Constructive Proof of Brouwer's Fixed Point Theorem for Sequentially Locally Non-constant and Uniformly Sequentially Continuous Functions

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***Abstract*—We present a constructive proof of Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions based on the existence of approximate fixed points. And we will show that our Brouwer's fixed point theorem implies Sperner's lemma for a simplex. Since the existence of approximate fixed points is derived from Sperner's lemma, our Brouwer's fixed point theorem is equivalent to Sperner's lemma.**

***Index Terms*—Brouwer's fixed point theorem, Sperner's lemma, sequentially locally non-constant functions, uniformly sequentially continuous functions, constructive mathematics.**

I. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved[1]. Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented an approximate version of Brouwer's theorem using Sperner's lemma. See [3] and [4]. Thus, Brouwer's fixed point theorem is constructively, in the sense of constructive mathematics á la Bishop, proved in its approximate version.

Also Dalen in [3] states a conjecture that a uniformly continuous function f from a simplex into itself, with property that each open set contains a point x such that $x \neq f(x)$, which means $|x - f(x)| > 0$, and also at every point x on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. We call such a property of functions *local non-constancy*. In this paper we present a partial answer to Dalen's conjecture.

Recently [5] showed that the following theorem is equivalent to Brouwer's fan theorem.

> Each uniformly continuous function f from a compact metric space X into itself with at most one fixed point and approximate fixed points has a fixed point.

In [3], [4] and [5] uniform continuity of functions is assumed. We consider a weaker uniform sequential continuity of functions according to [6]. In classical mathematics uniform continuity and uniform sequential continuity are equivalent. In constructive mathematics á la Bishop, however, uniform sequential continuity is weaker than uniform continuity[2]

And by reference to the notion of *sequentially at most one maximum* in [9] we require a condition that a function f is *sequentially locally non-constant*, and will show the following result.

> Each sequentially locally non-constant and uniformly sequentially continuous function f from an n-dimensional simplex into itself has a fixed point,

without the fan theorem[3]. Our sequential local non-constancy, the condition in [3] (local non-constancy) and the condition that a function has *at most one fixed point* in [5] are mutually different.

[11] constructed a computably coded continuous function f from the unit square into itself, which is defined at each computable point of the square, such that f has no computable fixed point. His map consists of a retract of the computable elements of the square to its boundary followed by a rotation of the boundary of the square. As pointed out by [12], since there is no retract of the square to its boundary, his map does not have a total extension.

In the next section we present Sperner's lemma. Its proof is omitted indicating references. In Section 3 we present our Brouwer's fixed point theorem and its proof. The first part of the proof proves the existence of an approximate fixed point of uniformly sequentially continuous functions using Sperner's lemma, and the second part proves the existence of an exact fixed point of sequentially locally non-constant and uniformly sequentially continuous functions. In Section 4 we will derive Sperner's lemma from Brouwer's fixed point theorem for uniformly sequentially continuous and sequentially locally non-constant functions.

II. Sperner's lemma

Let Δ denote an n-dimensional simplex. n is a finite natural number. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Fig. 1 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides

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[1] [1] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics á la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [2] or [3].

[2] Also in constructive mathematics sequential continuity is weaker than continuity, and uniform continuity (respectively, uniform sequential continuity) is stronger than continuity (respectively, sequential continuity) even in a compact space. See, for example, [7]. As stated in [8] all proofs of the equivalence between continuity and sequential continuity involve the law of excluded middle, and so the equivalence of them is non-constructive.

[3] In another paper [10] we have presented a partial answer to Dalen's conjecture with uniform continuity and sequential local non-constancy, that is, a proof of the existence of a fixed point for a uniformly continuous and sequentially locally non-constant functions.

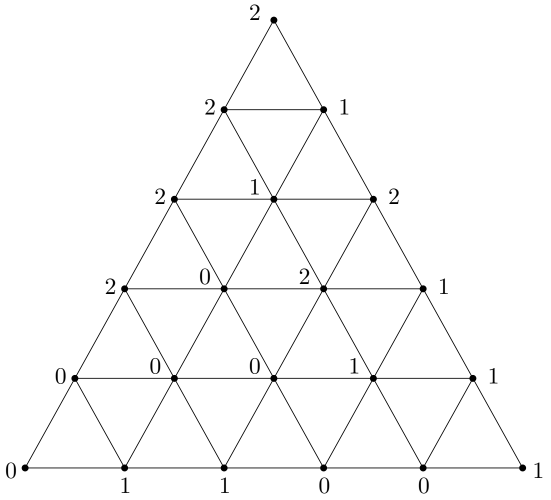

Fig. 1. Partition and labeling of 2-dimensional simplex

of Δ. m is a finite natural number. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the above-mentioned way, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers 0, 1, 2, $\dots$, n subject to the following rules.

