Terminal Cost Distribution in Discrete-Time Controlled System with Disturbance and Noise-Corrupted State Information

Valery Y. Glizer, Vladimir Turetsky, and Josef Shinar .

***Abstract*—Recursive formula for the terminal cost distribution in a scalar linear discrete-time system with disturbance and noise corrupted measurements is obtained. The system is subject to a linear saturated control strategy. The distributions of the initial state and the estimator error are assumed to be known. The disturbance is independent of the state/control and its distribution is known. The general result is applied to an interception problem with different types of disturbance. An illustrative numerical example confirms that the analytical method can replace extensive Monte Carlo simulations.**

***Index Terms*—linear discrete-time system, robust transferring strategy, noisy measurements, terminal state distribution.**

I. Introduction

VARIOUS real life control problems (including interception and navigation) can be formulated as a problem of transferring a system to a prescribed hyperplane in the state space at a prescribed time by bounded control in the presence of noise corrupted measurements and unknown bounded disturbance [1], [2], [3], [4]. This problem can be reduced to a scalar one, where the new state variable z is the distance between a current point on the trajectory of the uncontrolled system motion and the hyperplane. By this scalarization, the problem of transferring to a prescribed hyperplane becomes a problem of robust transferring to zero.

If perfect state information is available, several classes of deterministic feedback strategies $u = u(t, z(t))$ that robustly transfer a scalar system from some domain of initial positions to zero, are known. Among such robust transferring strategies are differential game based bang-bang strategies [1], [2], as well as various linear, saturated linear and weakly nonlinear strategies [3], [5], [6], [7], [8], [9].

In real life applications, the state information is corrupted by measurement noise and only part of the state variables can be directly measured. This fact impedes significantly the practical implementation of theoretically robust transferring strategies. Moreover, an estimator, reconstructing and filtering the state variables, becomes an indispensable component of the control loop. Due to the noisy measurements and the uncertain (random) disturbance the control function $u(t, z(t))$ receives, instead of the exact value of $z(t)$, a random estimator output $\hat{z}(t) = z(t) + \eta(t)$, where $\eta(t)$ is the estimation error. As the consequence, the terminal value of z becomes a random variable with an a-priori unknown distribution. In order to appreciate the extent of performance deterioration of a deterministic robust transferring strategy by using such a stochastic data, the distribution of the terminal value of z has be found.

In the current practice, such a distribution is obtained, for any given estimator/control strategy combination and specified disturbance and noise models, by a large set of Monte Carlo simulations [10], [11]. Such a-posteriori test is necessary for validation purpose, but cannot be applied for an insightful control system design. There is an obvious need for an analytical a-priori estimate of the control strategy performance as a part of the integrated control system design.

State estimates in the presence of deterministic information errors were obtained in [12] and [13]. In [13], such estimates are used to construct a robust control of a dynamic system with inexact state information. In interception problems, the scalar state variable z is the zero-effort miss distance and its terminal value is the actual miss distance itself. Under some general linear assumptions without taking into account the system dynamics, the miss distance distribution was investigated in [14]. In the case of linear interceptor strategies, the dependence of the miss distance on the measurement noises was analyzed, by means of the adjoint approach in [15] and [16]. Unfortunately, this approach can be applied only in the case of non saturated linear strategies.

In this paper, the system dynamics is modeled by a discrete-time scalar linear equation controlled by a *saturated* linear transferring strategy. Assuming that the distributions of initial state z_0, the estimation error η_n and disturbance v_n are known, a recurrence formula for the distribution of z_{n+1} is obtained. The random variable z_{n+1} is the linear combination of two dependent random variables - the state z_n at the previous time step and the control variable u_n (nonlinearly depending on z_n via the saturation function) and an independent random variable v_n. This makes the problem to be mathematically nontrivial. The disturbance free version of the problem was studied in [17].

II. Problem Statement

A. Original Control Problem

Consider the controlled system

$$\dot{X} = A(t)X + b(t)u + c(t)v + f(t), \tag{1}$$

where $X \in \mathbb{R}^n$ is the state vector; $t \in [t_0, t_f]$, $X(t_0) = X_0$, t_f is a fixed time instant, $t_0 \in [0, t_f)$; the matrix function

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$A(t)$ and the vector functions $b(t)$, $c(t)$, $f(t)$ are differentiable for a sufficient number of times on the interval $[0, t_f]$. The scalar control u and disturbance v are assumed to be measurable on $[t_0, t_f)$ and satisfying the constraints

$$|u(t)| \leq 1, \quad |v(t)| \leq 1, \quad t \in [t_0, t_f). \tag{2}$$

The target set is the hyperplane $\mathcal{D} = \{X \in \mathbb{R}^n \mid d^T X + d_0 = 0\}$, where $d \in \mathbb{R}^n$ is a prescribed non-zero vector, d_0 is a prescribed scalar, the superscript T denotes the transposition. The control objective is to guarantee $X(t_f) \in \mathcal{D}$ against any admissible disturbance function $v(t)$.

