New Results on the Game of N-Player Cutcake

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***Abstract*—The game of n-player Cutcake is the n-player variant of Cutcake, a classic combinatorial game. Even though determining the solution of Cutcake is trivial, solving n-player Cutcake is challenging because of the identification of queer games, i.e., games where no player has a winning strategy. New results about the classification of the instances of n-player Cutcake are presented.**

***Index Terms*—combinatorial game, Cutcake, n-player game, queer game.**

I. Introduction

THE game of Cutcake [1] is a classic two-player combinatorial game. Every instance of this game is defined as a set of rectangles of integer side-lengths with edges parallel to the x- and y-axes. The two players are often called Left and Right. A legal move for Left is to divide one of the rectangles into two rectangles of integer side-length by means of a single cut parallel to the x-axis and a legal move for Right is to divide one of the rectangles into two rectangles of integer side-length by means of a single cut parallel to the y-axis. Players take turns making legal moves until one of them cannot move. In the normal play convention, the first player unable to move is the loser. We recall that in the game of Cutcake the outcome for an $l \times r$ rectangle depends on the dimensions of l and r as shown in Table I. For example, in the 8×7 rectangle Left has a winning strategy and in the 3×4 rectangle Right has a winning strategy but the 7×4 rectangle is a zero-game.

TABLE I
OUTCOME CLASSES IN CUTCAKE

	Left starts	Right starts
$\lfloor \log_2 l \rfloor > \lfloor \log_2 r \rfloor$	Left wins	Left wins
$\lfloor \log_2 l \rfloor < \lfloor \log_2 r \rfloor$	Right wins	Right wins
$\lfloor \log_2 l \rfloor = \lfloor \log_2 r \rfloor$	Right wins	Left wins

The game of Cutblock (a three-player version of Cutcake) was introduced by Propp in [2]. Cincotti [3], [4] presents a classification of the instances of Cutcake with an arbitrary finite number of players. Every instance of n-player Cutcake is defined as a set of n-cubes (or hypercubes) of integer side-lengths. A legal move for player i with $1 \leq i \leq n$ is to divide one of the n-cubes into two n-cubes of integer side-length, i.e.,

$$\begin{aligned}[d_1, \ldots, d_i, \ldots, d_n] \quad \rightarrow \quad & [d_1, \ldots, d_{i_1}, \ldots, d_n] + \\ & [d_1, \ldots, d_{i_2}, \ldots, d_n]\end{aligned}$$

where $d_i > 1$, $d_{i_1} > 0$, $d_{i_2} > 0$, and $d_{i_1} + d_{i_2} = d_i$.

Players take turns making legal moves in a cyclic fashion

$$(p_1, p_2, \ldots, p_n, p_1, p_2, \ldots, p_n, p_1, p_2, \ldots)$$

where $(p_1, p_2, \ldots, p_n)$ is a permutation of $(1, \ldots, n)$. $p_i = j$ means that player j makes the ith move, e.g, $p_1 = 3$ means that player 3 makes the first move. When one of the n players is unable to move then that player leaves the game and the remaining $n-1$ players continue playing in the same mutual order as before. The remaining player is the winner.

The paper is organized as follows. In Section 2, we recall the main definitions of n-player partizan games. In Section 3, we report the previous results about the classification of n-player Cutcake. In the fourth section, we show our new results and in the last section future work is indicated.

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II. N-player partizan games

For the sake of self-containment, we recall in this section the basic definitions and main results concerning n-player partizan games. Such a theory is an extension of Conway's theory of partizan games [5] and, as a consequence, it is both a theory of games and a theory of numbers.

Definition 1: If $G_1,\ldots,G_n$ are any n sets of games previously defined, then $\{G_1|\ldots|G_n\}$ is a game. All games are constructed in this way.
Let

$$g = \{G_1|\ldots|G_n\}$$

be a game. We denote by g_1, ..., g_n, respectively, the typical elements of G_1, ..., G_n. Therefore, the game can be written as $g = \{g_1|\ldots|g_n\}$. The games g_1, g_2, ..., g_n will be called respectively the 1st, 2nd, ..., nth options of g. We introduce n different relations $(\geq_1, \ldots, \geq_n)$ representing players' evaluations of the games.

Definition 2: Let g and h be two games. We say that:

$$\begin{aligned} g \geq_i h \quad & \Longleftrightarrow \quad (\nexists g_j \in G_j)(h \geq_i g_j) \wedge \\ & \qquad (\nexists h_i \in H_i)(h_i \geq_i g), \\ & \qquad \forall j \in \{1, \ldots, n\}, j \neq i, \\ g \leq_i h \quad & \Longleftrightarrow \quad h \geq_i g, \end{aligned}$$

where $i \in \{1, \ldots, n\}$.
We write $g \ngeq_i h$ to mean that $g \geq_i h$ does not hold.