1) The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, $\dots$, a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.
2) If a vertex of K is contained in an $n-1$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face.
3) If a vertex of K is contained in an $n-2$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face. And so on for cases of lower dimension.
4) A vertex contained inside of Δ is labeled with an arbitrary number among 0, 1, $\dots$, n.

A small simplex of K which is labeled with the numbers 0, 1, $\dots$, n is called a *fully labeled simplex*. Sperner's lemma is stated as follows.

Lemma 1 (Sperner's lemma): If we label the vertices of K following the rules 1) $\sim$ 4), then there are an odd number of fully labeled simplices, and so there exists at least one fully labeled simplex.

Proof: About constructive proofs of Sperner's lemma see [13] or [14]. ■

Since n and partition of Δ are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n-dimensional simplex of K through finite steps.

III. Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions

Let $x = (x_0, x_1, \dots, x_n)$ be a point in an n-dimensional simplex Δ, and consider a function f from Δ to itself. Denote the i-th components of x and $f(x)$ by, respectively, x_i and $f_i(x)$ or f_i.

Uniform continuity, sequential continuity and uniform sequential continuity of functions are defined as follows;

Definition 1 (Uniform continuity): A function f is uniformly continuous in Δ if for any $x, x' \in \Delta$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{If } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \varepsilon.$$

δ depends on only ε.

Definition 2 (Sequential continuity): A function f is sequentially continuous at $x \in \Delta$ in Δ if for sequences $(x_n)_{n\geq 1}$ and $(f(x_n))_{n\geq 1}$ in Δ

$$f(x_n) \longrightarrow f(x) \text{ whenever } x_n \longrightarrow x.$$

Definition 3 (Uniform sequential continuity): A function f is uniformly sequentially continuous in Δ if for sequences $(x_n)_{n\geq 1}$, $(x'_n)_{n\geq 1}$, $(f(x_n))_{n\geq 1}$ and $(f(x'_n))_{n\geq 1}$ in Δ

$$|f(x_n) - f(x'_n)| \longrightarrow 0 \text{ whenever } |x_n - x'_n| \longrightarrow 0.$$

$|x_n - x'_n| \longrightarrow 0$ means

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ (|x_n - x'_n| < \varepsilon),$$

where ε is a real number, and n and N are natural numbers. Similarly, $|f(x_n) - f(x'_n)| \longrightarrow 0$ means

$$\forall \varepsilon > 0\ \exists N'\ \forall n \geq N'\ (|f(x_n) - f(x'_n)| < \varepsilon).$$

N' is a natural number. In classical mathematics uniform continuity and uniform sequential continuity of functions are equivalent. But in constructive mathematics á la Bishop uniform sequential continuity is weaker than uniform continuity.

On the other hand, the definition of local non-constancy of functions is as follows;

Definition 4: (*Local non-constancy of functions*)

1) At a point x on a face (boundary) of a simplex $f(x) \neq x$. This means $f_i(x) > x_i$ or $f_i(x) < x_i$ for at least one i.
2) In any open set of Δ there exists a point x such that $f(x) \neq x$.

The notion that φ has at most one fixed point in [5] is defined as follows;

Definition 5 (At most one fixed point): For all $x, y \in \Delta$, if $x \neq y$, then $f(x) \neq x$ or $f(y) \neq y$.

Next, by reference to the notion of *sequentially at most one maximum* in [9], we define the property of *sequential local non-constancy*.

First we recapitulate the compactness (total boundedness with completeness) of a set in constructive mathematics. Δ is totally bounded in the sense that for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to Δ[4]. An ε-approximation to Δ is a subset of Δ such that for each $x \in \Delta$ there exists y in that ε-approximation with $|x - y| < \varepsilon$.

[4]A set S is finitely enumerable if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S.

According to Corollary 2.2.12 of [15] we have the following result.

Lemma 2: For each $\varepsilon > 0$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^n H_i$.

