Variance Reduction with Control Variate for Pricing Asian Options in a Geometric Lévy Model

Wissem Boughamoura and Faouzi Trabelsi

Abstract—**This paper is concerned with Monte Carlo simulation of generalized Asian option prices where the underlying asset is modeled by a geometric (finite-activity) Lévy process. A control variate technique is proposed to improve standard Monte Carlo method. Some convergence results for plain Monte Carlo simulation of generalized Asian options are proven, and a detailed numerical study is provided showing the performance of the variance reduction compared to straightforward Monte Carlo method.**

Index Terms—**Lévy process, Asian options, Minimal-entropy martingale measure, Monte Carlo method, Variance Reduction, Control variate.**

I. Introduction

THe pricing of Asian options is a delicate and interesting topic in quantitative finance, and has been a topic of attention for many years now. Asian options are commonly traded: they were introduced to avoid a problem common for European options, where the speculators could drive up the gains from the option by manipulating the price of the underlying asset near to the maturity date.They have become very attractive for investors since they provide a customized cheap way to hedge periodic cash flows [24]. In spite of such advantages there is no closed form solution for the price of an Asian option, even when the underlying follows a Geometric Brownian model. As a remedy to this difficulty, many approaches and numerical approximations are proposed. For instance, in continuous markets, [27] provided tight analytic bounds for the Asian option price, [12] computed the Laplace transform of the Asian Option price, [17] used Monte Carlo simulations, [16] investigated a method based on binomial trees, [1] used Fast Fourier Transform technique. The authors of [9] introduced change of numeraire technique and those of [26] used it to reduce the PDE problem in two variables. They got a lower bound for the price. [15] used lattices to price fixed-strike European-style Asian options that are discretely monitored. [7] derived a closed-form solution for the price of an average strike as well as an average price geometric Asian option, by making use of the path integral formulation. [19] studied a certain one-dimensional, degenerate parabolic partial differential equation with a boundary condition which arises in pricing of Asian options. [25] considered the approximation of the optimal stopping problem associated with ultradiffusion processes in valuating Asian options. The value function is characterized as the solution of an ultraparabolic variational inequality.

For discontinuous markets, [8] generalized Laplace transform for continuously sampled Asian option where underlying asset is driven by a Lévy Process, [18] presented a binomial tree method for Asian options under a particular jump-diffusion model. But such Market may be incomplete. Therefore, it will be a big class of equivalent martingale measures (EMMs). For the choice of suitable one, many methods have been proposed: Esscher martingale measure [13, 3], Minimal martingale measure [10], which can be viewed as time-honored Esscher transform and minimal entropy martingale measure (MEMM) introduced in [11], [28] developped a simple network approach to American exotic option valuation under Lévy processes using the fast Fourier transform (FFT). Finally, [22] derived bounds for the price of a discretely monitored arithmetic Asian option when the underlying asset follows an arbitrary Lévy process.

In this paper, we placed in a financial market driven by a finite-activity Lévy process and we study the pricing of generalized Asian options under the MEMM. The choice of such measure is justified by the fact that it preserves the Lévy and the statistical properties of the process. To estimate prices, we propose a control variate variance reduction technique to improve the Monte Carlo method and illustrate numerically its improvement for particular examples of generalized Asian options. The use of Monte Carlo techniques is an established approach when dealing with this kind of problems both in theory and practice, and the paper makes a contribution to the literature.

Our paper is structured as follows: In section (II), we recall the main properties of a geometric (finite activity) Lévy process and give some particular generalized Asian options. In section (III), we introduce the minimal entropy martingale measure under which we study the properties of the stock price. In section (IV), we study the convergence of Monte Carlo price estimator in L^2 using the Reimann scheme and propose a particular control variate variance reduction technique to improve the Monte Carlo method. Such amelioration is illustrated numerically for particular examples of generalized Asian options in section (V). Finally, section (VI) deals with concluding remarks.

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Laboratory of Mathematical and Numerical Modelling in Engineering Science, National Engineering School of Tunis (ENIT-LAMSIN), B.P. 37, 1002 Tunis-Belvédère, Tunis El Manar University, Tunisia, and Higher Institute of Applied Mathematics and Computer Science of Kairouan, Avenue Assad Ibn Fourat, 3100 Kairouan, University of Kairouan, Tunisia. Email: wissemb1@gmail.com.

Laboratory of Mathematical and Numerical Modelling in Engineering Science, National Engineering School of Tunis (ENIT-LAMSIN), B.P. 37, 1002 Tunis-Belvédère, Tunis El Manar University, Tunisia, and Higher Institute of Computer Sciences and Mathematics of Monastir, Avenue de la Korniche, B.P. 223, 5000 Monastir, University of Monastir, Tunisia. E-mail: faouzi_trabelsi@yahoo.fr and faouzi.trabelsi@lamsin.rnu.tn.

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II. The framework

A. The market model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space. We consider a market which consists of a bond with price process given by $B_t = e^{rt}$, where r denotes the (constant) interest rate and a non-dividend paying risky asset (the stock or index). The price process $(S_t)_{0\le t\le T}$ of the risky asset is modeled by the following geometric Lévy process:

$$S_t = S_0 \exp(rt + L_t),\ 0 \le t \le T, \tag{1}$$

where L_t is a Lévy process given by:

$$L_t = \mu t + \sigma B_t + \sum_{j=1}^{N_t} U_j, \tag{2}$$

here μ is a drift, σ is the underlying volatility supposed to be positive and $(N_t)_{0\le t\le T}$ is a Poisson process with intensity λ. The jump sizes $(U_j)_{j\ge 1}$ are i.i.d. $\mathbb{P}$-integrable r.v. with a distribution F and independent of $(N_t)_{0\le t\le T}$. We denote by $\hat{S}_t = e^{-rt} S_t$ the discounted underlying price process.
We assume that the σ-fields generated respectively by $(B_t)_{0\le t\le T}$, $(N_t)_{0\le t\le T}$ and $(U_j)_{j\ge1}$ are independents and we take $\mathcal{F}_t := \sigma(B_s, N_s, U_j \mathbf{1}_{\{j\le N_s\}}; s \le t, j \ge 1), t \ge 0$ as filtration.

