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Abnormal S-fuzzy Subpolygroups

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***Abstract*—In this paper, the notion of idempotent fuzzy subpolygroup with respect to a s-norm is introduced and some related properties are investigated. Then, homomorphic image and inverse image of S-fuzzy subpolygroups are discussed. Next, some properties of direct product of S-fuzzy subpolygroups are presented. Finally, abnormal S-fuzzy subpolygroups are studied.**

***Index Terms*—polygroup, S-fuzzy subpolygroup, S-product, abnormal S-fuzzy subpolygroup**

I. Introduction

The concept of a hypergroup was introduced by Marty [24]. Since then, many mathematicians have studied this concept ([4], [6], [10], [12], [13], [5], [19], [20], [18]). The theory of hypergroups has found many applications in the domain of mathematics and elsewhere. For example, polygroups which form an important subclass of hypergroups, were studied by Comer [7], [8]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced for the first time in [6].

The notion of fuzzy set was first introduced by Zadeh [29]. Zahedi, Bolurian and Hasankhani in 1995 [30] introduced the concept of a fuzzy subpolygroup. Triangular norms were introduced by Schweizer and Sklar [26], [27] to model the distances in probabilistic metric spaces. In fuzzy sets theory, triangular norm (t-norm) and triangular co-norm (t-conorm or s-norm) are extensively used to model the logical connectives: conjunction (AND) and disjunction (OR), respectively. There are many applications of triangular norms in several fields of Mathematics, and artificial intelligence [23]. Dudek and Jun [15] introduced the notion of an idempotent fuzzy subquasigroup with respect to a t-norm and discussed some of its properties. Akram [2] introduced the notion of an idempotent fuzzy subquasigroup with respect to a s-norm.

In this paper, the notion of idempotent fuzzy subpolygroup with respect to a s-norm is introduced and some related properties are investigated. Then, homomorphic image and inverse image of S-fuzzy subpolygroups are discussed. Next, some properties of direct product of S-fuzzy subpolygroups are presented. Finally, abnormal S-fuzzy subpolygroups are studied.

II. Preliminaries

In this section, we first review elementary aspects that are necessary for this paper.

***Definition** 2.1:* Let P be a non-empty set and $f : P \times P \to \mathcal{P}^*(P)$ be a mapping, where $\mathcal{P}^*(P)$ is the set of all non-empty subsets of P. Then f is called a binary hyperoperation on P.

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***Definition** 2.2:* A polygroup is a system $\mathcal{P} = \{P, \cdot, e, -1\}$, where $e \in P$, -1 is a unitary operation on P, $\cdot$ maps $P \times P$ into the $\mathcal{P}^*(P)$, and the following axioms hold for all $x, y, z \in P$:
(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(2) $e \cdot x = x \cdot e = x$,
(3) $x \in y \cdot z$ implied $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.
The following elementary facts about polygroups follow easily from the axioms:

$$e^{-1} = e, e \in x \cdot x^{-1} \cap x^{-1} \cdot x, (x^{-1})^{-1} = x, (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}.$$

A non-empty subset K of a polygroup P is a subpolygroup of P if and only if (1) $a, b \in K$ implies $a \cdot b \subseteq K$, (2) $a \in K$ implies $a^{-1} \in K$.
Let P_1 and P_2 be polygroups and $f : P_1 \to P_2$ a mapping. Then, f is called a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in P_1$. Also, f is called a homomorphism of type 3 if $f^{-1}(f(x)f(y)) = (f^{-1}f(x))(f^{-1}f(y))$ (see [30]).

***Definition** 2.3:* A mapping $\mu : P \to [0, 1]$ is called a fuzzy set in a polygroup P. For any fuzzy set μ in P and any $t \in [0, 1]$ we define two sets $L(\mu, t) = \{x \in P \mid \mu(x) \leq t\}$ and $\mu_t = \{x \in P \mid \mu(x) \geq t\}$ which are called lower t-level cut and upper t-level cut of μ, respectively.

***Definition** 2.4:* [30] A fuzzy set μ defined on a polygroup P is called a fuzzy subpolygroup if for all $x, y \in P$,
(1) $\min\{\mu(x), \mu(y)\} \leq \mu(z)$, for all $z \in xy$,
(2) $\mu(x) \leq \mu(x^{-1})$, for all $x \in P$.
It is easy to see that, $\mu(x) = \mu(x^{-1})$. Moreover, from the above definition we can deduce that $\mu(e) \geq \mu(x)$ for every $x \in P$.

