Stochastic Symplectic Approximation for a Linear System with Additive Noises

Lijin Wang, Jialin Hong, and Rudolf Scherer

Abstract—**Two symplectic numerical integration methods, of mean-square order 1 and 2 respectively, for a linear stochastic oscillator with two additive noises are constructed via the stochastic generating function approach and investigated. They are shown by numerical tests to be efficient and superior to non-symplectic numerical methods.**

Index Terms—**stochastic numerical integration methods, mean-square order, stochastic Hamiltonian systems, symplectic methods, generating functions.**

I. Introduction

THE linear stochastic oscillator with two additive noises

$$\begin{aligned} dq(t) &= p(t)dt + \sigma dW_1(t), \quad q(0) = q_0, \\ dp(t) &= -q(t)dt + \gamma dW_2(t), \quad p(0) = p_0, \end{aligned} \tag{1}$$

where $p(t)$, $q(t)$ are scalar functions, σ, γ are constants, and $W_i(t)$ $(i = 1, 2)$ are independent standard Wiener processes, is a stochastic Hamiltonian system ([8]), for which the Hamiltonian functions are

$$H = \frac{1}{2}(p^2 + q^2), \quad H_1 = \sigma p, \quad H_2 = -\gamma q. \tag{2}$$

It is revealed that ([8]), the stochastic Hamiltonian systems with m noises

$$\begin{aligned} dp &= -\frac{\partial H}{\partial q}dt - \sum_{r=1}^{m} \frac{\partial H_r}{\partial q} \circ dW_r(t), \quad p(0) = p_0, \\ dq &= \frac{\partial H}{\partial p}dt + \sum_{r=1}^{m} \frac{\partial H_r}{\partial p} \circ dW_r(t), \quad q(0) = q_0, \end{aligned} \tag{3}$$

which will be reduced to the deterministic Hamiltonian systems if the diffusion coefficients vanish, possess the symplectic structure

$$dp(t) \wedge dq(t) = dp(0) \wedge dq(0), \quad \forall t \geq 0, \tag{4}$$

as their deterministic counterparts do. Note that the small circle $\circ$ before $dW_r(t)$ in (3) denotes stochastic differential equations of Stratonovich sense, and the differential d in (3) is taken with respect to time t, while that in (4) is taken in the phase space with respect to the initial phase point $(p(0), q(0))$.

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Numerical discretization methods for Hamiltonian systems that inherit the symplectic structure are called symplectic methods, characterized by

$$dp_{n+1} \wedge dq_{n+1} = dp_n \wedge dq_n, \quad \forall n \geq 0, \quad n \in \mathbb{Z}. \tag{5}$$

Such methods are shown to be superior to non-symplectic ones in simulating both the deterministic ([1],[10],[11] etc.) and the stochastic ([8],[9] etc.) Hamiltonian systems, especially over long time intervals, probably owing to their preservation of the qualitative property, the symplecticity of the underlying continuous differential equation systems. Stochastic symplectic methods, however, are far less developed than deterministic ones for which one can refer to [3] and references therein, where [2] is a pioneering work on deterministic generating function theory.

In the present paper, we construct two symplectic methods for the linear stochastic oscillator (1) via the stochastic generating function approach proposed in [5] and [12], the theory of which is based on the fact that, the phase flow $\varphi_t : (p(0), q(0)) \mapsto (p(t), q(t))$ of the stochastic Hamiltonian systems, which is a symplectic mapping for every $t > 0$ almost surely ([8]), can be generated by certain generating functions. As an argument of the theory, the generating function that generates the true solution of (1) is given. Meanwhile, we give a proof of the mean-square orders of the constructed methods, based on the empirical analysis in [13]. Numerical experiments testify efficiency, mean-square orders of the generated numerical methods, and compare them with the non-symplectic Euler-Maruyama method.

Section II constructs two numerical methods for the system (1) via the stochastic generating function approach, proves their symplecticity, mean-square orders, and gives the generating function that produces the true solution of (1). Section III are numerical experiments testing the behavior of the numerical methods. The conclusion is in Section IV.