The definition of sequential local non-constancy is as follows;

Definition 6: (Sequential local non-constancy of functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ in each H_i, $|f(x_n) - x_n| \longrightarrow 0$ and $|f(y_n) - y_n| \longrightarrow 0$, then $|x_n - y_n| \longrightarrow 0$.

Now we show the following lemma, which is based on Lemma 2 of [9].

Lemma 3: Let f be a uniformly continuous function from Δ into itself. Assume $\inf_{x\in H_i} |f(x)-x| = 0$ for some $H_i \subset \Delta$ defined above. If the following property holds:

> For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in H_i$, $|f(x) - x| < \varepsilon$ and $|f(y) - y| < \varepsilon$, then $|x - y| \leq \delta$.

Then, there exists a point $z \in H_i$ such that $f(z) = z$, that is, f has a fixed point.

Proof: Choose a sequence $(x_n)_{n\geq 1}$ in H_i such that $|f(x_n) - x_n| \longrightarrow 0$. Compute N such that $|f(x_n) - x_n| < \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|x_m - x_n| \leq \delta$. Since $\delta > 0$ is arbitrary, $(x_n)_{n\geq 1}$ is a Cauchy sequence in H_i, and converges to a limit $z \in H_i$. The continuity of f yields $|f(z) - z| = 0$, that is, $f(z) = z$. ■

Next we show

Lemma 4: If X is a totally bounded space, and φ is a uniformly sequentially continuous function of X into a metric space, then $\varphi(X)$ is totally bounded.

Proof: Consider a sequence of positive real numbers $(\varepsilon_m)_{m\geq 1}$ such that $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_m$ and $\varepsilon_m \longrightarrow 0$, and a sequence of ε_m-approximation $L_m = \{x^{1_m}, x^{2_m}, \dots, x^{n_m}\}$ to X. For each $x \in X$ and each ε_m, there exists a point $x^{i_m} \in L_m$ such that $|x - x^{i_m}| < \varepsilon_m$. Thus, we can construct a sequence $(x^{i_m})_{m\geq 1}$ such that $|x - x^{i_m}| \longrightarrow 0$. The uniform sequential continuity implies $|\varphi(x) - \varphi(x^{i_m})| \longrightarrow 0$. $|x - x^{i_m}| \longrightarrow 0$ means

$$\forall \varepsilon_m > 0\ \exists M\ \forall m \geq M\ (|x - x^{i_m}| < \varepsilon_m).$$

M is a natural number. Similarly, $|\varphi(x) - \varphi(x^{i_m})| \longrightarrow 0$ means

$$\forall \varepsilon_m > 0\ \exists M'\ \forall m \geq M'\ (|\varphi(x) - \varphi(x^{i_m})| < \varepsilon_m).$$

M' is a natural number. Let $m' \geq \max(M, M')$. Then, corresponding to an $\varepsilon_{m'}$-approximation to X there exists an $\varepsilon_{m'}$-approximation to $\varphi(X)$. Therefore, $\varphi(X)$ is totally bounded.

■

This is a modified version of Proposition 2.2.6 of [15]. From this lemma we see that φ has the infimum in X by Corollary 2.2.7 of [15]. Then, $|f(x) - x|$ has the infimum in Δ.

With these preliminaries we show the following theorem.

Theorem 1: (*Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions*) Any sequentially locally non-constant and uniformly sequentially continuous function from an n-dimensional simplex Δ to itself has a fixed point.

Proof:

1) First we show that we can partition Δ so that the conditions for Sperner's lemma are satisfied. We partition Δ according to the method in Sperner's lemma, and label the vertices of simplices constructed by partition of Δ. It is important how to label the vertices contained in the faces of Δ. Let K be the set of small simplices constructed by partition of Δ, $x = (x_0, x_1, \dots, x_n)$ be a vertex of a simplex of K, and denote the i-th component of $f(x)$ by f_i. Then, we label a vertex x according to the following rule,

$$\text{If } x_k > f_k \text{ or } x_k + \tau > f_k, \text{ we label } x \text{ with } k,$$

where τ is a positive number. If there are multiple k's which satisfy this condition, we label x conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

For example, let x be a point contained in an $n-1$-dimensional face of Δ such that $x_i = 0$ for one i among $0, 1, 2, \dots, n$ (its i-th coordinate is 0). With $\tau > 0$, we have $f_i > 0$ or $f_i < \tau$. In constructive mathematics for any real number x we can not prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$.