***Theorem** 2.5:* [3], [30] Let P be a polygroup and μ a fuzzy set of P. Then, μ is a fuzzy subpolygroup of P if and only if for every $t \in [0, 1]$, $\mu_t (\neq \emptyset)$ is a subpolygroup of P.
$\square$

***Definition** 2.6:* [15], [27] A t-norm is a mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following conditions:
(T_1) $T(\alpha, 1) = \alpha$,
(T_2) $T(\alpha, \beta) = T(\beta, \alpha)$,
(T_3) $T(\alpha, \beta) \leq T(\alpha, \gamma)$, whenever $\beta \leq \gamma$,
(T_4) $T(\alpha, T(\beta, \gamma)) = T(T(\alpha, \beta), \gamma)$, for all $\alpha, \beta, \gamma \in [0, 1]$.
A simple example of a t-norm is the function $T(\alpha, \beta) = \min\{\alpha, \beta\}$. Generally, $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$ and $T(\alpha, 0) = 0$ for all $\alpha, \beta \in [0, 1]$. Moreover, $([0, 1], T)$ is a commutative semigroup with 0 as the neutral element. In particular,

$$T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))$$

holds for all $\alpha, \beta, \gamma, \delta \in [0, 1]$.

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Let T_1 and T_2 be two t-norms. We say that T_1 dominates T_2 and write $T_1 \gg T_2$ if

$$T_1(T_2(\alpha,\beta),T_2(\gamma,\delta)) \geq T_2(T_1(\alpha,\gamma),T_1(\beta,\delta))$$

for all $\alpha,\beta,\gamma,\delta \in [0,1]$. Obviously, $T \geq T$ for all t-norms (for more details see [15], [23], [28]).

A fuzzy set μ on a polygroup P is called a T-fuzzy subpolygroup (see [19]) of P if for all $x,y \in P$,
(1) $\mu(z) \geq T(\mu(x),\mu(y))$, for all $z \in xy$,
(2) $\mu(x^{-1}) \geq \mu(x)$.

Definition *2.7:* A s-norm is a mapping $S : [0,1]\times[0,1] \to [0,1]$ that satisfies the following conditions:
(S_1) $S(x,0) = x$,
(S_2) $S(x,y) = S(y,x)$,
(S_3) $S(x,S(y,z)) = S(S(x,y),z)$,
(S_4) $S(x,y) \leq S(x,z)$, whenever $y \leq z$,
for all $x,y,z \in [0,1]$.
A simple example of a s-norm is the function $S(x,y) = \max\{x,y\}$. Generally, $S(x,y) \geq \max\{x,y\}$ and $S(x,1) = 1$ for all $x,y \in [0,1]$. Moreover, $([0,1],S)$ is a commutative semigroup with 1 as the neutral element. In particular,

$$S(S(x,y),S(z,t)) = S(S(x,z),S(y,t))$$

holds for all $x,y,z,t \in [0,1]$.
Let S_1 and S_2 be two s-norms. We say that S_1 surrenders S_2 and write $S_1 \ll S_2$ if

$$S_1(S_2(x,y),S_2(z,t)) \leq S_2(S_1(x,z),S_1(y,t))$$

for all $x,y,z,t \in [0,1]$. Obviously, $S \ll S$ for all s-norms (see [2]).

III. S-FUZZY SUBPOLYGROUPS

In what follows P is a polygroup and S is a s-norm unless otherwise specified.

Definition *3.1:* The set of all idempotents with respect to S, i.e., the set $E_S = \{x \in [0,1] \mid S(x,x) = x\}$, is a subsemigroup of $([0,1],S)$. If $Im(\mu) \subseteq E_S$, then the fuzzy set μ is called an idempotent with respect to the s-norms S (briefly, S-idempotent).

In this section, we introduce the notion of S-fuzzy subpolygroup and investigate some interesting properties concerning to the level subsets, union, image and preimage under homomorphisms and the relationship between T-fuzzy and S-fuzzy subpolygroups.

Definition *3.2:* A fuzzy set μ in P is called a fuzzy subpolygroup of P with respect to a s-norm S (briefly, S-fuzzy subpolygroup) if:
(1) $\mu(z) \leq S(\mu(x),\mu(y))$, for all $x,y \in P$ and $z \in xy$,
(2) $\mu(x) \leq \mu(x^{-1})$, for all $x \in P$.
If a S-fuzzy subpolygroup μ of P is an idempotent, we say that μ is an idempotent S-fuzzy subpolygroup of P.

Example *3.3:* Suppose that H and K are subgroups of a group G and $H \subseteq K$. Define a system $G//H = \Big\{\{HgH \mid g \in G\},*,H,^{-1}\Big\}$, where $(HgH)^{-1} = Hg^{-1}H, (Hg_1H)*(Hg_2H) = \{Hg_1hg_2H \mid h \in H\}$. The system $G//H$ is a polygroup and $K//H$ is a subpolygroup of $G//H$. Let S be a s-norm defined by $S(x,y) = \min\{x+y,1\}$ for all $x,y \in [0,1]$. Define a fuzzy set μ in $G//H$ by

$$\mu(HgH) = \begin{cases} 1, & if\ g \in K, \\ 0, & otherwise, \end{cases}$$

for every $HgH \in G//H$. It is easy to see that μ satisfies:
(1) $\mu(Hg_1hg_2H) \leq S(\mu(Hg_1H),\mu(Hg_2H))$, for all $Hg_1H, Hg_2H \in G//H$ and $Hg_1hg_2H \in (Hg_1H)*(Hg_2H)$.
(2) $\mu(HgH) \leq \mu((HgH)^{-1})$, for every $HgH \in G//H$.
Also, $Im(\mu) \subseteq E_S$. Hence, μ is an idempotent S-fuzzy subpolygroup of $G//H$.