Let $\tau_1, \cdots, \tau_j, \cdots$ be the dates of jumps of the stock price S_t; i.e. the jump times of $(N_t)_{0\le t\le T}$ and $V_1, \cdots, V_j, \cdots$ their proportions of jumps at these times, respectively; i.e.

$$\Delta S_{\tau_j} = S_{\tau_j} - S_{\tau_j^-} = S_{\tau_j^-} V_j \qquad \text{for all } j \ge 1,$$

where $V_j = e^{U_j} - 1$. The r.v.'s $(V_j)_{j\ge1}$ are i.i.d. with a common distribution G and takes its values in the interval $(-1, +\infty)$.

Recall that L_t is a Lévy process with the characteristic triplet $(\mu + \int_{|x|\le 1} x\nu_L(dx), \sigma^2, \nu_L)$, where $\nu_L(dx) = \lambda F(dx)$. The Lévy-Itô decomposition of L_t takes the form:

$$L_t = \mu + \int_{|x|\le 1} x\nu_L(dx)t + \sigma B_t$$
$$\int_{[0,t]}\int_{|x|>1} xJ_L(ds,dx) + \int_{[0,t]}\int_{|x|\le 1} x\widetilde{J}_L(ds,dx) \tag{3}$$

where $J_L(dt,dx) = \sum_{j\ge1} \delta_{(\tau_j, \Delta L_{\tau_j})}$ is the random Poisson measure of L on $[0,+\infty[\times\mathbb{R}$ with intensity $\nu_L(dt,dx) := \nu_L(dx)dt$ and $\widetilde{J}_L(dt,dx) := (J_L - \nu_L)(dt,dx)$. Set $X_t := L_t + rt$, then X_t is a Lévy process with characteristic triplets (b, σ^2, ν_L), where

$$b = \mu + \int_{|x|\le 1} x\nu_L(dx) + r. \tag{4}$$

According to proposition 8.22 of [6], there exists a Lévy process $\hat{X}$ such that S_t is the Doleans-Dade exponent of $\hat{X}_t$: $S = \mathcal{E}(\hat{X})$. The process $\hat{X}$ is given by

$$\begin{aligned}\hat{X}_t &= X_t + \frac{\sigma^2}{2}t + \sum_{0\le u\le t}(1 + \Delta X_u - e^{\Delta X_u}) \\ &= L_t + \left(r + \frac{\sigma^2}{2}\right)t + \sum_{0\le u\le t}(1 + \Delta L_u - e^{\Delta L_u})\end{aligned} \tag{5}$$

and satisfies the stochastic differential equation

$$S_t = S_0 + \int_0^t S_{u-} d\hat{X}_u.$$

B. Generalized Asian options

Let Y_t be the arithmetic average value of the underlying $(S_t)_{t\in[0,T]}$ on the life time interval $[0,t]$:

$$Y_t = \frac{1}{t}\int_0^t S_u du.$$

Consider an Asian option maturing at time T based on Y_T with an $\mathcal{F}_T$-measurable payoff H having the form:

$$H = \varphi(S_T, Y_T),$$

where φ satisfies the following Lipchitz assumption:

$(\mathbf{A_1})$ $\qquad \exists a > 0; \forall (S, S', Y, Y') \in \mathbb{R}_+^4,$

$|\varphi(S,Y) - \varphi(S',Y')| \le a[|S - S'| + |Y - Y'|].$

Such option is called *generalized Asian option*. As examples we can cite the payoffs $\varphi(S_T, Y_T) = |S_T - Y_T|$ of a straddle Asian option and the payoff $\varphi(S_T, Y_T) = (\zeta.(Y_T - K_1 S_T - K_2))^+$, which is a fixed strike Asian option, for $K_1 = 0$, and a floating strike Asian option for $K_2 = 0$. The constant $\zeta = \pm 1$ determines whether the option is call or put.

The price V_t of such option at time t depends on t, S_t and on the path that the asset price followed up to time t. In particular, we can not invoke the Markov Property to claim that V_t is a function of t and S_t because H is not a function of T and S_T ; H depends on the whole path of S.

To overcome this difficulty, we increase the state S_t by defining a second process Y_t following the stochastic differential equation:

$$dY_t = \frac{S_t - Y_t}{t}dt. \tag{6}$$

The process (S_t, Y_t) is governed by the pair of equations (1) and (6) constitutes a two-dimensional Markov process. Furthermore, the payoff $H = \varphi(S_T, Y_T)$ is a function of T and the final value (S_T, Y_T) of this process. Thus, there exists some function $\phi(t, S, Y)$ such that the generalized Asian option price under a particular probability $\mathbb{P}^*$ is given by

$$\phi(t, S_t, Y_t) = \mathbb{E}^*[e^{-r(T-t)}\varphi(S_T, Y_T)/\mathcal{F}_t].$$

III. Minimal entropy martingale measure and Geometric Lévy processes

A. Minimal entropy martingale measure

In this section, we denote by $\mathcal{P}(S)$ the set of all EMM of $\hat{S}$; i.e. the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that the process $\{\hat{S}_t, 0 \le t \le T\}$ is an $(\mathcal{F}_t, \mathbb{Q})$-martingale and $\mathbb{Q} \sim \mathbb{P}$.

Definition 1: The relative entropy of $\mathbb{Q} \in \mathcal{P}(S)$, with respect to $\mathbb{P}$, is defined by:

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \int_\Omega \log\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] d\mathbb{Q}, & \text{if } \mathbb{Q} \ll \mathbb{P}; \\ \infty & \text{otherwise.}\end{cases}$$

If an EMM $\mathbb{P}^*$ satisfies the following condition:

$$H(\mathbb{P}^*|\mathbb{P}) \le H(\mathbb{Q}|\mathbb{P}), \qquad \text{for all } \mathbb{Q} \in \mathcal{P}(S),$$

then, $\mathbb{P}^*$ is called the *minimal entropy martingale measure* (MEMM) of $(\hat{S}_t)_{0\le t\le T}$.
A general feature of the MEMM is that its statistical properties resemble the original process so the specification of the

prior is quite important. It has also the advantage that the Lévy property is preserved for geometric Lévy processes. The following theorem resumes the main properties and results concerning the MEMM.