Definition 3: Let g and h be two games. We say that:

$$\begin{aligned} g =_i h \quad & \Longleftrightarrow \quad (g \geq_i h) \wedge (g \leq_i h), \\ g >_i h \quad & \Longleftrightarrow \quad (g \geq_i h) \wedge (h \ngeq_i g), \\ g <_i h \quad & \Longleftrightarrow \quad (h >_i g), \\ g = h \quad & \Longleftrightarrow \quad (g =_i h), \forall i \in \{1, \ldots, n\}. \end{aligned}$$

Definition 4: Let g be a game. We say that:

$$\begin{aligned} g =_{i,j} 0 \quad & \Longleftrightarrow \quad (g =_i 0) \wedge (g =_j 0) \wedge (g <_k 0), \\ & \qquad \forall k \in \{1, \ldots, n\}, k \neq i, k \neq j, \\ g =_{(i)} 0 \quad & \Longleftrightarrow \quad (g =_i 0) \wedge (g <_k 0), \\ & \qquad \forall k \in \{1, \ldots, n\}, k \neq i, \\ g < 0 \quad & \Longleftrightarrow \quad (g <_k 0), \forall k \in \{1, \ldots, n\}. \end{aligned}$$

(Advance online publication: 9 November 2011)

TABLE II
OUTCOME CLASSES FOR NUMBERS

	$p_1 = i$	$p_1 = j$	$p_1 = k$
$g = 0$	Player p_n wins	Player p_n wins	Player p_n wins
$g >_i 0$	Player i wins	Player i wins	Player i wins
$g =_{i,j} 0$	Player j wins	Player i wins	The player (i or j) who moves last wins
$g =_{(i)} 0$	?	?[a]	?[a]
$g < 0$	?	?	?

[a]Let k be the first player in the sequence $(p_1, p_2, \ldots, p_n)$ having a legal move, i.e., such that the set $G_k \neq \emptyset$.
If $k \neq i$, then player i has a winning strategy.

TABLE III
OUTCOMES OF $x = g + h$.

	$h = 0$	$h >_i 0$	$h >_j 0$	$h =_{i,j} 0$	$h =_{i,k} 0$
$g = 0$	$x = 0$	$x >_i 0$	$x >_j 0$	$x =_{i,j} 0$	$x =_{i,k} 0$
$g >_i 0$	$x >_i 0$	$x >_i 0$	?	$x >_i 0$	$x >_i 0$
$g >_j 0$	$x >_j 0$	?	$x >_j 0$	$x >_j 0$	?
$g =_{i,j} 0$	$x =_{i,j} 0$	$x >_i 0$	$x >_j 0$	$x =_{i,j} 0$	$x =_{(i)} 0$
$g =_{i,k} 0$	$x =_{i,k} 0$	$x >_i 0$	?	$x =_{(i)} 0$	$x =_{i,k} 0$
$g =_{k,l} 0$	$x =_{k,l} 0$	?	?	$x < 0$	$x =_{(k)} 0$
$g =_{(i)} 0$	$x =_{(i)} 0$	$x >_i 0$	?	$x =_{(i)} 0$	$x =_{(i)} 0$
$g =_{(j)} 0$	$x =_{(j)} 0$	?	$x >_j 0$	$x =_{(j)} 0$	$x < 0$
$g < 0$	$x < 0$	?	?	$x < 0$	$x < 0$

Definition 5: We say that two games g and h are identical ($g \cong h$) if their sets are identical, that is, if G_i is identical to H_i, $\forall i \in \{1, \ldots, n\}$.

Definition 6: We define the sum of two games as follows

$$g + h = \{g_1 + h, g + h_1 | g_2 + h, g + h_2 | \ldots | g_n + h, g + h_n\}$$

A special case of games can be considered to define what we call *numbers*.

Definition 7: If G_1,G_2,...,G_n are any n sets of numbers previously defined, and

$$(\nexists (g_i, g_j) \in G_i \times G_j)(g_i \geq_i g_j), \forall i, j \in \{1, \ldots, n\}, i \neq j$$

then $\{G_1 | G_2 | \ldots | G_n\}$ is a number. All numbers are constructed in this way.

Order relations and arithmetic operations on numbers are defined analogously to those for games. The most important distinction between numbers and general games is that numbers are totally ordered but games are not, e.g., there exist games g and h for which we have neither $g \geq_i h$ nor $h \geq_i g$.

All numbers can be classified in $(n^2 + 3n + 4)/2$ classes as shown in Table II where $n \geq 3$ is the number of players, $i, j \in \{1, \ldots, n\}$, and $i \neq j$.