Proposition *3.4:* [2], [23] If a fuzzy set μ is an idempotent with respect to a s-norm S, then $S(x,y) = \max\{x,y\}$, for all $x,y \in Im(\mu)$. □

In the next results, we investigate the relationship between S-fuzzy subpolygroups and their level subsets. It is easy to prove two following propositions.

Proposition *3.5:* Let a fuzzy set μ on a polygroup P be an idempotent with respect to a s-norm S. If each non-empty level subset μ_t is a subpolygroup of P, then μ is a S-idempotent fuzzy subpolygroup of P. □

Proposition *3.6:* Let μ be a S-fuzzy subpolygroup of P and $\alpha \in [0,1]$.
(1) If $\alpha = 0$, then $L(\mu,\alpha)$ is either empty or a subpolygroup of P.
(2) If $S = \max$, then $L(\mu,\alpha)$ is either empty or a subpolygroup of P. □

Theorem *3.7:* Let S be a s-norm. If each non-empty level subset $L(\mu,\alpha)$ of μ is a subpolygroup of P, then μ is a S-fuzzy subpolygroup of P.

Proof. If there exist $x,y \in P$ and $z \in xy$ such that $\mu(z) > S(\mu(x),\mu(y))$, then by taking $t_0 = \frac{1}{2}\{\mu(z) + S(\mu(x),\mu(y))\}$, we have $S(\mu(x),\mu(y)) < t_0 < \mu(z)$ and $S(\mu(x),\mu(y)) \geq \max\{\mu(x),\mu(y)\}$, then $\max\{\mu(x),\mu(y)\} < t_0$, hence $x,y \in L(\mu,t_0)$. Since $L(\mu,t_0)$ is a subpolygroup of P, then $z \in L(\mu,t_0)$ and $\mu(z) \leq t_0$, a contradiction. Similarly, we can prove that $\mu(x) \leq \mu(x^{-1})$ for all $x \in P$. Therefore, μ is a S-fuzzy subpolygroup of P. □

Definition *3.8:* Let S be a s-norm and μ and ν two fuzzy sets in P. Then, the S-product of μ and ν denoted by $[\mu\cdot\nu]_S$, is defined $[\mu\cdot\nu]_S(x) = S(\mu(x),\nu(x))$, for all $x \in P$. Obviously, $[\mu\cdot\nu]_S$ is a fuzzy set in P such that $[\mu\cdot\nu]_S = [\nu\cdot\mu]_S$.

Theorem *3.9:* Let S be a s-norm and μ and ν S-fuzzy subpolygroups of P. If a s-norm S^* surrenders S, then the S^*-product $[\mu\cdot\nu]_{S^*}$ is a S-fuzzy subpolygroup of P.

Proof. For all $x, y \in P$ and $z \in xy$, we have

$$\begin{aligned}&[\mu\cdot\nu]_{S^*}(z) = S^*(\mu(z),\nu(z))\\&\leq S^*(S(\mu(x),\mu(y)),S(\nu(x),\nu(y)))\\&\leq S(S^*(\mu(x),\nu(x)),S^*(\mu(y),\nu(y)))\\&= S([\mu\cdot\nu]_{S^*}(x),[\mu\cdot\nu]_{S^*}(y)).\end{aligned}$$

Also, for every $x \in P$, we have $\mu(x) \leq \mu(x^{-1})$ and $\nu(x) \leq \nu(x^{-1})$. Then $S^*(\mu(x),\nu(x)) \leq S^*(\mu(x^{-1}),\nu(x^{-1}))$, which implies that $[\mu\cdot\nu]_{S^*}(x) \leq [\mu\cdot\nu]_{S^*}(x^{-1})$. Therefore, $[\mu\cdot\nu]_{S^*}$ is a S-fuzzy subpolygroup of P. □

Corollary *3.10:* The S-product of S-fuzzy subpolygroups is a S-fuzzy subpolygroup. □

Definition *3.11:* Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy sets in P. Then, the union $\bigvee\limits_{i\in I}\mu_i$ of $\{\mu_i \mid i \in I\}$ is defined by $(\bigvee_{i\in I}\mu_i)(x) = \sup\{\mu_i(x) \mid i \in I\}$, for each $x \in P$.