Theorem 1: [11] Suppose there exists a constant β^* such that the following assumptions hold:

$(\mathbf{A_2})$

$$\int_{x>1} e^x e^{\beta^*(e^x-1)}\nu(dx) = \lambda\mathbb{E}^{\mathbb{P}}\left[1_{U_1>1}e^{U_1}e^{\beta^*(e^{U_1}-1)}\right] < \infty$$

$$(\mathbf{A_3})\ \ \mu + \left(\frac{1}{2}+\beta^*\right)\sigma^2 + \int_{\mathbb{R}}(e^x-1)e^{\beta^*(e^x-1)}\nu_L(dx) = 0$$

then,

1) Define a probability measure $\mathbb{P}^*$ on $\mathcal{F}_T$ by:

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} := e^{\beta^*\hat{X}_t - b_0 t}, \quad \text{for every } t\in[0,T],$$

where $(\hat{X}_t)$ is defined by (5), and

$$\begin{aligned} b_0 &= \frac{\beta^*}{2}(1+\beta^*)\sigma^2 + \beta^* b \\ &+ \int_{\mathbb{R}}(e^{\beta^*(e^x-1)} - 1 - \beta^* x 1_{\{|x|\le 1\}})\nu_L(dx). \end{aligned}$$

where b is given by (4).

2) The probability measure $\mathbb{P}^*$ is actually in $\mathcal{P}(S)$. In addition, $\mathbb{P}^*$ is the MEMM of $(S_t)_{t\ge 0}$ and the relative entropy of $\mathbb{P}^*$ with respect to $\mathbb{P}$ has the form:

$$\mathbb{H}(\mathbb{P}^*|\mathbb{P}) = -T\left[\beta^*\mu + \frac{\beta^*(1+\beta^*)}{2}\sigma^2 \right.$$

$$\left. + \int_{\mathbb{R}}(e^{\beta^*(e^x-1)} - \beta^* 1_{\{|x|\le 1\}})\nu_L(dx)\right].$$

3) The Stochastic process (L_t) remains a Lévy process under $\mathbb{P}^*$ with the following Lévy-Itô decomposition:

$$\begin{aligned} L_t &= \mu^* t + \sigma B^*_t + \int_0^t\int_{|x|\le 1} x(J_L - \nu^*_L)(ds,dx) \\ &+ \int_0^t\int_{|x|>1} xJ_L(ds,dx), \end{aligned} \quad (7)$$

where $\mu^* = \mu + \beta^*\sigma^2 + \int_{|x|\le 1} xe^{\beta^*(e^x-1)}\nu_L(dx)$, $\nu^*_L = e^{\beta^*(e^x-1)}\nu_L(dx)$ and $B^*_t = B_t - \sigma\beta^* t$ is a $\mathbb{P}^*$-Brownian motion. The Lévy process L is a $\mathbb{P}^*$-square integrable martingale with characteristic triplet $(\mu^*, \sigma^2, \nu^*_L)$.

Remark 1: Assumption $(\mathbf{A_2})$ ensures that the first exponential moment of L_t under $\mathbb{P}^*$ is finite, i.e. $(\hat{S}_t)_{0\le t\le T}$ is $\mathbb{P}^*$-integrable. Consequently, the r.v. $V_1 = e^{U_1}+1$ is also $\mathbb{P}^*$-integrable.

Remark 2: Set

$$f(\beta) = \mu + \left(\frac{1}{2}+\beta\right)\sigma^2 + \int_{\mathbb{R}}(e^x-1)e^{\beta(e^x-1)}\nu_L(dx). \quad (8)$$

(i) The function f is an increasing continuous function with $\lim_{-\infty} f = -\infty$ and $\lim_{+\infty} f = +\infty$. Thus, the function (8) has a unique root β^*.

(ii) $f(0)$ and β^* have opposite sign (if $\beta^*\neq 0$).

B. The geometric Lévy process under the MEMM

The decomposition (7) of L can be rewritten as:

$$L_t = \mu^* t + \sigma B^*_t + \left(\sum_{j=1}^{N_t} U_j - t\lambda\kappa\right).$$

where $\kappa = \mathbb{E}^*[U_1 \mathbf{1}_{\{|U_1|\le 1\}}]$ and $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$. The expression of S_t is finally given by:

$$S_t = S_0\exp(\mu_0 t + \sigma B^*_t)\prod_{j=1}^{N_t}(1+V_j),$$

where

$$\begin{aligned} \mu_0 &= \mu + \beta^*\sigma^2 + \int_{|x|\le 1} xe^{\beta^*(e^x-1)}\nu_L(dx) - \lambda\kappa + r \\ &= \mu + \beta^*\sigma^2 + r. \end{aligned}$$

IV. Pricing of generalized Asian options with Monte Carlo method

Monte Carlo methods [4, 5, 17, 20] are known to be useful as the state space is large. This motivates us to use Monte Carlo simulations in order to approximate numerically the (initial) price of a generalized Asian option under the MEMM. To this end, we approximate the integral of the underlying asset price by the standard Reimann scheme. The Monte Carlo method will be improved by the use of a control variate variance reduction technique. We will choose the underlying price process itself (at maturity) as a control variate. In order to apply the control variate approach, we choose as control variable the underlying asset itself following [23] for European call option.

A. Standard Monte Carlo technique

In this section, we present briefly the Monte Carlo Method using the Riemann scheme. Since we are able to approximate $A_T = \int_0^T S_u du$, we consider the subdivision $(t_k)_{k\in\{0,1,\dots,n\}}$ of $[0,T]$ where $t_k = k\frac{T}{n} = kh$, $k=0,\dots,n$. The Reimann scheme can be described by:

$$A^n_T = h\sum_{k=0}^{n-1} S_{t_k}.$$

The price of the generalized Asian option with payoff $\varphi(S_T, Y_T)$ is approximated using the Monte Carlo Method by:

$$V_0 \approx \frac{e^{-rT}}{M}\sum_{j=1}^{M}\varphi\left(S^j_T, \frac{1}{n}\sum_{k=0}^{n-1} S^j_{t_k}\right),$$

where M is the number of Monte Carlo and $S^j = (S^j_0, S^j_{t_1}, \dots, S^j_T)$ denote a realization of $S = (S_0, S_{t_1}, \dots, S_T)$ in the j^{th} iteration.

The complexity of the last algorithm is $O\left(\frac{1}{Mn}\right)$, which is in fact true for every kind of Monte Carlo method. It involves two kinds of errors: the Monte Carlo error which is of order $O\left(\frac{1}{M}\right)$ and the time step error which is harder to evaluate and expressed in proposition (1). This convergence result is original and contributes to the literature of Asian option pricing.