II. Symplectic Methods for the Stochastic Oscillator

Applying the stochastic generating function theory given in [5] and [12] to the stochastic Hamiltonian system (1), the first kind of generating function $S^1(p_{n+1}, q_n, h)$ for this

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system should have the form of series expansion

$$\begin{aligned} S^1(p_{n+1}, q_n, h) &= \sigma p_{n+1}\Delta_n W_1 - \gamma q_n \Delta_n W_2 \\ &+ \frac{h}{2}(p_{n+1}^2 + q_n^2) \\ &- \sigma\gamma \int_{t_n}^{t_{n+1}} (W_1(s) - W_1(t_n)) \circ dW_2(s) \\ &- \gamma p_{n+1} \int_{t_n}^{t_{n+1}} (s - t_n) dW_2(s) \\ &+ \sigma q_n \int_{t_n}^{t_{n+1}} (W_1(s) - W_1(t_n)) ds \\ &+ \frac{h^2}{2} p_{n+1} q_n + \cdots, \end{aligned} \tag{6}$$

where $h =: t_{n+1} - t_n$ is the time step-size, $\Delta_n W_i =: W_i(t_{n+1}) - W_i(t_n)$, $(i = 1, 2)$, and this kind of generating function can create the symplectic mapping $(p_n, q_n) \mapsto (p_{n+1}, q_{n+1})$ via the relations

$$p_n = p_{n+1} + \frac{\partial S^1}{\partial q_n}, \quad q_{n+1} = q_n + \frac{\partial S^1}{\partial p_{n+1}}. \tag{7}$$

Approximating S^1 by truncating the series after the fourth term, i.e. the term in the third line of (6), and using the relations (7) produces the scheme

$$\begin{aligned} p_{n+1} &= p_n + \gamma \Delta_n W_2 - h q_n \\ q_{n+1} &= q_n + \sigma \Delta_n W_1 + h p_{n+1}, \end{aligned} \tag{8}$$

which is the symplectic Euler-Maruyama method given in [8], but here reattained via the generating function approach.

Truncating the series of S^1 after the seventh term, i.e. the term in the sixth line of (6), with application of the relations (7), gives the following scheme

$$\begin{aligned} p_{n+1} &= p_n + \gamma \Delta_n W_2 - h q_n \\ &\quad - \sigma \int_{t_n}^{t_{n+1}} (W_1(s) - W_1(t_n)) ds - \frac{h^2}{2} p_{n+1} \\ q_{n+1} &= q_n + \sigma \Delta_n W_1 + h p_{n+1} \\ &\quad - \gamma \int_{t_n}^{t_{n+1}} (s - t_n) dW_2(s) + \frac{h^2}{2} q_n, \end{aligned} \tag{9}$$

which is a new scheme that contains some additional higher order terms than (8).

In principle, methods of higher mean-square order can be obtained by involving sequentially and appropriately more terms of the series into the truncated S^1.

For any $t \in [0, T]$, $h > 0$ such that $t + h \leq T$, if we denote $X(t + h) = (Q, P)^T$, $X(t) = (q, p)^T$, then the true solution of (1) on the domain $t \in [0, T]$ can be expressed as ([8])

$$\begin{aligned} Q &= q \cos h + p \sin h + u_1(t), \\ P &= -q \sin h + p \cos h + u_2(t), \end{aligned} \tag{10}$$

with $X(0) = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$, and

$$\begin{aligned} u_1(t) &= \sigma \int_t^{t+h} \cos(t + h - s) dW_1(s) \\ &\quad + \gamma \int_t^{t+h} \sin(t + h - s) dW_2(s) \\ u_2(t) &= -\sigma \int_t^{t+h} \sin(t + h - s) dW_1(s) \\ &\quad + \gamma \int_t^{t+h} \cos(t + h - s) dW_2(s). \end{aligned} \tag{11}$$

Proposition 1. The mean-square order of (8) is 1, and that of (9) is 2.

Proof. We only give the proof for scheme (9), since that of (8) is similar.