By the transformation of the state variable of (1),

$$z = z(t, X) = d^T \Big(\Phi(t_f, t) X + \int\limits_t^{t_f} \Phi(t_f, \tau) f(\tau) d\tau \Big) + d_0, \tag{3}$$

the system (1) is reduced to the scalar one

$$\dot{z} = h_1(t) u + h_2(t) v, \quad z(t_0) = z_0, \tag{4}$$

where $h_1(t) = d^T \Phi(t_f, t) b(t)$, $h_2(t) = d^T \Phi(t_f, t) c(t)$, $z_0 = z(t_0, X_0)$, $\Phi(t, t_0)$ is the fundamental matrix of the homogeneous system $\dot{X} = A(t) X$. The control objective becomes to guarantee $z(t_f) = 0$.

It is assumed that the control is given by a saturated linear strategy

$$u(t, z) = \mathrm{sat}(K(t) z), \tag{5}$$

where

$$\mathrm{sat}(y) = \begin{cases} 1, & y > 1, \\ y, & |y| \leq 1, \\ -1, & y < -1, \end{cases} \tag{6}$$

the gain function $K(t)$ satisfies the conditions [9], guaranteeing that the linear strategy $u = K(t) z$ is robust transferring.

B. Discrete-Time Estimation Problem

Define the division of the interval $[0, t_f]$: $0 = t_0 < t_1 < \ldots < t_N = t_f$, where $t_{n+1} - t_n = \Delta t$, $n = 0, \ldots, N-1$. The discrete-time version of the system (4) is

$$z_{n+1} = z_n + b_n u_n + c_n v_n, \tag{7}$$

where for the simplest Euler approximation of the differential equation (4),

$$b_n = \Delta t h_1(t_n), \quad c_n = \Delta t h_2(t_n). \tag{8}$$

The control is

$$u_n = \mathrm{sat}(k_n(z_n + \eta_n)), \tag{9}$$

where $k_n = K(t_n)$ is the control gain and η_n is the estimation error. The probability density functions $f_{z_0}(x)$ of z_0 and $f_{\eta_n}(x)$ of η_n, $n = 0, 1, \ldots, N-1$, are assumed to be known. For any n, the random value v_n is independent of z_n and u_n. Its probability density function $f_{v_n}(x)$ is assumed to be known. The problem is to obtain the probability density function $f_{z_N}(x)$.

Denote

$$w_{1n} \triangleq z_n + b_n u_n, \tag{10}$$

$$w_{2n} \triangleq c_n v_n. \tag{11}$$

Since v_n is independent of z_n and u_n, the random values w_{1n} and w_{2n} are independent. Thus, due to (7) and (10) – (11), the convolution formula [18] can be applied:

$$f_{z_{n+1}}(x) = \int\limits_{-\infty}^{\infty} f_{w_{1n}}(x - \xi) f_{w_{2n}}(\xi) d\xi, \tag{12}$$

where $f_{w_{1n}}(x)$ and $f_{w_{2n}}(x)$ are the probability density functions of w_{1n} and w_{2n}, respectively. However, since the random variables z_n and u_n are dependent, the distribution function of w_{1n} cannot be calculated by using the convolution formula.

III. Solution

Since the probability density function $f_{w_{2n}}(x)$ is assumed to be known, the calculation of $f_{z_{n+1}}(x)$, due to (12), is reduced to the calculation of $f_{w_{1n}}(x)$.

A. Calculation of $f_{w_{1n}}(x)$

Due to (7) – (9), the distribution function of w_{1n} is

$$F_{w_{1n}}(x) = P(w_{1n} < x) = p_1 P(k_n(z_n + \eta_n) > 1) + p_2 P(|k_n(z_n + \eta_n)| \leq 1) + p_3 P(k_n(z_n + \eta_n) < -1), \tag{13}$$

where p_1, p_2 and p_3 are the conditional probabilities

$$p_1 = P(z_n + b_n < x \;\big|\; k_n(z_n + \eta_n) > 1), \tag{14}$$

$$p_2 = P((1 + b_n k_n) z_n + b_n k_n \eta_n < x \;\big|\; |k_n(z_n + \eta_n)| \leq 1), \tag{15}$$

$$p_3 = P(z_n - b_n < x \;\big|\; k_n(z_n + \eta_n) < -1). \tag{16}$$

Thus, the problem is reduced to calculating the conditional probabilities (14) – (16).