Let us then study the L^2 convergence of the Reimann scheme. We show that it is of order $O\left(\frac{1}{n}\right)$. We have the following result:

Proposition 1: (**Convergence in** L^2) Under the MEMM, we have:

$$\begin{aligned}&\sqrt{\mathbb{E}^*\left|Y_T-Y_T^n\right|^2} \qquad (9)\\ &= \frac{S_0}{\sqrt{6}}\sqrt{\left(3(1-e^{rT})+\left(2-\frac{r}{c}\right)\left(e^{cT}-1\right)\right)}\frac{1}{n}\\ &+\frac{S_0T}{4\sqrt{6}}\frac{\left(e^{rT}-1+(r-1)(e^{cT}-1)\right)}{\sqrt{3(1-e^{rT})+\left(2-\frac{r}{c}\right)\left(e^{cT}-1\right)}}\frac{1}{n^2}\\ &+\quad o\left(n^{-2}\right).\end{aligned}$$

Proof: The proof is put in Appendix (B) and based of the following Lemma the proof of which is put in Appendix (A):

Lemma 1: For all $0 \le u \le v$, we have $\mathbb{E}^*[\hat{S}_u^2] = S_0^2 e^{c.u}$ and $\mathbb{E}^*[S_uS_v] = S_0^2 e^{rv} e^{u(c-r)}$ where

$$c = \sigma^2 + \int_{\mathbb{R}} (e^x-1)^2 \nu_L^*(dx). \qquad (10)$$

■

B. Variance reduction with control variate technique

Variance reduction techniques are used to increase accuracy in the estimated variable by decreasing sample standard deviation, instead of large samples. The method of control variate, first introduced by [2], takes the advantages of random variables with known expected value and positively correlated with the variable under consideration.

The price of a generalized Asian option with payoff $\varphi(S_T, Y_T)$ is approximated by

$$V_0 \approx \mathbb{E}^*[Z_n],$$

where Z_n is given by expression

$$Z_n = e^{-rT}\varphi\left(S_T, \frac{h}{T}\sum_{k=0}^{n-1} S_{t_k}\right), \qquad (11)$$

with φ a function satisfying hypothesis $(\mathbf{A_1})$.

For n large enough, we have to determine the mean of the random variable Z_n through simulation. In order to apply the control variate approach, we choose as control variable the price at maturity of the underlying asset itself following [23] for European call option. Each discretized sample path $S^j(n) = (S_0, S_{t_1}^j, \ldots, S_{t_{n-1}}^j, S_T^j)$ generates a sample of Z_n^j. The control variate estimator $\bar{V}_{CV}$ of $\mathbb{E}^*[Z_n]$ is given by:

$$\begin{aligned}\bar{V}_{CV} &= \bar{Z}_n - \bar{S}_T + \mathbb{E}^*[S_T]\\ &= \frac{1}{M}\sum_{j=1}^{M}\left[e^{-rT}\varphi\left(S_T^j, h_n\sum_{i=0}^{n-1}S_{t_i}^j\right) - S_T^j + S_0e^{rT}\right].\end{aligned}$$

We can show that the control variate estimator $\overline{V}_{CV}$ is *unbiased and consistent* (see [14, 23]). Its variance is given by

$$\text{Var}(\bar{V}_{CV}) = \frac{1}{M}\left[\sigma_{Z_n}^2 + \sigma_{S_T}^2 - 2\rho_{S_TZ_n}\sigma_{Z_n}\sigma_{S_T}\right].$$

This shows that the control variate estimator $\bar{V}_{CV}$, will have lower variance than $\bar{Z}_n$ if and only if

$$\frac{\text{Cov}(S_T, Z_n)}{\text{Var}(S_T)} > \frac{1}{2}.$$

To fully take advantage of control variate, a parameter ς is introduced and optimized to minimize the variance of $\bar{V}_{CV}$. The parameterized control variate estimator is defined as follows:

$$\bar{V}_{CV}(\varsigma) = \bar{Z}_n - \varsigma(\bar{S}_T - S_0e^{rT}).$$

The resulting variance is:

$$\text{Var}(\bar{V}_{CV}(\varsigma)) = \frac{1}{M}\left[\sigma_{Z_n}^2 + \varsigma^2\sigma_{S_T}^2 - 2\varsigma\rho_{S_TZ_n}\sigma_{Z_n}\sigma_{S_T}\right]. \qquad (12)$$

The optimal parameter ς_n^* that minimizes this variance is given by:

$$\varsigma_n^* = \frac{\sigma_{Z_n}}{\sigma_{S_T}}\rho_{S_TZ_n} = \frac{\text{Cov}(S_T, Z_n)}{\text{Var}(S_T)}. \qquad (13)$$

Inserting (13) in (12), we get

$$\text{Var}(\bar{V}_{CV}(\varsigma_n^*)) = \left(1-\rho_{S_TZ_n}^2\right)\frac{\sigma_{Z_n}^2}{M}. \qquad (14)$$

The variance reduction is then highly dependent on the correlation between the simulated variable Z_n and the control variate S_T. In practice, $\text{Cov}(S_T, Z_n)$ is never known and has to be simulated. Although $\text{Var}(S_T)$ is known explicitly, it has to be simulated since its expression involves some integrals. The sample estimators

$$\begin{aligned}\mathsf{S}_{S_TS_T} &= \frac{1}{M-1}\sum_{j=1}^{M}(S_T^j - \bar{S}_T)^2\\ \mathsf{S}_{S_TZ_n} &= \frac{1}{M-1}\sum_{j=1}^{M}(S_T^j - \bar{S}_T)(Z_n^j - \bar{Z}_n)\end{aligned}$$

give the estimator of ς_n^* as follows:

$$\hat{\varsigma}_n^* = \mathsf{S}_{S_TZ_n}\mathsf{S}_{S_TS_T}^{-1}.$$

Thus,

$$\text{Var}(\bar{V}_{CV}(\hat{\varsigma}_n^*)) = \frac{1}{M}\left[\text{Var}(Z_n) + \hat{\varsigma}_n^*\text{Var}(S_T)\left(\hat{\varsigma}_n^* - 2\varsigma_n^*\right)\right].$$

As $\hat{\varsigma}_n^*$ converges almost surely to ς_n^*, $\hat{\varsigma}_n^* - 2\varsigma_n^* < 0$ $a.s.$, for M large enough. Consequently, we get

$$\text{Var}(\bar{V}_{CV}(\hat{\varsigma}_n^*) < \frac{1}{M}\text{Var}(Z_n) = \text{Var}(\bar{Z}_n),$$

for M large enough. In order to estimate ς_n^*, there are two possible scenarios:

(1) We can first estimate ς_n^* by a small number of simulations, and next use many additional simulations to approximate the control variate estimator itself.

(2) Another alternative is to use the totality of simulations to estimate ς_n^*.