Table III and Table IV show all possibilities when we sum two numbers. The entries '?' are unrestricted and indicate that different outcomes are possible.

For further details, please refer to [6].

TABLE IV
OUTCOMES OF $x = g + h$.

	$h =_{k,l} 0$	$h =_{(i)} 0$	$h =_{(j)} 0$	$h < 0$
$g = 0$	$x =_{k,l} 0$	$x =_{(i)} 0$	$x =_{(j)} 0$	$x < 0$
$g >_i 0$	?	$x >_i 0$	?	?
$g >_j 0$	?	?	$x >_j 0$	?
$g =_{i,j} 0$	$x < 0$	$x =_{(i)} 0$	$x =_{(j)} 0$	$x < 0$
$g =_{i,k} 0$	$x =_{(k)} 0$	$x =_{(i)} 0$	$x < 0$	$x < 0$
$g =_{k,l} 0$	$x =_{k,l} 0$	$x < 0$	$x < 0$	$x < 0$
$g =_{(i)} 0$	$x < 0$	$x =_{(i)} 0$	$x < 0$	$x < 0$
$g =_{(j)} 0$	$x < 0$	$x < 0$	$x =_{(j)} 0$	$x < 0$
$g < 0$	$x < 0$	$x < 0$	$x < 0$	$x < 0$

TABLE V
CLASSIFICATION OF CUTBLOCK

	$g = [d_1, \ldots, d_n]$
$g = 0$	$[1, \ldots, 1]$
$g >_i 0$	$\lfloor \log_2 d_i \rfloor > \sum_{j \neq i} \lfloor \log_2 d_j \rfloor$
$g =_{i,j}$	$\lfloor \log_2 d_i \rfloor = \lfloor \log_2 d_j \rfloor$, $d_k = 1, k \neq i, j$
$g =_{(i)} 0$	$\lfloor \log_2 d_i \rfloor = \sum_{j \neq i} \lfloor \log_2 d_j \rfloor$, $\lfloor \log_2 d_i \rfloor > \lfloor \log_2 d_j \rfloor, j \neq i$
$g < 0$	otherwise

TABLE VI
OUTCOMES FOR A_1 AND A_2

$A >_i 0$	$A_1 >_i 0$	$A_2 >_i 0$
$A =_{i,j} 0$	$A_1 >_i 0$	$A_2 =_{i,j} 0$
$A =_{i,j} 0$	$A_1 >_i 0$	$A_2 >_i 0$
$A =_{(i)} 0$	$A_1 >_i 0$	$A_2 =_{(i)} 0$
$A =_{(i)} 0$	$A_1 >_i 0$	$A_2 >_i 0$

III. PREVIOUS RESULTS

Cincotti [4] presents a classification of the instances of n-player Cutcake using an n-player extension of partizan games as shown in Table V.

The case $g =_{(i)} 0$ is particular interesting. By previous results, we know that when player i makes the first move, either player i has a winning strategy or the game is queer, i.e., no player has a winning strategy. In a previous work [7], we presented some sufficient conditions to guarantee a win for player i when $n = 3$.

In the next section, we generalize the previous result for $n \geq 4$. Moreover, we give some sufficient conditions to guarantee a win for player i when $g < 0$.

IV. NEW RESULTS

Theorem 1: Let g be a general instance of n-player Cutcake where every n-cube $[d_1, \ldots, d_n]$ satisfies one of the following conditions:

$$\lfloor \log_2 d_i \rfloor \geq \sum_{j \neq i} \lfloor \log_2 d_j \rfloor \quad (1)$$

$$\lfloor \log_2 d_i \rfloor = \sum_{j \neq i} \lfloor \log_2 d_j \rfloor - 1, d_j = 2^{t_j}, t_j \geq 1, j \neq i \quad (2)$$

If the number of n-cubes satisfying the second condition is at most $n - 2$ and player i has just moved in g, then player i has a winning strategy.

Proof: If player j, with $j \neq i$, moves in an n-cube A satisfying the first condition, then he/she will create at least a new n-cube $A_1 >_i 0$ as shown in Table VI. If player j, with $j \neq i$, moves in an n-cube B satisfying the second condition, then he/she will create two new n-cubes B_1 and B_2 satisfying the first condition as shown in Table VII.