1) Calculation of p_1 and p_3: By using (14) and the formula for the probability of the product of dependent events,

$$p_1 = \tilde{p}_1 P(z_n < x - b_n) \Big/ P(z_n + \eta_n > 1/k_n), \tag{17}$$

where

$$\tilde{p}_1 = P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n < x - b_n\Big). \tag{18}$$

Let calculate the conditional probability $\tilde{p}_1$.

First, instead of the event $z_n < x - b_n$, consider the event $z_n \in (a, x - b_n)$, where a is a negative number with sufficiently large absolute value:

$$\tilde{p}_{1a} = P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n \in (a, x - b_n)\Big). \tag{19}$$

Note that

$$\tilde{p}_1 = \lim_{a \to -\infty} \tilde{p}_{1a}. \tag{20}$$

Let divide the interval $(a, x - b_n)$ into M subintervals of equal length $\Delta x = (x - b_n - a)/M$: $x_j = a + j\Delta x$, $j = 0, 1, \ldots, M$. Then, since the events $z_n \in (x_j, x_{j+1})$, $j = 0, \ldots, M-1$, are mutually exclusive,

$$\tilde{p}_{1a} \approx \sum_{j=0}^{M-1} P\Big(z_n \in (x_j, x_{j+1}) \;\Big|\; z_n \in (a, x - b_n)\Big) \times$$

$$P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n \in (x_j, x_{j+1})\Big). \tag{21}$$

Let start with calculating the first conditional probability under the sum in (21). Note that

$$P\Big(\Big[z_n \in (x_j, x_{j+1})\Big] \& \Big[z_n \in (a, x - b_n)\Big]\Big) =$$

$$P\Big(z_n \in (x_j, x_{j+1}) \;\Big|\; z_n \in (a, x - b_n)\Big) P(z_n \in (a, x - b_n)). \tag{22}$$

At the other hand,

$$P\Big(\Big[z_n \in (x_j, x_{j+1})\Big] \& \Big[z_n \in (a, x - b_n)\Big]\Big) =$$

$$\underbrace{P\Big(z_n \in (a, x - b_n) \;\Big|\; z_n \in (x_j, x_{j+1})\Big)}_{=1} P(z_n \in (x_j, x_{j+1}))$$

$$= P(z_n \in (x_j, x_{j+1})). \tag{23}$$

From (22) – (23),

$$P\Big(z_n \in (x_j, x_{j+1}) \;\Big|\; z_n \in (a, x - b_n)\Big) =$$

$$P(z_n \in (x_j, x_{j+1})) \Big/ P(z_n \in (a, x - b_n)). \tag{24}$$

For sufficiently small Δx, the second conditional probability under the sum in (21) can be approximated as

$$P\Big(z_n + \eta_n > 1/k_n \Big| z_n \in (x_j, x_{j+1})\Big) \approx$$

$$P\Big(z_n + \eta_n > 1/k_n \;\Big|\; z_n = \bar{x}_j\Big) = P\Big(\eta_n > 1/k_n - \bar{x}_j\Big), \tag{25}$$

where $\bar{x}_j = (x_j + x_{j+1})/2$.

Due to (21) and (24) – (25),

$$\tilde{p}_{1a} \approx \frac{1}{P(z_n \in (a, x - b_n))} \sum_{j=0}^{M-1} P\Big(z_n \in (x_j, x_{j+1})\Big) \times$$

$$P\Big(\eta_n > 1/k_n - \bar{x}_j\Big) =$$

$$\frac{1}{\int\limits_a^{x-b_n} f_{z_n}(y)dy} \sum_{j=0}^{M-1} \int\limits_{x_j}^{x_{j+1}} f_{z_n}(y)dy \int\limits_{1/k_n - \bar{x}_j}^{\infty} f_{\eta_n}(y)dy, \tag{26}$$

where $f_{z_n}(y)$ and $f_{\eta_n}(y)$ are the probability density functions of the random variables z_n and η_n, respectively. Since

$$\int\limits_{x_j}^{x_{j+1}} f_{z_n}(y)dy \approx f_{z_n}(\bar{x}_j)\Delta x, \tag{27}$$

the equation (26) can be rewritten as

$$\tilde{p}_{1a} \approx \frac{1}{\int\limits_a^{x-b_n} f_{z_n}(y)dy} \sum_{j=0}^{M-1} \left[f_{z_n}(\bar{x}_j) \int\limits_{1/k_n - \bar{x}_j}^{\infty} f_{\eta_n}(y)dy \right] \Delta x. \tag{28}$$

Hence,

$$\tilde{p}_{1a} = \frac{\lim\limits_{M\to\infty} \sum\limits_{j=0}^{M-1} \left[f_{z_n}(\bar{x}_j) \int\limits_{1/k_n - \bar{x}_j}^{\infty} f_{\eta_n}(y)dy \right] \Delta x}{\int\limits_a^{x-b_n} f_{z_n}(y)dy} =$$

$$\frac{\int\limits_a^{x-b_n} \left[f_{z_n}(s) \int\limits_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int\limits_a^{x-b_n} f_{z_n}(y)dy}. \tag{29}$$