The second approach has the disadvantage that a bias arises in the estimation of $\mathbb{E}^*[Z_n]$ since $\hat{\varsigma}_n^*$, S_T and Z_n are dependent. This bias does not allow to calculate the variance directly from (14). Besides, the r.v.

$$\frac{\bar{V}_{CV}(\hat{\varsigma}_n^*) - \mathbb{E}^*[Z_n]}{\sigma(\bar{V}_{CV}(\hat{\varsigma}_n^*)}$$

is not a t-distribution and consequently confidence intervals based on t-distrubution can not be used directly.

According to [21], the variance of the control variate estimator in the case where (S_T, Z_n) follows a multivariate normal distribution, is given by the expression:

$$\text{Var}(\bar{V}_{CV}(\hat{\varsigma}_n^*)) = \frac{M-2}{M-3}\left(1-\rho^2_{S_T Z_n}\right)\frac{\sigma^2_{Z_n}}{M}.$$

which can be used to derive a valid confidence interval.

Since the normality assumption of (S_T, Z_n) is not satisfied, we have to proceed by approximation. So that an asymptotically valid variance and confidence interval for Z_n is obtained, since $\hat{\varsigma}_n^*$ converges to the true value ς_n^*. Hence, as the sample size M is large enough, we can use the variance approximation

$$\text{Var}(\bar{V}_{CV}(\hat{\varsigma}_n^*)) \approx \frac{M-2}{M-3}\left(1-\rho^2_{S_T Z_n}\right)\frac{\sigma^2_{Z_n}}{M}.$$

V. Numerical computations

In this section, we approximate the price of particular generalized Asian options (fixed strike call and straddle) under the MEMM $\mathbb{P}^*$. First, we will characterize the MEMM in terms of the Lévy measure ν. We consider two cases:

- ν_L is supported by a finite set.
- ν_L is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

In a second part, we compare the approximation of the price using Monte Carlo with and without variance reduction technique in each case.

A. Numerical approximation of β^*

The root β^* of the function (8) characterizes the MEMM. From now on, we take these parameters: $\mu = 0.1$, $\sigma = 0.3$ and $\lambda = 1$ and we focus on the following cases:

1) Case 1: ν_L is supported by a finite set:

$$\nu_L = \lambda\sum_{i=1}^N p_i\delta_{a_i} \text{ where } \sum_{i=1}^N p_i = 1, p_i \in]0,1[,\ a_i \in \mathbb{R},$$

and $N \in \mathbb{N}^*$. It is easy to see that:

$$\nu_L^* = \lambda^*\sum_{i=1}^N p_i^* e^{\beta^*(e^{a_i}-1)}\delta_{a_i},$$

where $\lambda^* = \lambda\sum_{j=1}^N p_j e^{\beta^*(e^{a_j}-1)}$ and $p_i^* = \frac{p_i e^{\beta^*(e^{a_i}-1)}}{\sum_{j=1}^N p_j e^{\beta^*(e^{a_j}-1)}}$. Note that $\sum_{i=1}^N p_i^* = 1$. Under $\mathbb{P}$, $U_1 \sim \frac{\nu_L}{\lambda} = \sum_{i=1}^N p_i\delta_{a_i}$ and under $\mathbb{P}^*$, $U_1 \sim \frac{\nu_L^*}{\lambda^*} = \sum_{i=1}^N p_i^*\delta_{a_i}$. Therefore,

$$\kappa = \mathbb{E}^*[U_1\mathbf{1}_{\{|U_1|\le 1\}}] = \sum_{i=1}^N p_i^* a_i\mathbf{1}_{\{|a_i|\le 1\}}.$$

Function (8) becomes

$$f(\beta) = \mu + \left(\frac{1}{2}+\beta\right)\sigma^2 + \lambda^*\sum_{i=1}^n p_i^*(e^{a_i}-1).$$

To generate atoms $a = (a_1, a_2, \ldots, a_N)$, for example in the range $[-1.5, 1.5]$, we use the MATLAB code:

```
a=rand(1,N)*3-1.5;
```

and their corresponding masses $p = (p_1, p_2, \ldots, p_N)$ can be generated by the code:

```
R=rand(1,N); p=R/sum(R);
```

We will take the case of $N = 5$ atoms.2 The root of the function (8) is found using Dichotomy method.

For the atoms

$$a = (1.0222, -0.7372, 0.9429, -0.7694, 1.2878) \quad (15)$$

with the corresponding masses

$$p = (0.2422, 0.0822, 0.1631, 0.2253, 0.2872), \quad (16)$$

we obtain

$$\beta^* = -0.9300 \text{ and } f(0) = 1.4216.$$

Thus $p^* = (0.0754, 0.2175, 0.0618, 0.6045, 0.0407)$, $\lambda^* = 0.6139$ and $\mu^* = 0.0958$. If we take $a = (-0.4500, -0.9102, -0.7467, 0.3481, -0.0801)$ with the same masses, then,

$$\beta^* = 0.0253 \text{ and } f(0) = -0.0060.$$

2) Case 2: ν_L is absolutely continuous: We take the case, as in Merton's model, where $\nu_L = \lambda F$ with F is the standard normal distribution. We use a combination of Monte Carlo and Dichotomy methods. For the same parameters of μ, σ, r and λ, we obtain:

$$\beta^* = -0,4986 \quad \text{and} \quad \mu^* = 0.0972.$$

B. Approximation of Asian option prices

In order to give numerical simulations of the prices of generalized Asian options using Monte Carlo method (with and without variance reduction technique), we take as model's parameters: $\mu = 0.1$, $\sigma = 0.3$, $\lambda = 1$, $T = 1$, $r = 0.1$, $S_0 = 100$ and $K = 100$. As in the previous subsection, we focus on fixed strike Asian call and Asian straddle.

1) Case 1: ν_L is supported by a finite set: We consider the case of subsection V-A1, where ν_L is supported by the set a given by Eq. (15) and with probability masses p of Eq. (16). The drift value of $\hat{S}$ under $\mathbb{P}^*$ is $\mu^* = 0.0958$.

Table I shows that the variance reduction with control variate technique enhances the Monto Carlo approximation. For the fixed strike call case, we see that, even if we use $M = 10000$ loops with a variance reduction technique, we obtain more accuracy for the confidence intervals than the classical one with 20000 loops. That's means less material resources are used and less time is consumed. However in the case of straddle, we obtain a similar results (up to 0.05%) when 50% of loops are used. But, for the same number of Monte Carlo M, the improvement of the estimator become more and more noticeable.

