{h}{\varphi}(\exp(\varphi T)-1)$.

At $t=T$, we have

$$E(V^*(T)) = v_0\exp(\delta T)+$$

$$\alpha\left(\begin{array}{l}\dfrac{\exp(\varphi T)(\exp(\delta T)-\exp(-(\varphi-\omega)T))}{\varphi+\delta-\omega} \\ +\dfrac{(\exp(\delta T)-\exp(\omega T))}{\omega-\delta}\end{array}\right)$$

$$-\int_0^T E(C^*(s))\exp(\delta(T-s))ds.$$

It is easy to observe that the expected optimal final fund is the sum of the fund that one would have by investing the whole portfolio always in the risk-free asset and the risky asset plus the term

$$\alpha\left(\begin{array}{l}\dfrac{\exp(\varphi T)(\exp(\delta T)-\exp(-(\varphi-\omega)T))}{\varphi+\delta-\omega} \\ +\dfrac{(\exp(\delta T)-\exp(\omega T))}{\omega-\delta}\end{array}\right)$$

that depends both on the goodness of the risky asset with respect to the risk-free asset, less the sum of the consumption from the beginning up to the terminal period. Hence, the higher the Sharpe ratio of the risky asset, θ, the higher the expected optimal final wealth, everything else being equal. But,

$$C^*(t) = V\psi \exp\left[\frac{1}{(\gamma-1)^2}\left(-\rho + r + \frac{\gamma\theta^2}{(1-\gamma)}\right)(T-t)\right]\times$$

$$\left(K \exp\left[\frac{1}{(\gamma-1)}\left(-\rho + r\gamma + \frac{\gamma\theta^2}{2(1-\gamma)}\right)(T-t)\right] + \lambda\right)^{\frac{1}{\gamma-1}}$$

Taking the expectation of bothsides of the above equation, we obtain the following

$$E\left(C^*(t)\right) = E(V(t))\psi \exp[z_1(T-t)]\times$$

$$\left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}} \tag{56}$$

where, $z_1 = \frac{1}{(\gamma-1)^2}\left(-\rho + r + \frac{\gamma\theta^2}{(1-\gamma)}\right)$

and $z_2 = \frac{1}{(\gamma-1)}\left(-\rho + r\gamma + \frac{\gamma\theta^2}{2(1-\gamma)}\right)$

Substituting (55) into (56), we obtain the following

$$E\left(C^*(t)\right) =$$

$$\begin{pmatrix} v_0 \exp(\delta t) + \frac{\alpha \exp(\varphi T)(\exp(\delta t))}{\varphi + \delta - \omega} - \\ \frac{\exp(-(\varphi-\omega)t)}{\varphi+\delta-\omega} + \frac{\alpha(\exp(\delta t) - \exp(\omega t))}{\omega - \delta} \\ -\int_0^t E\left(C^*(s)\right)\exp(\delta(t-s))ds \end{pmatrix}\psi \times$$

$$\exp[z_1(T-t)]\left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}}$$

Therefore,

$$E\left(C^*(t)\right) + \psi \exp[z_1(T-t)]\times$$

$$\left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}}\times$$

$$\int_0^t E\left(C^*(s)\right)\exp(\delta(t-s))ds =$$

$$\begin{pmatrix} \frac{\alpha \exp(\varphi T)(\exp(\delta t) - \exp(-(\varphi-\omega)t))}{\varphi+\delta-\omega} + \\ \frac{\alpha(\exp(\delta t) - \exp(\omega t))}{\omega-\delta} + v_0 \exp(\delta t) \end{pmatrix}\psi \times$$

$$\exp[z_1(T-t)]\left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}}$$

It implies that

$$\int_0^t \begin{Bmatrix} dE\left(C^*(t)\right) + \psi \exp[z_1(T-t)]\times \\ \left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}}\times \\ E\left(C^*(s)\right)\exp(\delta(t-s)) \end{Bmatrix} ds =$$

$$\begin{pmatrix} \frac{\alpha \exp(\varphi T)(\exp(\delta t) - \exp(-(\varphi-\omega)t))}{\varphi+\delta-\omega} \\ + \frac{\alpha(\exp(\delta t) - \exp(\omega t))}{\omega-\delta} + v_0 \exp(\delta t) \end{pmatrix}\psi \times$$

$$\exp[z_1(T-t)]\left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}} \tag{57}$$

with $C(0) = 0$.