Theorem *3.12:* If $\{\mu_i \mid i \in I\}$ is a family of S-fuzzy subpolygroups of a polygroup P, then $\bigvee\limits_{i\in I}\mu_i$ is again a S-fuzzy subpolygroup of P.

Proof. For $x, y \in P$ and $z \in xy$, we have

$$\begin{aligned}&(\bigvee_{i\in I}\mu_i)(z) = \sup\{\mu_i(z) \mid i \in I\}\\&\leq \sup\left\{S(\mu_i(x),\mu_i(y)) \mid i \in I\right\}\\&= S(\sup\{\mu_i(x) \mid i \in I\},\sup\{\mu_i(y) \mid i \in I\})\\&= S((\bigvee_{i\in I}\mu_i)(x),(\bigvee_{i\in I}\mu_i)(y)).\end{aligned}$$

Also, for each $x \in P$, we have

$$\begin{aligned}&(\bigvee_{i\in I}\mu_i)(x) = \sup\{\mu_i(x) \mid i \in I\}\\&\leq \sup\{\mu_i(x^{-1}) \mid i \in I\} = (\bigvee_{i\in I}\mu_i)(x^{-1}).\end{aligned}$$

Therefore, $\bigvee\limits_{i\in I}\mu_i$ is a S-fuzzy subpolygroup of P. □

Definition *3.13:* Let f be a mapping on P. If ν is a fuzzy set in $f(P)$, then the fuzzy set $f^{-1}(\nu) = \nu\circ f$ in P is called the preimage of ν under f.

Theorem *3.14:* An onto homomorphism preimage of an (idempotent) S-fuzzy subpolygroup is an (idempotent) S-fuzzy subpolygroup.

Proof. Let $f : P_1 \to P_2$ be an onto homomorphism of polygroups. If ν is a S-fuzzy subpolygroup of P_2 and $f^{-1}(\nu)$ is the preimage of ν under f, then for all $x, y \in P_1$ and $z \in xy$ $(f(z) \in f(x)f(y))$ we have

$$\begin{aligned}f^{-1}(\nu)(z) &= (\nu\circ f)(z) = \nu(f(z))\\&\leq S(\nu(f(x)),\nu(f(y)))\\&= S(\nu\circ f(x),\nu\circ f(y))\\&= S(f^{-1}(\nu)(x),f^{-1}(\nu)(y)).\end{aligned}$$

Also, for each $x \in P_1$, we have

$$\begin{aligned}f^{-1}(\nu)(x) &= (\nu\circ f)(x) = \nu(f(x))\\&\leq \nu(f(x)^{-1}) = \nu(f(x^{-1}))\\&= (\nu\circ f)(x^{-1}) = f^{-1}(\nu)(x^{-1}).\end{aligned}$$

Moreover, if ν is S-idempotent, then for each $x \in Im(f^{-1}(\nu))$, there exists $y \in P_1$ such that $f^{-1}(\nu)(y) = x$. Then, $S(x,x) = S(f^{-1}(\nu)(y),f^{-1}(\nu)(y)) = S((\nu\circ f)(y),(\nu\circ f)(y))$. Since ν is S-idempotent, $\nu(f(y)) \in Im(\nu) \subseteq E_S$, then $S(\nu(f(y)),\nu(f(y))) = \nu(f(y)) = (\nu\circ f)(y) = f^{-1}(\nu)(y) = x$. Therefore, $f^{-1}(\nu)$ is idempotent. □

Proposition *3.15:* Let S and S^* be s-norms such that S^* surrenders S. If $h : P_1 \to P_2$ be an onto homomorphism of polygroups, then for any S-fuzzy subpolygroups μ and ν of P_2, we have $h^{-1}([\mu\cdot\nu]_{S^*}) = \Big[h^{-1}(\mu)\cdot h^{-1}(\nu)\Big]_{S^*}$.

Proof. By Theorem 3.14, $h^{-1}(\mu)$, $h^{-1}(\nu)$ and $h^{-1}([\mu\cdot\nu]_{S^*})$ are S-fuzzy subpolygroups of P_1. For each $x \in P_1$, we have

$$\begin{aligned}&\Big[h^{-1}([\mu\cdot\nu]_{s^*})\Big](x) = [\mu\cdot\nu]_{S^*}(h(x))\\&= S^*(\mu(h(x)),\nu(h(x)))\\&= S^*([h^{-1}(\mu)](x),[h^{-1}(\nu)](x))\\&= \Big[h^{-1}(\mu)\cdot h^{-1}(\nu)\Big]_{S^*}(x),\end{aligned}$$

which completes the proof. □

Definition *3.16:* Let μ be a fuzzy set in P and f be a mapping defined on P. Then the fuzzy set μ^f in $f(P)$ defined by

$$\mu^f(y) = \inf_{x\in f^{-1}(y)}\mu(x), \quad \forall y \in f(P),$$

is called the image of μ under f.
A fuzzy set μ in P has the $\inf$ property if for any subset $A \subseteq P$, there exist $a_0 \in A$ such that $\mu(a_0) = \inf\limits_{a\in A}\mu(a)$.