Table II reveal the contribution of variance reduction technique to the Monte Carlo approximation. We see that the confidence band is decreasing when the number of Monte Carlo is increasing, for a call as for a straddle. This result is predictable since $\text{Var}(\bar{V}_{CV}(\varsigma_n^*))$ and $\text{Var}(\bar{Y}_n)$ tend to zero as M goes to ∞. Furthermore, the approximation of the price using variance reduction technique improve the confidence intervals for the two cases, since $\text{Var}(\bar{V}_{CV}(\varsigma_n^*)) < \text{Var}(\bar{Y}_n)$.

2) Case 2: ν_L is absolutely continuous: We consider the case 2 of subsection V-A2. We found:

$$\beta^* \simeq -0.4986, \quad \kappa = -0.4173, \quad \text{and} \quad \mu^* = 0.0972.$$

As in table I, table III shows that the variance reduction with control variate technique improves the Monte Carlo approximation, with the same interpretations for the fixed strike call case as well for the straddle one.

Also as in table II, table IV shows that the control variate variance reduction technique improves the Monte Carlo method. The two tables are commented with the same arguments.

C. Convergence of Monte Carlo schemes

To be able to get a real price, we use zero strike Asian call option, with the same other parameters as above. The value of the option's price is $V_0 = 94.1733$. Fig. (1) and Fig. (2) below illustrates the convergence of the Monte Carlo method, when the Lévy measure is absolutely continuous, without and with variance reduction respectively.

Fig. (1) shows the convergence of the no variance reduction estimator as the number of Monte Carlo increases. We see that the band of the confidence interval becomes, as the Monte Carlo number M increases, close to the real price. This show the convergence of the no variance reduction method. But to obtain a more accuracy estimated value, the number M should be taken very large.

However, in Fig. (2) we clearly see that the Monte Carlo method with variance reduction technique improves the approximation of the real price. Indeed, the convergence of the control variate variance reduction estimator to the real price becomes faster and more accurate that the no variance reduction one, as the Monte Carlo number M becomes large. For instance, with $M = 10000$ iterations, we observe that the approximated variance reduction price is very close to the real price, while the approximated no variance reduction one deviates in a significant way, for the same Monte Carlo number, from the exact option's value.

VI. Concluding Remarks

This paper is focused on Asian option pricing in Geometric Lévy market under the minimal entropy martingale measure. This choice of such EMM is appropriate, since it preserves statistical properties of the market measure as much as possible. The main contribution is a control variate variance reduction technique to improve the Monte Carlo method and an illustration of its improvement for particular examples of generalized Asian options. The convergence of Monte Carlo Asian price estimator in L^2 using the Reimann scheme is also a contribution to the literature. There exits many other schemes in the literature, for example trapezoidal scheme, Simpson scheme etc ... In fact there are no options written on the integral of S_t in practice, but only on some (possibly weighted) average of S_t. Thus the Reimann scheme approximation of the integral represents a possible "market option", whereas the integral itself is an approximation of this. Hence, the integral of S_t over some period is only an approximation of an Asian option which has very many sampling points. The introduction of other numerical integration schemes may be questionable namely from this point of view.

A possible continuation of this work is to compare our approach with other existing methods (e.g. binomial tree method [18]) and to extend the work to the case of infinite activity market models.

Appendix A
Proof of lemma 1

The process $\hat{S}$ is a $\mathbb{P}^*$-martingale. Therefore, for all $t \geq 0$, $\mathbb{E}^*[\hat{S}_t] = S_0$. From the Lévy-Khintchine representation of the Lévy process, we get

$$\mathbb{E}^*[S_u^2] = S_0^2 \mathbb{E}^*[e^{2L_u}] = S_0^2 e^{uc},$$

where $c = 2\mu^* + 2\sigma^2 + \int_{|x|<1}(e^{2x} - 1 - 2x)\nu^*(dx) + \int_{|x|\geq 1}(e^{2x}-1)\nu^*(dx)$. Taking account of the condition $(\mathbf{A_3})$, we can deduce that c is given by expression (10). For all $0 \leq u \leq v$, we have

$$\mathbb{E}^*\left[S_u S_v\right] = \mathbb{E}^*\left[S_u^2 \frac{S_v}{S_u}\right] = \mathbb{E}^*\left[S_u^2 \frac{S_{v-u}}{S_0}\right]$$
$$= \frac{1}{S_0}\mathbb{E}^*\left[S_u^2\right]\mathbb{E}^*[S_{v-u}] = e^{r(v-u)}\mathbb{E}^*[S_u^2] = S_0^2 e^{rv} e^{u(c-r)}.$$

Appendix B
Proof of Proposition 1

We have

$$\begin{aligned}
&\mathbb{E}^*\left[(A_T - A_T^n)^2\right] \\
=\ &\mathbb{E}^*\left[\left(\int_0^T S_u du - h\sum_{k=0}^{n-1} S_{t_k}\right)^2\right] \\
=\ &\mathbb{E}^*\left[\left(\int_0^T S_u du\right)^2\right] + h^2\mathbb{E}^*\left[\left(\sum_{j=0}^{n-1} S_{t_j}\right)^2\right] \\
&-2h\mathbb{E}^*\left[\sum_{j,k=0}^{n-1}\int_{t_j}^{t_{j+1}} S_u S_{t_k} du\right].
\end{aligned}$$

Each part is computed as follows: First we have

$$\begin{aligned}
&\mathbb{E}^*\left[\left(\int_0^T S_u du\right)^2\right] \\
&= \int_0^T\int_0^T \mathbb{E}^*[S_u S_v](1_{u<v} + 1_{v<u})dudv \\
=\ &2\int_0^T\int_0^T \mathbb{E}^*[S_u S_v]1_{u<v}dudv \\
=\ &2S_0^2\int_0^T e^{rv}\left[\int_0^v e^{u(c-r)}du\right]dv \\
&= 2\frac{S_0^2}{c-r}\left(\frac{e^{c.T}-1}{c} - \frac{e^{r.T}-1}{r}\right).
\end{aligned}$$