When $f_i > 0$, from $\sum_{j=0}^n x_j = 1$, $\sum_{j=0}^n f_j = 1$ and $x_i = 0$,

$$\sum_{j=0, j\neq i}^n x_j > \sum_{j=0, j\neq i}^n f_j.$$

Then, for at least one j (denote it by k) we have $x_k > f_k$, and we label x with k, where k is one of the numbers which satisfy $x_k > f_k$. Since $f_i > x_i = 0$, i does not satisfy this condition. Assume $f_i < \tau$. $x_i = 0$ implies $\sum_{j=0, j\neq i}^n x_j = 1$. Since $\sum_{j=0, j\neq i}^n f_j \leq 1$, we obtain

$$\sum_{j=0, j\neq i}^n x_j \geq \sum_{j=0, j\neq i}^n f_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j\neq i}^n (x_j + \tau) > \sum_{j=0, j\neq i}^n f_j.$$

There is at least one $j (\neq i)$ which satisfies $x_j + \tau > f_j$. Denote it by k, and we label x with k. k is one of the numbers other than i such that $x_k + \tau > f_k$ is satisfied. i itself satisfies this condition ($x_i + \tau > f_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i. We have proved that we can label the vertices contained in an $n-1$-dimensional face of Δ such that $x_i = 0$ for one i among $0, 1, 2, \dots, n$ with the numbers other than i. By similar procedures we can show that we can label the vertices contained in an $n-2$-dimensional face of Δ such that $x_i = 0$ for two i's among $0, 1, 2, \dots, n$ with the numbers other than those i's, and so on. Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K.

Consider a sequence $(\Delta_m)_{m\geq 1}$ of partitions of Δ, and a sequence of fully labeled simplices $(\delta_m)_{m\geq 1}$. The larger m, the finer partition. The larger m, the smaller the diameter of a fully labeled simplex. Let $x_m^0, x_m^1, \dots$ and x_m^n be the vertices of a fully labeled simplex δ_m. We name these vertices so that $x_m^0, x_m^1, \dots, x_m^n$ are labeled, respectively, with 0, 1, $\dots$, n. The values of f at theses vertices are $f(x_m^0), f(x_m^1), \dots$ and $f(x_m^n)$. We can consider sequences of vertices of fully labeled simplices. Denote them by $(x_m^0)_{m\geq 1}$, $(x_m^1)_{m\geq 1}$, $\dots$, and $(x_m^n)_{m\geq 1}$. And consider sequences of the values of f at vertices of fully labeled simplices. Denote them by $(f(x_m^0))_{m\geq 1}$, $(f(x_m^1))_{m\geq 1}$, $\dots$, and $(f(x_m^n))_{m\geq 1}$. By the uniform sequential continuity of f

$$|(f(x_m^i)) - (f(x_m^j))|_{m\geq 1} \longrightarrow 0$$
$$\text{whenever } |(x_m^i) - (x_m^j)|_{m\geq 1} \longrightarrow 0,$$

for $i \neq j$. $|(x_m^i)_{m\geq 1} - (x_m^j)_{m\geq 1}| \longrightarrow 0$ means

$$\forall \varepsilon > 0\ \exists M\ \forall m \geq M\ (|x_m^i - x_m^j| < \varepsilon)\ i \neq j,$$

and $|(f(x_m^i))_{m\geq 1} - (f(x_m^j))_{m\geq 1}| \longrightarrow 0$ means

$$\forall \varepsilon > 0\ \exists M'\ \forall m \geq M'\ (|f(x_m^i) - f(x_m^j)| < \varepsilon)\ i \neq j.$$

Consider a fully labeled simplex δ_l in partition of Δ such that $l \geq \max(M, M')$. Denote vertices of δ_l by x^0, x^1, $\dots$, x^n. We name these vertices so that $x^0, x^1, \dots, x^n$ are labeled, respectively, with 0, 1, $\dots$, n. Then, $|x^i - x^j| < \varepsilon$ and $|f(x^i) - f(x^j)| < \varepsilon$. About x^0, from the labeling rules we have $x_0^0 + \tau > f(x^0)_0$. About x^1, also from the labeling rules we have $x_1^1 + \tau > f(x^1)_1$ which implies $x_1^1 > f(x^1)_1 - \tau$. $|f(x^0) - f(x^1)| < \varepsilon$ means $f(x^1)_1 > f(x^0)_1 - \varepsilon$. On the other hand, $|x^0 - x^1| < \varepsilon$ means $x_1^0 > x_1^1 - \varepsilon$. Thus, from