By virtue of (20),

$$\tilde{p}_1 = \frac{\int\limits_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int\limits_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int\limits_{-\infty}^{x-b_n} f_{z_n}(y)dy}. \tag{30}$$

Due to (17) and (30),

$$p_1 = \frac{\int\limits_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int\limits_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int\limits_{1/k_n}^{\infty} f_{z_n+\eta_n}(y)dy}. \tag{31}$$

The random variables z_n and η_n are independent. Therefore,

$$f_{z_n+\eta_n}(y) = f_{z_n}(y) * f_{\eta_n}(y) = \int\limits_{-\infty}^{\infty} f_{z_n}(y - s) f_{\eta_n}(s)ds. \tag{32}$$

Finally,

$$p_1 = \frac{\int\limits_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int\limits_{1/k_n - s}^{\infty} f_{\eta_n}(y)dy \right] ds}{\int\limits_{-1/k_n}^{\infty} \left[\int\limits_{-\infty}^{\infty} f_{z_n}(y - s) f_{\eta_n}(s)ds \right] dy}. \tag{33}$$

The calculation of p_3 is similar to the calculation of p_1, resulting in

$$p_3 = \frac{\int\limits_{-\infty}^{x+b_n} \left[f_{z_n}(s) \int\limits_{-\infty}^{-1/k_n - s} f_{\eta_n}(x)dx \right] ds}{\int\limits_{-\infty}^{-1/k_n} \left[\int\limits_{-\infty}^{\infty} f_{z_n}(x - s) f_{\eta_n}(s)ds \right] dx}. \tag{34}$$

2) Calculation of p_2*:* Consider the case $b_n \geq 0$. By definition of the conditional probability,

$$p_2 = \frac{P\Big((z_n,\eta_n) \in R(x) \& (z_n,\eta_n) \in Q\Big)}{P\Big((z_n,\eta_n) \in Q\Big)} =$$

$$P\Big((z_n,\eta_n) \in S(x)\Big) \Big/ P\Big((z_n,\eta_n) \in Q\Big), \qquad (35)$$

where (see Fig. 1)

$$R(x) \triangleq \{(z_n,\eta_n): \quad \eta_n < -Az_n + B(x)\}, \qquad (36)$$

$$A = 1 + \frac{1}{b_n k_n},\ B(x) = \frac{x}{b_n k_n},$$

$$Q \triangleq \{(z_n,\eta_n): \quad -z_n - 1/k_n \leq \eta_n \leq -z_n + 1/k_n\}, \qquad (37)$$

$$S(x) \triangleq R(x) \bigcap Q. \qquad (38)$$

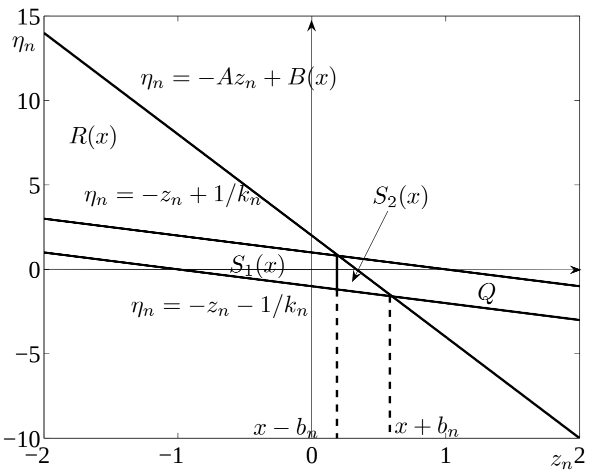

Fig. 1. Sets $R(x)$ and Q for $b_n > 0$

The straight line $\eta_n = -Az_n + B(x)$ (the upper boundary of the set $R(x)$) intersects the straight lines $\eta_n = -z_n \pm 1/k_n$ (boundaries of the set Q) at $z_n = x \pm b_n$. Thus, the set $S(x)$ can be represented as

$$S(x) = S_1(x) \bigcup S_2(x), \quad S_1(x) \bigcap S_2(x) = \emptyset, \qquad (39)$$

where

$$S_1(x) = \Big\{(z_n,\eta_n): \quad z_n < x - b_n,$$

$$-z_n - 1/k_n \leq \eta_n \leq -z_n + 1/k_n\Big\}, \qquad (40)$$

$$S_2(x) = \Big\{(z_n,\eta_n): \quad z_n \in [x - b_n, x + b_n],$$

$$-z_n - 1/k_n \leq \eta_n \leq -Az_n + B(x)\Big\}. \qquad (41)$$

Therefore,

$$P\Big((z_n,\eta_n) \in S(x)\Big) =$$

$$P\Big((z_n,\eta_n) \in S_1(x)\Big) + P\Big((z_n,\eta_n) \in S_2(x)\Big). \qquad (42)$$