Next we have using Lemma (1):

$$h^2\mathbb{E}^*\left[\left(\sum_{j=0}^{n-1}S_{t_j}\right)^2\right]$$

$$= 2h^2\sum_{0\le j<k\le n-1}\mathbb{E}^*[S_{t_j S_{t_k}}]+h^2\sum_{j=0}^{n-1}\mathbb{E}^*[S_{t_j}^2]$$

$$= 2h^2S_0^2\sum_{0\le j<k\le n-1}e^{rt_k}e^{t_j(c-r)}$$

$$+ h^2S_0^2\sum_{j=0}^{n-1}e^{ct_j}.$$

and finally we have

$$- 2h\mathbb{E}^*\left[\sum_{0\le j<k\le n-1}\int_{t_j}^{t_{j+1}}S_uS_{t_k}du\right]$$

$$= -2h\sum_{j,k=0}^{n-1}\int_{t_j}^{t_{j+1}}\mathbb{E}^*[S_uS_{t_k}]du$$

$$= -2h\left[\sum_{0\le j<k\le n-1}\int_{t_j}^{t_{j+1}}\mathbb{E}^*[S_uS_{t_k}]du\right.$$

$$+ \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\mathbb{E}^*[S_uS_{t_j}]du$$

$$\left.+ \sum_{0\le k<j\le n-1}\int_{t_j}^{t_{j+1}}\mathbb{E}^*[S_uS_{t_k}]du\right]$$

$$= -2hS_0^2\left[\sum_{0\le j<k\le n-1}\int_{t_j}^{t_{j+1}}e^{rt_k}e^{(c-r)u}du\right.$$

$$+ \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}e^{ru}e^{(c-r)t_j}du$$

$$\left.+ \sum_{0\le k<j\le n-1}\int_{t_j}^{t_{j+1}}e^{ru}e^{(c-r)t_k}du\right]$$

$$= -2hS_0^2\left(\frac{e^{(c-r)h}-1}{c-r}+\frac{e^{rh}-1}{r}\right)$$

$$\times \sum_{0\le j<k\le n-1}e^{rt_k}e^{(c-r)t_j}$$

$$- \frac{2hS_0^2}{r}(e^{rh}-1)\sum_{j=0}^{n-1}e^{ct_j}.$$

Thus

$$\mathbb{E}^*\left|A_T-A_T^n\right|^2 = 2\frac{S_0^2}{c-r}\left(\frac{e^{c.T}-1}{c}-\frac{e^{r.T}-1}{r}\right)$$

$$+ hS_0^2\left(h-\frac{2}{r}(e^{rh}-1)\right)\sum_{j=0}^{n-1}e^{ct_j}$$

$$+ 2hS_0^2\left(h-\frac{e^{(c-r)h}-1}{c-r}-\frac{e^{rh}-1}{r}\right)$$

$$\times \sum_{0\le j<k\le n-1}e^{rt_k}e^{(c-r)t_j}.$$

On the other hand, we have the following obvious result:

Lemma 2: For all $c,r\in\mathbb{R}^*$ sucht that $c\neq r$

1) $\displaystyle\sum_{0\le j<k\le n-1}e^{rt_k}e^{(c-r)t_j} = \frac{e^{cT}-1}{(e^{rh}-1)(e^{hc}-1)} + \frac{1-e^{(c-r)T}}{(e^{rh}-1)(e^{(c-r)h}-1)}.$

2) $\displaystyle\sum_{0\le j\le n-1}e^{ct_j}=\frac{e^{cT}-1}{e^{hc}-1}.$

Using Lemma (2), we get the Taylor expansion.

$$\mathbb{E}^*\left|A_T-A_T^n\right|^2$$

$$= S_0^2\left(3(1-e^{rT})+\left(2-\frac{r}{c}\right)(e^{cT}-1)\right)\frac{h^2}{6}$$

$$+ S_0^2\left(e^{rT}-1+(r-1)(e^{cT}-1)\right)\frac{h^3}{12}+o(h^3).$$

Therefore the expression (9) follows. Proposition 1 is thus proved.

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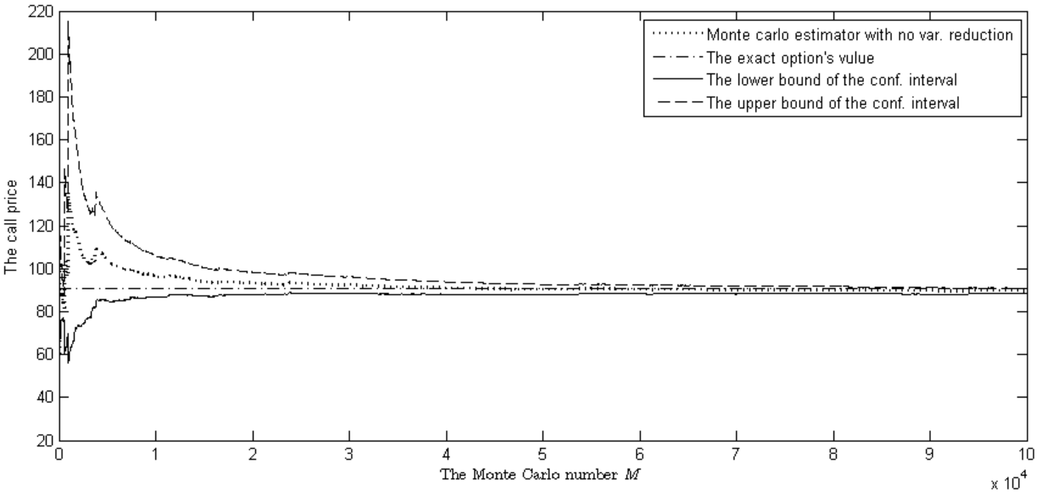

Fig. 1. Price approximation of zero-strike call when $\frac{\nu_L}{\lambda}$ follow a standard normal law with no variance reduction.

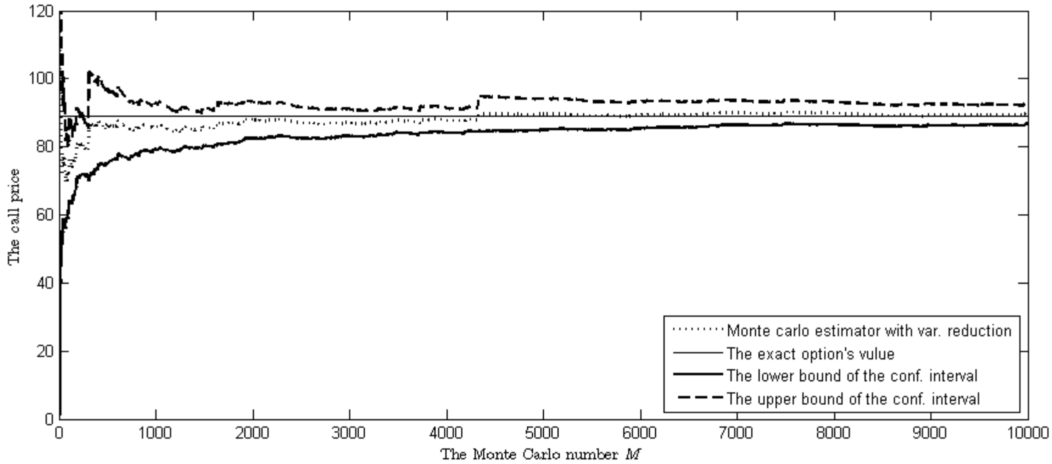

Fig. 2. Price approximation of zero-strike call when $\frac{\nu_L}{\lambda}$ follow a standard normal law with variance reduction.