At $t = T$, we have

$$\int_0^T \begin{Bmatrix} dE\left(C^*(T)\right) + \psi(K+\lambda)^{\frac{1}{\gamma-1}} E\left(C^*(s)\right)\times \\ \exp(\delta(T-s)) \end{Bmatrix} ds =$$

$$\begin{pmatrix} v_0 \exp(\delta t) + \frac{\alpha \exp(\varphi T)(\exp(\delta T))}{\varphi+\delta-\omega} \\ -\frac{\exp(-(\varphi-\omega)T)}{\varphi+\delta-\omega} \\ + \frac{\alpha(\exp(\delta T) - \exp(\omega T))}{\omega-\delta} \end{pmatrix} \times \psi(K+\lambda)^{\frac{1}{\gamma-1}}$$

Taking the limit of bothsides of (57) as t tends to infinity, we have the following

$$\lim_{t\to\infty} \int_0^t \begin{Bmatrix} dE\left(C^*(t)\right) + \psi \exp[z_1(T-t)]\times \\ \left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}}\times \\ E\left(C^*(s)\right)\exp(\delta(t-s)) \end{Bmatrix} ds =$$

$$\lim_{t\to\infty} \begin{pmatrix} \begin{pmatrix} \frac{\alpha \exp(\varphi T)(\exp(\delta t) - \exp(-(\varphi-\omega)t))}{\varphi+\delta-\omega} \\ + \frac{\alpha(\exp(\delta t) - \exp(\omega t))}{\omega-\delta} + v_0 \exp(\delta t) \end{pmatrix}\psi \times \\ \exp[z_1(T-t)]\left(K \exp[z_2(T-t)] + \lambda\right)^{\frac{1}{\gamma-1}} \end{pmatrix}$$

$$\int_0^\infty \begin{Bmatrix} dE\left(C^*(\infty)\right) + \psi \exp[z_1(T-\infty)]\times \\ \left(K \exp[z_2(T-\infty)] + \lambda\right)^{\frac{1}{\gamma-1}}\times \\ E\left(C^*(s)\right)\exp(\delta(\infty-s)) \end{Bmatrix} ds =$$

$$\lim_{t\to\infty}\left(\left(\begin{array}{l}\dfrac{\alpha\exp(\varphi T)\left(\exp(\delta t)-\exp(-(\varphi-\omega)t)\right)}{\varphi+\delta-\omega}\\+\dfrac{\alpha\left(\exp(\delta t)-\exp(\omega t)\right)}{\omega-\delta}+v_0\exp(\delta t)\end{array}\right)\psi\times\atop \exp[z_1(T-t)]\left(K\exp[z_2(T-t)]+\lambda\right)^{\frac{1}{\gamma-1}}\right)$$

$$\int_0^\infty\left\{dE\left(C^*(\infty)\right)+E\left(C^*(s)\right)\exp\left(\delta(\infty-s)\right)\right\}ds=0.$$

This shows that consumption terminates, when the value of the PPM's accumulations terminates after a very large t. It implies that

$$\int_0^\infty E\left(C^*(s)\right)\exp\left(\delta(\infty-s)\right)ds=-C_\infty.$$

This shows that at infinity, the sum of consumption would be negative, which is an intuitive result.

VI. CONCLUSION

We considered the optimal portfolio and consumption strategies for a defined contributory pension plan. We found an explicit solution to the non-linear partial differential equation that arises from our problem. We also found that part of the portfolio value is proportional to the ratio of the present value of expected discounted future contributions to the optimal portfolio value at time t. We found an interesting result that tells us that as the market evolve, part of the portfolio value should be transferred to the cash account at time t in order to offset unforeseen shocks that may occur to the PPM's salary and the investment in future time. We also found that the consumption of a PPM terminates, when the value of the PPM"s accumulations terminates.

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