$$x_1^0 > x_1^1 - \varepsilon,\ x_1^1 > f(x^1)_1 - \tau,\ f(x^1)_1 > f(x^0)_1 - \varepsilon$$

we obtain

$$x_1^0 > f(x^0)_1 - 2\varepsilon - \tau$$

By similar arguments, for each i other than 0,

$$x_i^0 > f(x^0)_i - 2\varepsilon - \tau. \tag{1}$$

For $i = 0$ we have $x_0^0 + \tau > f(x^0)_0$. Then,

$$x_0^0 > f(x^0)_0 - \tau \tag{2}$$

Adding (1) and (2) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j\neq k}^{n} x_j^0 > \sum_{j=0, j\neq k}^{n} f(x^0)_j - 2(n-1)\varepsilon - n\tau.$$

From $\sum_{j=0}^n x_j^0 = 1$, $\sum_{j=0}^n f(x^0)_j = 1$ we have $1 - x_k^0 > 1 - f(x^0)_k - 2(n-1)\varepsilon - n\tau$, which is rewritten as

$$x_k^0 < f(x^0)_k + 2(n-1)\varepsilon + n\tau.$$

Since (1) implies $x_k^0 > f(x^0)_k - 2\varepsilon - \tau$, we have

$$f(x^0)_k - 2\varepsilon - \tau < x_k^0 < f(x^0)_k + 2(n-1)\varepsilon + n\tau.$$

Thus,

$$|x_k^0 - f(x^0)_k| < 2(n-1)\varepsilon + n\tau \tag{3}$$

is derived. On the other hand, adding (1) from 1 to n yields

$$\sum_{j=1}^{n} x_j^0 > \sum_{j=1}^{n} f(x^0)_j - 2n\varepsilon - n\tau.$$

From $\sum_{j=0}^n x_j^0 = 1$, $\sum_{j=0}^n f(x^0)_j = 1$ we have

$$1 - x_0^0 > 1 - f(x^0)_0 - 2n\varepsilon - n\tau. \tag{4}$$

Then, from (2) and (4) we get

$$|x_0^0 - f(x^0)_0| < 2n\varepsilon + n\tau. \tag{5}$$

From (3) and (5) we obtain the following result,

$$|x_i^0 - f(x^0)_i| < 2n\varepsilon + n\tau \text{ for all } i.$$

Thus,

$$|x^0 - f(x^0)| < n(n+1)(2\varepsilon + \tau). \tag{6}$$

Since n is finite, x^0 is an approximate fixed point of f[5]. And since $\varepsilon > 0$ and τ are arbitrary,

$$\inf_{x\in\Delta} |x - f(x)| = 0.$$

By Lemma 2 we have $\inf_{x\in H_i} |x - f(x)| = 0$ for some $H_i \subset \Delta$ defined in that lemma.

2) Choose a sequence $(\xi_m)_{m\geq 1}$ in H_i such that $|f(\xi_m) - \xi_m| \longrightarrow 0$. In view of Lemma 3 it is enough to prove that the following condition holds.

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in H_i$, $|f(x) - x| < \varepsilon$ and $|f(y) - y| < \varepsilon$, then $|x - y| \leq \delta$.

Assume that the set

$$K = \{(x, y) \in H_i \times H_i :\ |x - y| \geq \delta\}$$

is nonempty and compact[6]. Since the mapping $(x, y) \longrightarrow \max(|f(x) - x|, |f(y) - y|)$ is uniformly sequentially continuous, by Lemma 4 we can construct an increasing binary sequence $(\lambda_m)_{m\geq 1}$ such that

$$\lambda_m = 0 \Rightarrow \inf_{(x,y)\in K} \max(|f(x) - x|, |f(y) - y|) < 2^{-m},$$

$$\lambda_m = 1 \Rightarrow \inf_{(x,y)\in K} \max(|f(x) - x|, |f(y) - y|) > 2^{-m-1}.$$

It suffices to find m such that $\lambda_m = 1$. In that case, if $|f(x) - x| < 2^{-m-1}$ and $|f(y) - y| < 2^{-m-1}$, we have $(x, y) \notin K$ and $|x - y| \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_m = 0$, choose $(x_m, y_m) \in K$ such that $\max(|f(x_m) - x_m|, |f(y_m) - y_m|) < 2^{-m}$, and if $\lambda_m = 1$, set $x_m = y_m = \xi_m$. Then, $|f(x_m) - x_m| \longrightarrow 0$ and $|f(y_m) - y_m| \longrightarrow 0$, so $|x_m - y_m| \longrightarrow 0$. Computing M such that $|x_M - y_M| < \delta$, we must have $\lambda_M = 1$. We have completed the proof.