Denote with $(P_n, Q_n)^T =: Y_n$ the true solution of the oscillator at time t_n, with $(p_{n+1}, q_{n+1})^T =: y_{n+1}$ the one-step approximation resulted from the numerical scheme (9) starting from (P_n, Q_n) with time step-size h, and with $(P_{n+1}, Q_{n+1})^T =: Y_{n+1}$ the true solution at time $t_{n+1} = t_n + h$, calculated from the iteration formula (10) with $t = t_n$.

We have

$$\begin{aligned} &y_{n+1} - Y_{n+1} \\ &= \begin{pmatrix} (\frac{2}{2+h^2} - \cos h) P_n + (\frac{-2h}{2+h^2} + \sin h) Q_n + R_1 \\ (\frac{2h}{2+h^2} - \sin h) P_n + (\frac{4+h^4}{4+2h^2} - \cos h) Q_n + R_2 \end{pmatrix} \end{aligned} \tag{12}$$

with

$$\begin{aligned} R_1 &= \frac{2}{2 + h^2} \left(\gamma \Delta_n W_2 - \sigma \int_{t_n}^{t_n + h} (W_1(s) - W_1(t_n)) ds \right) \\ &\quad - u_2(t_n) \\ R_2 &= \frac{2h}{2 + h^2} \left(\gamma \Delta_n W_2 - \sigma \int_{t_n}^{t_n + h} (W_1(s) - W_1(t_n)) ds \right) \\ &\quad + \sigma \Delta_n W_1 - \gamma \int_{t_n}^{t_n + h} (s - t_n) dW_2(s) - u_1(t_n) \end{aligned} \tag{13}$$

Note that $\mathbf{E}(u_i) = 0$, $\mathbf{E}(\Delta_n W_i) = 0$, $i = 1, 2$, and that all the integrals in R_1 and R_2 have zero expectations, we have $\mathbf{E}(R_i) = 0$ for $i = 1, 2$. Thus

$$\begin{aligned} &|\mathbf{E}(y_{n+1} - Y_{n+1})|^2 \\ &= \left[(\frac{2}{2 + h^2} - \cos h) \mathbf{E} P_n + (\frac{-2h}{2 + h^2} + \sin h) \mathbf{E} Q_n \right]^2 \\ &+ \left[(\frac{2h}{2 + h^2} - \sin h) \mathbf{E} P_n + (\frac{4 + h^4}{4 + 2h^2} - \cos h) \mathbf{E} Q_n \right]^2. \end{aligned} \tag{14}$$

A straight-forward calculation with using the Taylor expansions

$$\begin{aligned} \sin h &= h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots, \\ \cos h &= 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots \end{aligned} \tag{15}$$

yields

$$\begin{aligned}
&|\mathbf{E}(y_{n+1} - Y_{n+1})|^2 \\
&= \frac{4}{9}\frac{h^6}{(2+h^2)^2}((\mathbf{E}P_n)^2 + (\mathbf{E}Q_n)^2) + O(h^7).
\end{aligned} \tag{16}$$

Now we calculate $\mathbf{E}|y_{n+1} - Y_{n+1}|^2$. Note that for $i = 1, 2$, $\mathbf{E}(\Delta_n W_i)^2 = h$, and that W_1 and W_2 are independent. Using the definitions and properties of W and Itô stochastic integrals, we can calculate that