TABLE I
PRICE CONFIDENCE INTERVALS OF FIXED STRIKE ASIAN CALL AND ASIAN STRADDLE IN TERMS OF TIME STEP NUMBER n, WHEN ν_L IS SUPPOTED BY $N = 5$ ATOMS.

		Without V.R. (M=20000)		With V.R. (M=10000)		With V.R. (M=20000)	
	n	Price	Conf. Int.(95%)	Price	Conf. Int.(95%)	Price	Conf. Int.(95%)
call	10	19.918	[19.618 20.217]	16.860	[16.634 17.0868]	16.948	[16.789 17.107]
	50	13.299	[13.049 13.550]	11.433	[11.202 11.6642]	11.346	[11.184 11.509]
	100	12.568	[12.324 12.813]	10.743	[10.514 10.9721]	10.709	[10.546 10.872]
	500	11.968	[11.726 12.209]	10.559	[10.327 10.7919]	10.626	[10.460 10.792]
	1000	11.868	[11.630 12.106]	9.989	[9.757 10.2207]	10.365	[10.200 10.531]
straddle	10	11.123	[10.972 11.275]	9.194	[8.983 9.406]	9.314	[9.164 9.464]
	50	12.609	[12.402 12.816]	11.619	[11.338 11.900]	11.718	[11.517 11.919]
	100	13.567	[13.348 13.786]	9.061	[8.782 9.339]	9.064	[8.867 9.261]
	500	13.892	[13.676 14.108]	9.743	[9.468 10.018]	9.658	[9.464 9.851]
	1000	14.052	[13.829 14.275]	9.541	[9.264 9.818]	9.664	[9.466 9.862]

TABLE II

PRICE CONFIDENCE INTERVALS OF FIXED STRIKE ASIAN CALL AND ASIAN STRADDLE IN TERMS OF MONTE CARLO NUMBER M, WHEN ν_L IS SUPPOTED BY $N = 5$ ATOMS ($n = 100$).

		Without Variance Reduction		With Variance Reduction	
	M	Price	Conf. Int.(95%)	Price	Conf. Int.(95%)
call	100	14.571	[10.410 18.732]	12.1077	[9.277 14.937]
	1000	12.497	[11.425 13.570]	11.212	[10.475 11.949]
	10000	12.737	[12.391 13.084]	11.043	[10.811 11.275]
	20000	12.561	[12.316 12.806]	10.968	[10.806 11.131]
	50000	12.575	[12.421 12.730]	11.246	[11.139 11.352]
straddle	100	15.880	[12.742 19.018]	11.753	[8.788 14.718]
	1000	13.650	[12.641 14.658]	11.691	[10.752 12.630]
	10000	13.414	[13.113 13.718]	9.609	[9.333 9.885]
	20000	13.567	[13.348 13.786]	9.064	[8.867 9.261]
	50000	13.386	[13.251 13.521]	9.454	[9.332 9.576]

TABLE III

PRICE CONFIDENCE INTERVALS OF FIXED STRIKE ASIAN CALL AND ASIAN STRADDLE IN TERMS OF TIME STEP NUMBER n, WHEN ν_L CORRESPONDS TO MERTON'S MODEL.

		Without V.R. (M=20000)		With V.R. (M=10000)		With V.R. (M=20000)	
	n	Price	Conf. Int. (95%)	Price	Conf. Int.(95%)	Price	Conf. Int.(95%)
call	10	19.646	[19.349 19.943]	17.144	[16.928 17.353]	17.085	[16.932 17.247]
	50	13.312	[13.062 13.563]	11.630	[11.398 11.861]	11.671	[11.507 11.835]
	100	12.439	[12.194 12.684]	11.116	[10.875 11.356]	11.270	[11.098 11.443]
	500	11.984	[11.746 12.221]	10.253	[10.022 10.484]	10.201	[10.038 10.364]
	1000	11.867	[11.630 12.103]	9.722	[9.486 9.958]	9.820	[9.652 9.988]
straddle	10	11.154	[11.002 11.307]	8.869	[8.653 9.085]	8.817	[8.666 8.969]
	50	12.480	[12.273 12.687]	9.435	[9.157 9.712]	9.296	[9.105 9.487]
	100	13.383	[13.166 13.600]	8.737	[8.462 9.011]	8.740	[8.546 8.935]
	500	13.790	[13.578 14.002]	6.660	[6.381 6.939]	6.717	[6.519 6.915]
	1000	14.214	[13.990 14.439]	8.660	[8.378 8.941]	8.652	[8.456 8.849]

TABLE IV

PRICE CONFIDENCE INTERVALS OF FIXED STRIKE ASIAN CALL AND ASIAN STRADDLE IN TERMS OF MONTE CARLO NUMBER M,WHEN ν_L CORRESPONDS TO MERTON'S MODEL ($n = 100$).

		Without Variance Reduction		With Variance Reduction	
	M	Price	Conf. Int.(95%)	Price	Conf. Int.(95%)
call	100	16.371	[12.381 20.362]	9.714	[6.741 12.687]
	1000	18.162	[16.831 19.494]	8.681	[7.756 9.606]
	10000	12.313	[11.969 12.656]	8.975	[8.682 9.268]
	20000	11.013	[10.780 11.246]	10.965	[10.799 11.130]
	50000	12.555	[12.401 12.709]	11.251	[11.145 11.357]
straddle	100	11.165	[8.706 13.624]	10.285	[7.876 12.694]
	1000	12.949	[12.010 13.887]	9.469	[8.604 10.335]
	10000	13.456	[13.157 13.756]	7.334	[7.059 7.609]
	20000	13.383	[13.166 13.600]	8.740	[8.546 8.935]
	50000	13.466	[13.331 13.602]	9.388	[9.265 9.511]