Similarly to the calculation of p_1, by discretization and limiting,

$$P\Big((z_n,\eta_n) \in S_1(x)\Big) =$$

$$\int\limits_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int\limits_{-s-1/k_n}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds, \qquad (43)$$

$$P\Big((z_n,\eta_n) \in S_2(x)\Big) =$$

$$\int\limits_{x-b_n}^{x+b_n} \left[f_{z_n}(s) \int\limits_{-s-1/k_n}^{-As+B(x)} f_{\eta_n}(y)dy \right] ds. \qquad (44)$$

By virtue of (35), (37) and (43) – (44),

$$p_2 = \frac{1}{C_n} \left\{ \int\limits_{-\infty}^{x-b_n} \left[f_{z_n}(s) \int\limits_{-s-1/k_n}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds + \right.$$

$$\left. \int\limits_{x-b_n}^{x+b_n} \left[f_{z_n}(s) \int\limits_{-s-1/k_n}^{-As+B(x)} f_{\eta_n}(y)dy \right] ds \right\}, \qquad (45)$$

where

$$C_n = \int\limits_{-1/k_n}^{1/k_n} \left[\int\limits_{-\infty}^{\infty} f_{z_n}(y-s) f_{\eta_n}(s)ds \right] dy. \qquad (46)$$

If $b_n < 0$, similarly to (45), it is obtained that

$$p_2 = \frac{1}{C_n} \left\{ \int\limits_{-\infty}^{x+b_n} \left[f_{z_n}(s) \int\limits_{-s-1/k_n}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds + \right.$$

$$\left. \int\limits_{x+b_n}^{x-b_n} \left[f_{z_n}(s) \int\limits_{-As+B(x)}^{-s+1/k_n} f_{\eta_n}(y)dy \right] ds \right\}. \qquad (47)$$

3) Final expression of $f_{w_{1n}}(x)$*:* For $b_n \geq 0$, by substituting (31), (34) and (45) into (13) and by simplifying the obtained expression,

$$F_{w_{1n}}(x) = \int\limits_{-\infty}^{x-b_n} f_{z_n}(s)ds +$$

$$\int\limits_{x-b_n}^{x+b_n} \left[f_{z_n}(s) \int\limits_{-\infty}^{-As+B(x)} f_{\eta_n}(y)dy \right] ds. \qquad (48)$$

Similarly, for $b_n < 0$,

$$F_{w_{1n}}(x) = \int\limits_{-\infty}^{x+b_n} f_{z_n}(s)ds +$$

$$\int\limits_{x+b_n}^{x-b_n} \left[f_{z_n}(s) \int\limits_{-As+B(x)}^{\infty} f_{\eta_n}(y)dy \right] ds. \qquad (49)$$

Differentiating (48) and (49) with respect to x yields the same expression for the probability density function of w_{1n}:

$$f_{w_{1n}}(x) = f_{z_n}(x-b_n) + f_{z_n}(x+b_n) \int\limits_{-\infty}^{-x-1/k_n-b_n} f_{\eta_n}(y)dy -$$

$$f_{z_n}(x-b_n) \int_{-\infty}^{-x+1/k_n+b_n} f_{\eta_n}(y)dy+$$

$$\frac{1}{b_n k_n} \int_{x-b_n}^{x+b_n} [f_{z_n}(s) f_{\eta_n}(-As+B(x))]ds. \quad (50)$$

B. Calculation of $F_{z_N}(x)$

Once $f_{w_{1n}}(x)$ has been calculated, the probability density function $f_{z_{n+1}}(x)$ is obtained by using (12). The probability density function $f_{z_N}(x)$ of a terminal state is computed by applying the recurrence formulae (50) and (12) N times. Finally, the probability function $F_{z_N}(x)$ is

$$F_{z_N}(x) = \int_{-\infty}^{x} f_{z_N}(\xi) d\xi. \quad (51)$$

IV. Interception Problem

A. Problem Outline

As an example, a planar engagement between two point-mass objects (pursuer and evader) is considered. It is assumed that the dynamics of each object is expressed by a first-order transfer function with the time constants τ_p and τ_e, respectively. The velocities V_p and V_e and the bounds of the lateral acceleration commands $a_p^{\max}$ and $a_e^{\max}$ of the objects are constant. The geometry of such planar engagement is presented in Fig. 2.