■

[5]In another paper [16] we have shown the existence of an approximate fixed point of each uniformly continuous function in a locally-convex space.

[6]See Theorem 2.2.13 of [15].

IV. From Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions to Sperner's lemma

In this section we will derive Sperner's lemma from Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions. Let Δ be an n-dimensional simplex. Denote a point on Δ by x. Consider a function f from Δ to itself. Partition Δ in the way depicted in Fig. 1. Let K denote the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers 0, 1, 2, ..., n subject to the same rules as those in Lemma 1. Now we derive Sperner's lemma expressed in Lemma 1 from Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions.

Denote the vertices of an n-dimensional simplex of K by $x^0, x^1, \dots, x^n$, the j-th coordinate of x^i by x^i_j, and denote the label of x^i by $l(x^i)$. Let τ be a positive number which is smaller than $x^i_{l(x^i)}$ for all x^i, and define a function $f(x^i)$ as follows[7]:

$$f(x^i) = (f_0(x^i), f_1(x^i), \dots, f_n(x^i)),$$

and

$$f_j(x^i) = \begin{cases} x^i_j - \tau & \text{for } j = l(x^i), \\ x^i_j + \frac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases} \tag{7}$$

f_j denotes the j-th component of f. From the labeling rules we have $x^i_{l(x^i)} > 0$ for all x^i, and so $\tau > 0$ is well defined. Since $\sum_{j=0}^n f_j(x^i) = \sum_{j=0}^n x^i_j = 1$, we have

$$f(x^i) \in \Delta.$$

We extend f to all points in the simplex by convex combinations on the vertices of the simplex. Let z be a point in the n-dimensional simplex of K whose vertices are $x^0, x^1, \dots, x^n$. Then, z and $f(z)$ are expressed as follows;

$$z = \sum_{i=0}^n \lambda_i x^i, \text{ and } f(z) = \sum_{i=0}^n \lambda_i f(x^i),\ \lambda_i \geq 0,\ \sum_{i=0}^n \lambda_i = 1.$$

We verify that f is uniformly sequentially continuous. Consider sequences $(z(n))_{n\geq 1}$, $(z'(n))_{n\geq 1}$, $(f(z(n)))_{n\geq 1}$ and $(f(z'(n)))_{n\geq 1}$ such that $|z(n) - z'(n)| \longrightarrow 0$. Denote each component of $z(n)$ by $z(n)_j$ and so on. When $|z(n)-z'(n)| \longrightarrow 0$, $|z(n)_j - z'(n)_j| \longrightarrow 0$ for each j. Then, since $\tau > 0$ is finite, we have $|f(z(n)) - f(z'(n))| \longrightarrow 0$, and so f is uniformly sequentially continuous.

Next we verify that f is sequentially locally non-constant.

1) Assume that a point z is contained in an $n-1$-dimensional small simplex δ^{n-1} constructed by partition of an $n-1$-dimensional face of Δ such that its i-th coordinate is $z_i = 0$. Denote the vertices of δ^{n-1} by x^j, $j = 0, 1, \dots, n-1$ and their i-th coordinate by x^j_i. Then, we have

$$f_i(z) = \sum_{j=0}^{n-1} \lambda_j f_i(x^j),\ \lambda_j \geq 0,\ \sum_{j=0}^{n-1} \lambda_j = 1.$$

Since all vertices of δ^{n-1} are not labeled with i, (7) means $f_i(x^j) > x^j_i$ for all $j = \{0, 1, \dots, n-1\}$. Then, there exists no sequence $(z(m))_{m\geq 1}$ such that $|f(z(m)) - z(m)| \longrightarrow 0$ in an $n-1$-dimensional face of Δ.