Theorem *3.17:* An onto homomorphism of type 3 image of a fuzzy subpolygroup with the $\inf$ property is a fuzzy subpolygroup.

Proof. See [30]. □

Definition *3.18:* Let T be a t-norm. Then, it is easy to verify that the map $S : [0,1]\times[0,1] \to [0,1]$ defined by $S(x,y) = 1 - T(1-x,1-y)$ is a s-norm.
By assumption of this definition, we have the following theorem.

***Theorem** 3.19:* A fuzzy set μ of P is a T-fuzzy subpolygroup of P if and only if its complement μ^c is a S-fuzzy subpolygroup of P.

Proof. Let μ be a T-fuzzy subpolygroup of P. For $x, y \in P$ and $z \in xy$, we have

$$\begin{aligned}\mu^c(z) &= 1 - \mu(z) \leq 1 - T(\mu(x), \mu(y)) \\ &= 1 - T(1 - \mu^c(x), 1 - \mu^c(y)) = S(\mu^c(x), \mu^c(y)).\end{aligned}$$

Similarly, we can prove that $\mu^c(x^{-1}) \geq \mu^c(x)$ for all $x \in P$. Hence, μ^c is a S-fuzzy subpolygroup of P.
Conversely, Assume that μ^c is a S-fuzzy subpolygroup of P. For $x, y \in P$ and $z \in xy$, we have

$$\begin{aligned}\mu(z) &= 1 - \mu^c(z) \geq 1 - S(\mu^c(x), \mu^c(y)) \\ &= 1 - S(1 - \mu(x), 1 - \mu(y)) = T(\mu(x), \mu(y))\end{aligned}$$

Similarly, we can prove that $\mu(x) \leq \mu(x^{-1})$. Hence, μ is a S-fuzzy subpolygroup of P. □

IV. Direct product of S-fuzzy subpolygroups

In this section, we introduce the notion of direct product of S-fuzzy subpolygroups. Specially, we investigate the relationship between direct product and S-product of S-fuzzy subpolygroups.

***Definition** 4.1:* Let S be a s-norm. If μ_1 and μ_2 are two fuzzy sets on P_1 and P_2, respectively, then μ defined on $P_1 \times P_2$ by $\mu(x_1, x_2) = S(\mu_1(x_1), \mu_2(x_2))$ is a fuzzy set on $P_1 \times P_2$, which is denoted by $\mu_1 \times \mu_2$.

***Theorem** 4.2:* Let P_1 and P_2 be polygroups and $P = P_1 \times P_2$ be the direct product polygroup of P_1 and P_2. Let λ be a S-fuzzy subpolygroup of P_1 and μ a S-fuzzy subpolygroup of P_2. Then, $\nu = \lambda \times \mu$ is a S-fuzzy subpolygroup of $P_1 \times P_2$. Moreover, if λ and μ are S-idempotent, then so is $\nu = \lambda \times \mu$.

Proof. Let (x_1, x_2), $(y_1, y_2) \in P_1 \times P_2$ and $(z_1, z_2) \in (x_1, x_2)(y_1, y_2)$. Then,

$$(\lambda \times \mu)(z_1, z_2) = S(\lambda(z_1), \mu(z_2)), \;\; z_1 \in x_1 y_1, z_2 \in x_2 y_2.$$

Also, we have

$$\begin{aligned}(\lambda \times \mu)(z_1, z_2) &\leq S(S(\lambda(x_1), \lambda(y_1)), S(\mu(x_2), \mu(y_2))) \\ &= S(S(\lambda(x_1), \mu(x_2)), S(\lambda(y_1), \mu(y_2))) \\ &= S((\lambda \times \mu)(x_1, x_2), (\lambda \times \mu)(y_1, y_2)).\end{aligned}$$

Similarly, we can prove that $(\lambda \times \mu)(x_1, x_2) \leq (\lambda \times \mu)(x_1, x_2)^{-1}$ for all $(x_1, x_2) \in P_1 \times P_2$. Assume that λ and μ are S-idempotent, then for each $(x, y) \in Im(\lambda \times \mu)$, there exists $(z, t) \in P_1 \times P_2$ such that $(\lambda \times \mu)(z, t) = (x, y)$. We have

$$\begin{aligned}S((x, y), (x, y)) &= S((\lambda \times \mu)(z, t), (\lambda \times \mu)(z, t)) \\ &= S(S(\lambda(z), \mu(t)), S(\lambda(z), \mu(t))) \\ &= S(S(\lambda(z), \lambda(z)), S(\mu(t), \mu(t))) \\ &= S(\lambda(z), \mu(t)) = S((\lambda \times \mu)(z, t)) \\ &= (x, y).\end{aligned}$$

Therefore, $\lambda \times \mu$ is a S-idempotent fuzzy subpolygroup of $P_1 \times P_2$. □