By hypothesis, the number of n-cubes satisfying the second condition is at most $n - 2$. Therefore, at least one player j, with $j \neq i$, will move in an n-cube satisfying the first condition and, as a consequence, at least one n-cube

$$[e_1, \ldots, e_i, \ldots, e_n] >_i 0$$

TABLE VII
OUTCOMES FOR B_1 AND B_2

$B < 0$	$B_1 =_{(i)} 0$	$B_2 =_{(i)} 0$
$B < 0$	$B_1 >_i 0$	$B_2 =_{(i)} 0$
$B < 0$	$B_1 =_{i,j} 0$	$B_2 =_{i,j} 0$
$B < 0$	$B_1 >_i 0$	$B_2 =_{i,j} 0$

TABLE VIII
OUTCOMES FOR $[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n]$ AND $[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n]$

$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] >_i 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] >_i 0$
$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] =_{(i)} 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] >_i 0$
$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] =_{i,j} 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] >_i 0$
$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] = 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] >_i 0$
$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] =_{(i)} 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] =_{(i)} 0$
$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] =_{i,j} 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] =_{i,j} 0$
$[e_1,\dots,\lfloor e_i/2\rfloor,\dots,e_n] = 0$	$[e_1,\dots,\lceil e_i/2\rceil,\dots,e_n] = 0$

will be created. If player i moves in

$$[e_1,\dots,e_i,\dots,e_n] >_i 0$$

then he/she is always able to create two new n-cubes satisfying the first condition as shown in Table VIII.

At the end of the first round, i.e., after that all the players have made one move, all the hypothesis of the theorem are still satisfied and therefore, by the inductive hypothesis, player i has a winning strategy. ∎

Corollary 1: Let $[d_1,\dots,d_n] =_{(i)} 0$ be an n-cube of n-player Cutcake where $d_j = 2^{t_j}, t_j \geq 1, j \neq i, n \geq 4$. Then, when player i makes the first move, he/she has a winning strategy.

Proof: If player i moves

$$[d_1,\dots,d_i,\dots,d_n] \rightarrow [d_1,\dots,\lfloor d_i/2\rfloor,\dots,d_n] + [d_1,\dots,\lceil d_i/2\rceil,\dots,d_n]$$

then

$$\lfloor \log_2 \lfloor d_i/2\rfloor\rfloor = \sum_{j\neq i}\lfloor \log_2 d_j\rfloor - 1$$

and

$$[d_1,\dots,\lfloor d_i/2\rfloor,\dots,d_n] < 0$$

Moreover, either

$$\lfloor \log_2 \lceil d_i/2\rceil\rfloor = \sum_{j\neq i}\lfloor \log_2 d_j\rfloor - 1$$

and

$$[d_1,\dots,\lceil d_i/2\rceil,\dots,d_n] < 0$$

or

$$\lfloor \log_2 \lceil d_i/2\rceil\rfloor = \sum_{j\neq i}\lfloor \log_2 d_j\rfloor$$

and

$$[d_1,\dots,\lceil d_i/2\rceil,\dots,d_n] =_{(i)} 0$$

In both cases, by Theorem 1, player i has a winning strategy. We observe that the hypothesis $n \geq 4$ is necessary only when we have two n-cubes satisfying the second condition; otherwise the hypothesis $n \geq 3$ is sufficient. ∎

Theorem 2: Let $[d_1,\dots,d_n] < 0$ be an n-cube of n-player Cutcake where

$$\lfloor \log_2 d_i\rfloor = \sum_{j\neq i}\lfloor \log_2 d_j\rfloor - 1, d_{p_1} = 2^t, t \geq 1$$

If $p_1 \neq i$ and $p_2 \neq i$, then player i has a winning strategy.

Proof: In the beginning, player p_1 moves in

$$[d_1,\dots,d_n]$$

and he/she will create two new n-cubes A_1 and A_2. As shown in Table IX, the game $A_1 + A_2$ has three different possible outcomes. In any of these three cases, when player p_2 moves in $A_1 + A_2$, the game will become $>_i 0$ therefore, player i has a winning strategy. ∎

TABLE IX
OUTCOMES FOR A_1 AND A_2

$A_1 =_{(i)} 0$	$A_2 =_{(i)} 0$	$A_1 + A_2 =_{(i)} 0$
$A_1 =_{(i)} 0$	$A_2 >_i 0$	$A_1 + A_2 >_i 0$
$A_1 =_{p_2,i} 0$	$A_2 =_{p_2,i} 0$	$A_1 + A_2 =_{p_2,i} 0$

V. FUTURE WORK

In this paper, we presented some sufficient conditions to guarantee a win for player i in the game $g =_{(i)} 0$ when he/she makes the first move. Moreover, we gave some sufficient conditions to guarantee a win for player i in the game $g < 0$ when he/she makes neither the first nor the second move.

Future work will concern the resolution of the following open problems:

- To find some sufficient conditions to identify queer games in the case $g =_{(i)} 0$ when player i makes the first move.
- To find some sufficient conditions to identify queer games in the case $g < 0$.

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