2) Let z be a point in an n-dimensional simplex δ^n. Assume that no vertex of δ^n is labeled with i. Then

$$f_i(z) = \sum_{j=0}^n \lambda_j f_i(x^j) = z_i + \left(1 + \frac{1}{n}\right)\tau. \tag{8}$$

Then, there exists no sequence $(z(m))_{m\geq 1}$ such that $|f(z(m)) - z(m)| \longrightarrow 0$ in δ^n.

3) Assume that z is contained in a fully labeled n-dimensional simplex δ^n, and rename vertices of δ^n so that a vertex x^i is labeled with i for each i. Then,

$$\begin{aligned} f_i(z) &= \sum_{j=0}^n \lambda_j f_i(x^j) = \sum_{j=0}^n \lambda_j x^j_i + \sum_{j\neq i} \lambda_j \frac{\tau}{n} - \lambda_i \tau \\ &= z_i + \left(\frac{1}{n}\sum_{j\neq i} \lambda_j - \lambda_i\right)\tau \text{ for each } i. \end{aligned}$$

Consider sequences $(z(m))_{m\geq 1} = (z(1), z(2), \dots)$, $(z'(m))_{m\geq 1} = (z'(1), z'(2), \dots)$ such that $|f(z(m)) - z(m)| \longrightarrow 0$ and $|f(z'(m)) - z'(m)| \longrightarrow 0$. Let $z(m) = \sum_{i=0}^n \lambda(m)_i x^i$ and $z'(m) = \sum_{i=0}^n \lambda'(m)_i x^i$. Then, we have

$$\frac{1}{n}\sum_{j\neq i} \lambda(m)_j - \lambda(m)_i \longrightarrow 0, \text{ and}$$
$$\frac{1}{n}\sum_{j\neq i} \lambda'(m)_j - \lambda'(m)_i \longrightarrow 0 \text{ for all } i.$$

Therefore, we obtain

$$\lambda(m)_i \longrightarrow \frac{1}{n+1}, \text{and } \lambda'(m)_i \longrightarrow \frac{1}{n+1}.$$

These mean

$$|z(m) - z'(m)| \longrightarrow 0.$$

Thus, f is sequentially locally non-constant, and it has a fixed point. Let z^* be a fixed point of f. We have

$$z^*_i = f_i(z^*) \text{ for all } i. \tag{9}$$

Suppose that z^* is contained in a small n-dimensional simplex δ^*. Let $z^0, z^1, \dots, z^n$ be the vertices of δ^*. Then, z^* and $f(z^*)$ are expressed as

$$z^* = \sum_{i=0}^n \lambda_i z^i \text{ and } f(z^*) = \sum_{i=0}^n \lambda_i f(z^i),\ \lambda_i \geq 0,\ \sum_{i=0}^n \lambda_i = 1.$$

(7) implies that if only one z^k among $z^0, z^1, \dots, z^n$ is labeled with i, we have

$$\begin{aligned} f_i(z^*) &= \sum_{j=0}^n \lambda_j f_i(z^j) = \sum_{j=0}^n \lambda_j z^j_i + \sum_{j\neq k} \lambda_j \frac{\tau}{n} - \lambda_k \tau \\ &= z^*_i \ (z^*_i \text{ is the } i\text{−th coordinate of } z^*). \end{aligned}$$

This means

$$\frac{1}{n}\sum_{j\neq k}^n \lambda_j - \lambda_k = 0.$$

Then, (9) is satisfied with $\lambda_k = \frac{1}{n+1}$ for all k. If no z^j is labeled with i, we have (8) with $z = z^*$ and then (9) can not

[7]We refer to [17] about the definition of this function.

be satisfied. Thus, one and only one z^j must be labeled with i for each i. Therefore, δ^* must be a fully labeled simplex, and so the existence of a fixed point of f implies the existence of a fully labeled simplex.

We have completely proved Sperner's lemma.

V. Concluding Remarks

As a future research program we are studying the following themes.

1) An application of Brouwer's fixed point theorem for sequentially locally non-constant functions to economic theory and game theory, in particular, the problem of the existence of an equilibrium in a competitive economy with excess demand functions which have the property that is similar to sequential local non-constancy, and the existence of Nash equilibrium in a strategic game with payoff functions which satisfy sequential local non-constancy.
2) A generalization of the result of this paper to Kakutani's fixed point theorem for multi-valued functions with property of sequential local non-constancy and its application to economic theory.

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