***Theorem** 4.3:* Let λ and μ be fuzzy sets in P such that $\lambda \times \mu$ is a S-fuzzy subpolygroup of $P \times P$. The following statements hold.
(1) Either $\lambda(e) \leq \lambda(x)$ or $\mu(e) \leq \mu(x)$, for all $x \in P$.
(2) If μ is S-idempotent and $\lambda(e) \leq \lambda(x)$, for all $x \in P$, then either $\mu(e) \leq \lambda(x)$ or $\mu(e) \leq \mu(x)$.
(3) If λ is S-idempotent and $\mu(e) \leq \mu(x)$, for all $x \in P$, then either $\lambda(e) \leq \lambda(x)$ or $\lambda(e) \leq \mu(x)$.

Proof. (1) Assume that $\lambda(x) < \lambda(e)$ and $\mu(y) < \mu(e)$ for some $x, y \in P$. Then, $(\lambda \times \mu)(x, y) = S(\lambda(x), \mu(y)) < S(\lambda(e), \mu(e)) = (\lambda \times \mu)(e, e)$. This implies that $(\lambda \times \mu)(x, y) < (\lambda \times \mu)(e, e)$, which is a contradiction.

(2) Assume that $\lambda(x) < \mu(e)$ and $\mu(y) < \mu(e)$ for some $x, y \in P$. Then,

$$(\lambda \times \mu)(x, y) = S(\lambda(x), \mu(y)) < S(\mu(e), \mu(e)) = \mu(e).$$

Since $\lambda(e) \leq \lambda(x) < \mu(e)$, then $\mu(e) \leq S(\lambda(e), \mu(e)) = (\lambda \times \mu)(e, e)$. Therefore,

$$(\lambda \times \mu)(x, y) < \mu(e) \leq S(\lambda(e), \mu(e)) = (\lambda \times \mu)(e, e),$$

which is a contradiction.

(3) Assume that $\lambda(e) > \lambda(x)$ and $\lambda(e) > \mu(y)$ for some $x, y \in P$. Then,

$$(\lambda \times \mu)(x, y) = S(\lambda(x), \mu(y)) < S(\lambda(e), \lambda(e)) = \lambda(e).$$

Since $\mu(e) \leq \mu(y) < \lambda(e)$, then $\lambda(e) \leq S(\lambda(e), \mu(e))$. Therefore,

$$(\lambda \times \mu)(x, y) < \lambda(e) \leq S(\lambda(e), \mu(e)) = (\lambda \times \mu)(e, e),$$

which is a contradiction. □

***Theorem** 4.4:* Let μ and ν be fuzzy sets in P such that $\mu \times \nu$ is a S-fuzzy subpolygroup of $P \times P$. The following statements hold.
(1) If $\nu(x) \geq \mu(e)$ for all $x \in P$, then ν is a S-fuzzy subpolygroup of P.
(2) If $\mu(x) \geq \mu(e)$ for all $x \in P$ and $\nu(y) < \mu(e)$ for some $y \in P$, then μ is a S-fuzzy subpolygroup of P.

Proof. (1) If for all $x \in P$, $\nu(x) \geq \mu(e)$, then for all $x, y \in P$ and $z \in xy$, we have

$$\begin{aligned}\nu(z) &\leq S(\mu(e), \nu(z)) = (\mu \times \nu)(e, z) \\ &\leq S((\mu \times \nu)(e, x), (\mu \times \nu)(e, y)) \\ &= S(S(\mu(e), \nu(x)), S(\mu(e), \nu(y))) \\ &= S(S(\mu(e), \mu(e))), S(\nu(x), \nu(y))) \\ &\leq S(S(\nu(x), \nu(y))), S(\nu(x), \nu(y))) \\ &= S(\nu(x), \nu(y)).\end{aligned}$$

Similarly, we can prove that $\nu(x) \leq \nu(x^{-1})$ for all $x \in P$. Therefore, ν is a S-fuzzy subpolygroup of P.

(2) Assume that $\mu(x) \geq \mu(e)$ for all $x \in P$ and $\nu(y) < \mu(e)$ for some $y \in P$. Then, $\nu(e) \leq \nu(y) < \mu(e) \leq \mu(x)$, it follows that $\nu(e) < \mu(x)$ for all $x \in P$. For all $x, y \in P$ and $z \in xy$, then $\mu(z) > \nu(e)$. Now, we have $(\mu \times \nu)(x, e) = S(\mu(x), \nu(e)) \geq \mu(x)$ and $(\mu \times \nu)(y, e) = S(\mu(y), \nu(e)) \geq \mu(y)$. Therefore,

$$\begin{aligned}
\mu(z) &\leq S(\mu(z), \nu(e)) = (\mu \times \nu)(z, e) \\
&\leq S((\mu \times \nu)(x, e), (\mu \times \nu)(y, e)) \\
&= S(S(\mu(x), \nu(e)), S(\mu(y), \nu(e))) \\
&= S(S(\nu(e), \nu(e)), S(\mu(x), \mu(y))) \\
&\leq S(S(\mu(x), \mu(y)), S(\mu(x), \mu(y))) \\
&= S(\mu(x), \mu(y)).
\end{aligned}$$