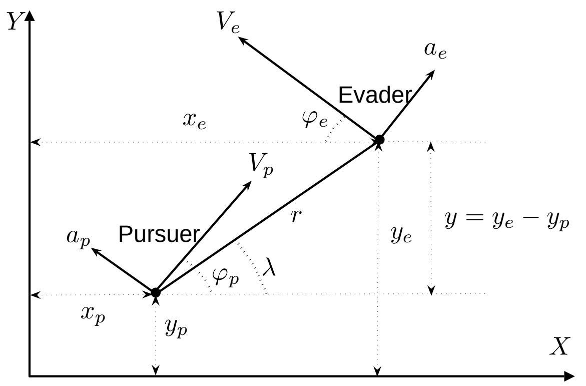

Fig. 2. Interception geometry

Assuming that the aspect angles φ_p and φ_e are small, the engagement is modeled by the system (1), where X_1 is the relative separation between the objects, normal to the initial line-of-sight; X_2 is the relative normal velocity; X_3 and X_4 are the lateral accelerations of the evader and the pursuer, respectively; $t_f = r_0/(V_p+V_e)$, where r_0 is the initial range between the objects;

$$A(t) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_e & 0 \\ 0 & 0 & 0 & -1/\tau_p \end{bmatrix}, \quad (52)$$

$$b(t) \equiv (0,0,0,a_p^{\max}/\tau_p)^T, \quad c(t) \equiv (0,0,a_e^{\max}/\tau_e,0)^T, \quad (53)$$

$$f(t) \equiv 0, \;\; X_0 = (0, X_{20}, 0, 0)^T, \;\; X_{20} = V_e\varphi_e(0) - V_p\varphi_p(0). \quad (54)$$

The controls of the pursuer u and the evader v are the normalized lateral acceleration commands, satisfying the constraints (2). The objective of the pursuer is to nullify the miss distance $|X_1(t_f)|$, i.e. in the target hyperplane, $d = (1,0,0,0)^T$, $d_0 = 0$.

In the scalarized system (4), $z_0 = t_f X_{20}$,

$$h_1(t) = -h(t;\tau_p,a_p^{\max}), \quad h_2(t) = h(t;\tau_e,a_e^{\max}), \quad (55)$$

where

$$h(t;\tau,a^{\max}) = \tau a^{\max}\Psi((t_f-t)/\tau), \quad (56)$$

$$\Psi(\xi) = \exp(-\xi) + \xi - 1 > 0, \quad \xi > 0. \quad (57)$$

The new state variable z is the well-known zero-effort miss distance [15]. The target point is $(t_f, 0)$.

The pursuer strategy is given by (5) with the gain function

$$K(t) = 2/(t_f-t)^3. \quad (58)$$

The scalar system is approximated by the discrete-time equation (7), where the coefficients $b_n < 0$ and $c_n > 0$ are given by (8), while the functions $h_1(t)$ and $h_2(t)$ are given by (55).

B. Types of Disturbance

In this example four types of different disturbances, representing evader maneuvers, are considered.

1) Constant disturbance: In this case, the evader employs the constant (deterministic) strategy $v(t) \equiv V = \mathrm{const}$, and the probability function of w_{2n}, given by (11), is

$$F_{w_{2n}}(x) = \begin{cases} 0, & x \le c_n V, \\ 1, & x > c_n V, \end{cases} \quad (59)$$

yielding

$$f_{w_{2n}}(x) = \delta(x - c_n V), \quad (60)$$

where $\delta(x)$ is the Dirac delta function. Thus, due to (12),

$$f_{z_{n+1}}(x) = f_{w_{1n}}(x - c_n V). \quad (61)$$

2) Bang-bang disturbance: In this case, the evader employs the bang-bang strategy

$$v(t) = \begin{cases} 1, & t \in [0, t_{sw}], \\ -1, & t \in (t_{sw}, t_f), \end{cases} \quad (62)$$

with a fixed switch time $t_{sw} \in (0, t_f)$. In the discrete model (7), it is assumed that $t_{sw} = \Delta t n_{sw}$, where $n_{sw} \in \{0, \ldots, N-1\}$, i.e.

$$w_{2n} = \begin{cases} c_n, & n \le n_{sw} \\ -c_n, & n > n_{sw}. \end{cases} \quad (63)$$

Thus, the probability function of w_{2n} is

$$F_{w_{2n}}(x) = \begin{cases} 0, & x \le c_n \\ 1, & x > c_n, \end{cases} \quad \text{for } n \le n_{sw}, \quad (64)$$

and

$$F_{w_{2n}}(x) = \begin{cases} 0, & x \le -c_n \\ 1, & x > -c_n, \end{cases} \quad \text{for } n > n_{sw}. \quad (65)$$

The equations (64) – (65) yield

$$f_{w_{2n}}(x) = \begin{cases} \delta(x - c_n), & n \le n_{sw} \\ \delta(x + c_n), & n > n_{sw}. \end{cases} \tag{66}$$

Consequently, by using (12),

$$f_{z_{n+1}}(x) = \begin{cases} f_{w_{1n}}(x - c_n), & n \le n_{sw} \\ f_{w_{1n}}(x + c_n), & n > n_{sw}. \end{cases} \tag{67}$$

3) Random switch bang-bang disturbance: In this case, the evader also employs the bang-bang strategy (62), but the switch time t_{sw} is random, uniformly distributed over the interval $[0, t_f]$. In the discrete model (7), it is assumed that $t_{sw} = \Delta t n_{sw}$, where n_{sw} can accept any value from the set $\{0, 1, \dots, N-1\}$ with the probability $p = \dfrac{1}{N}$.