Similarly, we can prove that $\mu(x) \leq \mu(x^{-1})$ for all $x \in P$. Therefore, μ is a S-fuzzy subpolygroup of P. $\square$

***Theorem** 4.5:* The relationship between S-fuzzy subpolygroups $\mu \times \nu : P \times P \longrightarrow I$ and $[\mu \cdot \nu]_S$ can be viewed via the following diagram

$$\begin{tikzcd}
P \arrow[r, "d"] \arrow[d, "{[\mu \cdot \nu]_S}"'] \arrow[dr, phantom, "\mu \times \nu"] & P \times P \arrow[d, shift right, "\mu"'] \arrow[d, shift left, "\nu"] \\
I & I \times I \arrow[l, "S"]
\end{tikzcd}$$

where $I = [0, 1]$ and $d : P \to P \times P$ is defined by $d(x) = (x, x)$. Also, $[\mu \cdot \nu]_S$ is the preimage of $\mu \times \nu$ under d.

Proof. For every $x \in P$, we have

$$\begin{aligned}
d^{-1}(\mu \times \nu)(x) &= ((\mu \times \nu) \circ d)(x) \\
&= (\mu \times \nu)(d(x)) = (\mu \times \nu)(x, x) \\
&= S(\mu(x), \nu(x)) = [\mu \cdot \nu]_S(x).
\end{aligned}$$

Therefore, $[\mu \cdot \nu]_S$ is the preimage of $\mu \times \nu$ under d. $\square$

***Definition** 4.6:* If ν is a fuzzy set in P, the weakest S-fuzzy relation on P, that is S-fuzzy relation on ν, is μ_ν given by $\mu_\nu(x, y) = S(\nu(x), \nu(y))$ for all $x, y \in P$.

***Theorem** 4.7:* Let ν be a fuzzy set in P and μ_ν the weakest S-fuzzy relation on P. Then, ν is a S-fuzzy subpolygroup of P if and only if μ_ν is a S-fuzzy subpolygroup of $P \times P$.

Proof. Suppose that ν is a S-fuzzy subpolygroup of P. Let (x_1, x_2), $(y_1, y_2) \in P \times P$ and $(z_1, z_2) \in (x_1, x_2)(y_1, y_2)$. Then, we have

$$\begin{aligned}
\mu_\nu(z_1, z_2) &= S(\nu(z_1), \nu(z_2)) \\
&\leq S(S(\nu(x_1), \nu(y_1)), S(\nu(x_2), \nu(y_2))) \\
&= S(S(\nu(x_1), \nu(x_2)), S(\nu(y_1), \nu(y_2))) \\
&= S(\mu_\nu(x_1, x_2), \mu_\nu(y_1, y_2)).
\end{aligned}$$

Let $(x_1, x_2) \in P \times P$, then

$$\begin{aligned}
\mu_\nu(x_1, x_2) &= S(\nu(x_1), \nu(x_2)) \leq S(\nu(x_1^{-1}), \nu(x_2^{-1})) \\
&= \mu_\nu(x_1^{-1}, x_2^{-1}) = \mu_\nu((x_1, x_2)^{-1}).
\end{aligned}$$

Therefore, μ_ν is a S-fuzzy subpolygroup of $P \times P$. Conversely, suppose that μ_ν is a S-fuzzy subpolygroup of $P \times P$. For any $x_1, x_2, y_1, y_2 \in P$ and $z_1 \in x_1 y_1$ and $z_2 \in x_2 y_2$, we have

$$\begin{aligned}
S(\nu(z_1), \nu(z_2)) &= \mu_\nu(z_1, z_2) \\
&\leq S(\mu_\nu(x_1, y_1), \mu_\nu(x_2, y_2)) \\
&= S(S(\nu(x_1), \nu(y_1)), S(\nu(x_2), \nu(y_2))).
\end{aligned}$$

In particular, putting $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $z_1 = z_2 = z$, we have $\nu(z) \leq S(\nu(x), \nu(y))$, for all $x, y \in P$ and $z \in xy$. Similarly, we can prove that $\nu(x) \leq \nu(x^{-1})$ for all $x \in P$. Therefore, ν is a S-fuzzy subpolygroup of P. $\square$

V. Abnormal S-fuzzy subpolygroups

In this section, we introduce the notion of abnormal S-fuzzy subpolygroup. By this notion, we show that, under certain conditions, S-fuzzy subpolygroups are two-valued.

***Definition** 5.1:* A S-fuzzy subpolygroup of P is said to be abnormal if there exists $x \in P$ such that $\mu(x) = 0$. Note that, if S-fuzzy subpolygroup μ of P is abnormal, then $\mu(e) = 0$. Hence, μ is abnormal if and only if $\mu(e) = 0$.

***Theorem** 5.2:* Let μ be a S-fuzzy subpolygroup of P and μ^+ a fuzzy set in P defined by $\mu^+(x) = \mu(x) - \mu(e)$ for all $x \in P$. Then, μ^+ is an abnormal S-fuzzy subpolygroup of P containing μ.

Proof. For all $x, y \in P$ and $z \in xy$, we have

$$\begin{aligned}
\mu^+(z) &= \mu(z) - \mu(e) \\
&\leq S(\mu(x), \mu(y)) - \mu(e) \\
&= S(\mu(x) - \mu(e), \mu(y) - \mu(e)) \\
&= S(\mu^+(x), \mu^+(y)).
\end{aligned}$$

For every $x \in P$, we have

$$\begin{aligned}
\mu^+(x) &= \mu(x) - \mu(e) \\
&\leq \mu(x^{-1}) - \mu(e) = \mu^+(x^{-1}).
\end{aligned}$$

We have $\mu^+(e) = \mu(e) - \mu(e) = 0$. Therefore, μ^+ is an abnormal S-fuzzy subpolygroup of P. Clearly, $\mu \subseteq \mu^+$. $\square$

***Corollary** 5.3:* If μ is a S-fuzzy subpolygroup of P satisfying $\mu^+(x) = 1$ for some $x \in P$, then $\mu(x) = 1$. $\square$

***Theorem** 5.4:* Let μ and ν be S-fuzzy subpolygroups of P. If $\nu \subseteq \mu$ and $\mu(e) = \nu(e)$, then $P_\mu \subseteq P_\nu$, where $P_\mu = \{x \in P \mid \mu(x) = \mu(e)\}$.

Proof. Assume that $\nu \subseteq \mu$ and $\mu(e) = \nu(e)$. If $x \in P_\mu$, then $\nu(x) \leq \mu(x) = \mu(e) = \nu(e)$. Noticing that $\nu(e) \leq \nu(x)$ for all $x \in P$, we have $\nu(x) = \nu(e)$, then $x \in P_\nu$. $\square$

***Corollary** 5.5:* If μ and ν are abnormal S-fuzzy subpolygroups of P satisfying $\nu \subseteq \mu$, then $P_\mu \subset P_\nu$. $\square$

***Theorem** 5.6:* A S-fuzzy subpolygroup of P is abnormal if and only if $\mu^+ = \mu$.

Proof. The sufficiency is obvious. Assume that μ is an abnormal S-fuzzy subpolygroup of P and $x \in P$, then $\mu^+(x) = \mu(x) - \mu(e) = \mu(x)$. □

***Theorem** 5.7:* If μ is a S-fuzzy subpolygroup of P, then $(\mu^+)^+ = \mu$.

Proof. Assume that $x \in P$, then

$$\begin{aligned}(\mu^+)^+(x) &= \mu^+(x) - \mu^+(e)\\ &= (\mu(x) - \mu(e)) - (\mu(e) - \mu(e))\\ &= \mu(x) - \mu(e) = \mu^+(x).\end{aligned}$$

□

***Theorem** 5.8:* Let μ be a non-constant S-fuzzy subpolygroup of P, which is minimal in the poset of abnormal S-fuzzy subpolygroups of P under sets inclusion. Then, μ takes only two values 0 and 1.

Proof. Note that, $\mu(e) = 0$. Let $x \in P$ be such that $\mu(x) \neq 0$. It is sufficient to show that $\mu(x) = 1$. Assume that there exists $a \in P$ such that $0 < \mu(a) < 1$. Define a fuzzy set ν by $\nu(x) = \frac{1}{2}(\mu(x) + \mu(a))$ for each $x \in P$. Then, clearly ν is well-defined and for all $x, y \in P$ and $z \in xy$ we have

$$\begin{aligned}\nu(z) &= \frac{1}{2}(\mu(z) + \mu(a))\\ &\leq \frac{1}{2}(S(\mu(x), \mu(y)) + \mu(a))\\ &= S(\frac{1}{2}(\mu(x) + \mu(a)), \frac{1}{2}(\mu(y) + \mu(a)))\\ &= S(\nu(x), \nu(y)).\end{aligned}$$

Similarly, we can prove that $\nu(x) \leq \nu(x^{-1})$ for all $x \in P$. Hence, ν^+ is an abnormal S-fuzzy subpolygroup of P. Noticing that $\mu^+(e) = 0 < \nu^+(a) = \frac{1}{2}\mu(a) < \mu(a)$, we know that ν^+ is non-constant. From $\nu^+(a) < \mu(a)$, it follows that μ is not minimal, which is a contradiction. Therefore, the proof is completed. □

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