Let calculate the probability

$$p_n^+ \triangleq P(w_{2n} = c_n). \tag{68}$$

Due to (62),

$$p_n^+ = P(n \le n_{sw}) = 1 - F_{n_{sw}}(n), \tag{69}$$

where $F_{n_{sw}}(x)$ is the probability function of n_{sw}:

$$F_{n_{sw}}(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{1}{N}, & 0 < x \le 1, \\ \dfrac{2}{N}, & 1 < x \le 2, \\ \dots \\ 1, & x > N-1, \end{cases} \tag{70}$$

yielding

$$F_{n_{sw}}(n) = \frac{n}{N}, \quad n = 0, 1, \dots, N-1, \tag{71}$$

and, by (69),

$$p_n^+ = 1 - \frac{n}{N}. \tag{72}$$

Therefore, the disturbance term w_{2n} has a random value

$$w_{2n} = \begin{cases} c_n, & p = p_n^+, \\ -c_n, & p = 1 - p_n^+, \end{cases} \tag{73}$$

where p is the probability.

Thus, the probability function of w_{2n} is

$$F_{w_{2n}}(x) = \begin{cases} 0, & x \le -c_n, \\ \dfrac{n}{N}, & -c_n < x \le c_n, \\ 1, & x > c_n. \end{cases} \tag{74}$$

By differentiating (74), the probability density function is

$$f_{w_{2n}}(x) = \frac{n}{N}\delta(x + c_n) + \left(1 - \frac{n}{N}\right)\delta(x - c_n). \tag{75}$$

Equation (12) along with (75) yields

$$f_{z_{n+1}}(x) = \frac{n}{N} f_{w_{1n}}(x + c_n) + \left(1 - \frac{n}{N}\right) f_{w_{1n}}(x - c_n). \tag{76}$$

4) Random value disturbance: In this case, it is assumed that for any $t \in [0, t_f]$, the disturbance $v(t)$ has a random value, uniformly distributed on the interval $[-1, 1]$. Thus, in the discrete model (7), the random variable w_{2n} is uniformly distributed on the interval $[-c_n, c_n]$, yielding the probability density function

$$f_{w_{2n}}(x) = \begin{cases} \dfrac{1}{2c_n}, & x \in (-c_n, c_n], \\ 0, & x \notin (-c_n, c_n]. \end{cases} \tag{77}$$

The latter, along with (12), leads to

$$f_{z_{n+1}}(x) = \int_{-c_n}^{c_n} f_{w_{1n}}(x - \xi) d\xi. \tag{78}$$

C. Numerical Illustration

In this subsection the analytical results are compared to the outcome of extensive Monte Carlo simulations. It is assumed that the initial value z_0 and the estimation errors η_n are gaussian: $z_0 \sim \mathcal{N}(0.5, 0.1)$, $\eta_n \sim \mathcal{N}(\mu_n, \sigma_n)$, $n = 0, \dots, N-1$. The set of such values of μ_n and σ_n were extracted from a realistic Monte Carlo simulation with noisy line-of-sight measurements and an estimator in the control loop. For the sake of comparison, the same set of values were chosen for all the four types of disturbance. The data for the comparisons are $t_f = 1$ s, $N = 10$, $\Delta t = 0.1$ s, $\tau_p = 0.2$ s, $a_p^{\max} = 30$ m/s^2, $\tau_e = 0.2$ s, $a_e^{\max} = 15$ m/s^2.

The cumulative distribution function of the miss distance $|z_N|$ is calculated as

$$F_{|z_N|}(x) = F_{z_N}(x) - F_{z_N}(-x), \tag{79}$$

where $F_{z_N}(x)$ is given by (51).

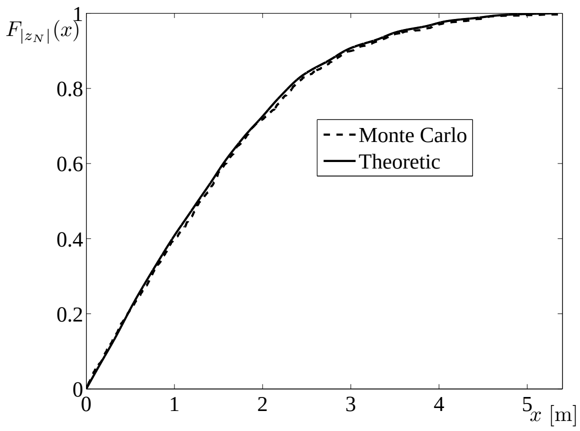

Fig. 3. Simulative and theoretic distribution functions of $|z_N|$ for constant disturbance

In Figs. 3 – 6, the cumulative distributions of $|z_N|$, obtained by Monte Carlo simulation of (4) and by using (50), (51) and (79), are depicted and compared to the results of Monte Carlo simulation of (4) for the four different types of disturbance.

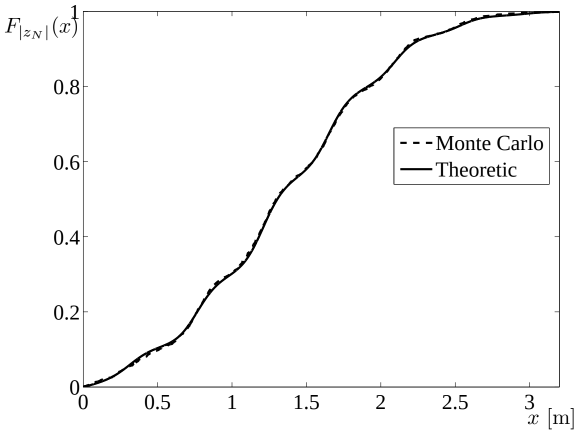

Fig. 4. Simulative and theoretic distribution functions of $|z_N|$ for bang-bang disturbance, $n_{sw} = 5$

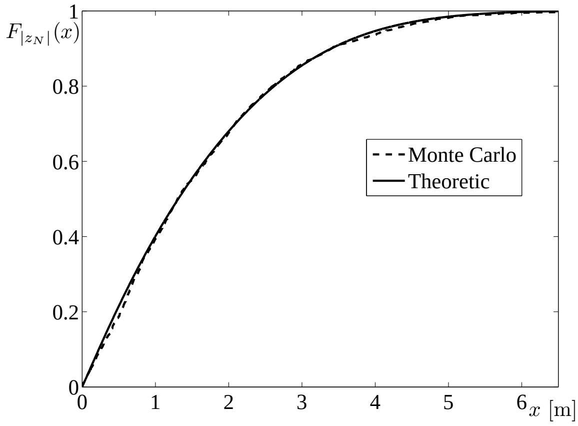

Fig. 5. Simulative and theoretic distribution functions of $|z_N|$ for bang-bang disturbance with random switch

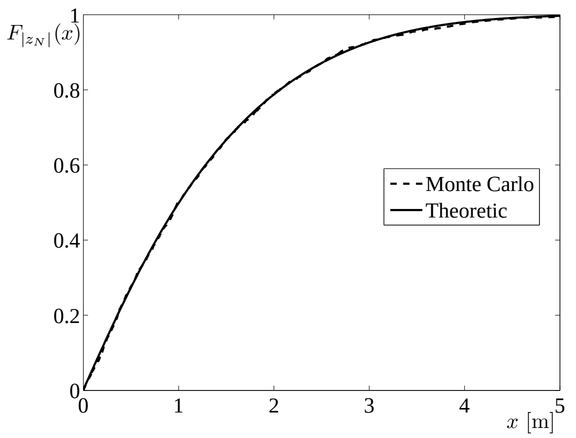

Fig. 6. Simulative and theoretic distribution functions of $|z_N|$ for random value disturbance

For all cases 2000 runs of Monte Carlo simulations were performed. In these simulations there was no estimator in the loop and the estimation errors η were the same as in the analytical expressions. It is seen that in all cases the two curves are very close.

V. Conclusions

The problem of evaluating the probability distribution of the final state of a scalar discrete-time system is solved. In this problem, it is assumed that the state information is corrupted by an error with known distribution and the initial state distribution is also known. Moreover, the system is subject to an additive random disturbance with known distribution. The control is realized by a saturated linear strategy. The formulation is motivated by various real life control problems, such as the interception problem, where validating robust transferring deterministic strategies in realistic stochastic environment is of a high practical importance.

The problem is mathematically nontrivial, because the evaluation of the sum of two dependent random variables is required. The solution is based on proper discretization of some conditional probabilities. The resulting formula allows to obtain the final state distribution without carrying out a great amount of Monte Carlo simulation runs.

The general result is used to compare the miss distance distribution in an interception problem with four different types of disturbance using a given set of estimation errors with the outcome of a large set of Monte Carlo simulations. The numerical examples confirm that the large number of Monte Carlo runs can be replaced by using analytic formulae.

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