$$\begin{aligned}
&\mathbf{E}\left(\int_{t_n}^{t_n+h}(W_1(s) - W_1(t_n))ds\right)^2 = \frac{h^3}{3}, \\
&\mathbf{E}(u_1(t_n)^2 + u_2(t_n)^2) = (\sigma^2+\gamma^2)h, \\
&\mathbf{E}(\Delta_n W_2 \cdot u_2) = \gamma \sin h, \\
&\mathbf{E}(\Delta_n W_1 \cdot u_1) = \sigma \sin h, \\
&\mathbf{E}(\Delta_n W_2 \cdot u_1) = \gamma(1 - \cos h), \\
&\mathbf{E}\left(\int_{t_n}^{t_n+h}(W_1(s) - W_1(t_n))ds \cdot u_2\right) \\
&= \sigma(h\cos h - \sin h), \\
&\mathbf{E}\left(\int_{t_n}^{t_n+h}(W_1(s) - W_1(t_n))ds \cdot u_1\right) \\
&= \sigma(h \sin h + \cos h - 1), \\
&\mathbf{E}\left(\int_{t_n}^{t_n+h}(s-t_n)dW_2(s) \cdot u_1\right) = \gamma(h - \sin h), \\
&\mathbf{E}\left(\int_{t_n}^{t_n+h}(s-t_n)dW_2(s)\right)^2 = \frac{h^3}{3}, \\
&\mathbf{E}\left(\Delta_n W_2 \cdot \int_{t_n}^{t_n+h}(s-t_n)dW_2(s)\right) = \frac{h^2}{2}, \\
&\mathbf{E}\left(\Delta_n W_1 \cdot \int_{t_n}^{t_n+h}(W_1(s) - W_1(t_n))ds\right) = \frac{h^2}{2}.
\end{aligned} \tag{17}$$

Using these results, together with (14)-(16), also noticing that (P_n, Q_n) are independent to R_i $(i = 1, 2)$, we obtain

$$\mathbf{E}|y_{n+1} - Y_{n+1}|^2 = \frac{8}{15}\frac{\gamma^2+\sigma^2}{(2+h^2)^2}h^5 + O(h^6). \tag{18}$$

The results (16) and (18) implies that, the mean-square order of the scheme (9) is 2 (see THEOREM 1.1 in [7]). This ends the proof. □

Proposition 2. The numerical methods (8) and (9) for the stochastic oscillator (1) are both symplectic.

Proof. Symplecticity of (8) is already assured by [8], therefore we only prove that for (9). In fact, the proof for (8) can just follow the same way.

Rewrite (9) into the following convenient form

$$\begin{aligned}
p_{n+1} &= \frac{2}{2+h^2}p_n - \frac{2h}{2+h^2}q_n + \frac{2}{2+h^2}\beta_1, \\
q_{n+1} &= \frac{2h}{2+h^2}p_n + \frac{4+h^4}{2(2+h^2)}q_n + \frac{2}{2+h^2}\beta_2,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\beta_1 &= \gamma\Delta_n W_2 - \sigma\int_{t_n}^{t_{n+1}}(W_1(s) - W_1(t_n))ds, \\
\beta_2 &= h\beta_1 + (1 + \frac{h^2}{2})\alpha_1,
\end{aligned} \tag{20}$$

with

$$\alpha_1 = \sigma\Delta_n W_1 - \gamma\int_{t_n}^{t_{n+1}}(s-t_n)dW_2(s). \tag{21}$$

Consequently,

$$\begin{aligned}
dp_{n+1} \wedge dq_{n+1} &= \left(\frac{4+h^4}{(2+h^2)^2} + \frac{4h^2}{(2+h^2)^2}\right)dp_n \wedge dq_n \\
&= dp_n \wedge dq_n.
\end{aligned} \tag{22}$$

□

It is not difficult to check that, the true solution (10) is a symplectic mapping $(p, q) \mapsto (P, Q)$ which can be generated by the function

$$\begin{aligned}
S(q, Q, h) &= (Q - u_1)(u_2 - q\csc h) \\
&\quad + \frac{1}{2}(q^2 + (Q-u_1)^2)\cot h
\end{aligned} \tag{23}$$

via the relations

$$p = -\frac{\partial S}{\partial q}, \quad P = \frac{\partial S}{\partial Q}. \tag{24}$$

Note that S and S^1 are two different kinds of generating functions with different assignments of independent variables. Actually, they can be transformed to each other through coordinate transformation (see e.g. [2], [3], [12]), and each of them satisfies a corresponding stochastic Hamilton-Jacobi PDE, by solving which the series expansion of them such as (6) can be obtained. For example, given the stochastic Hamiltonian system (3), the stochastic Hamilton-Jacobi PDE for $S^1(P, q, t)$ should be

$$\begin{aligned}
&dS_t^1 = H(P, q + \frac{\partial S^1}{\partial P})dt + \sum_{r=1}^{m} H_r(P, q + \frac{\partial S^1}{\partial P}) \circ dW_r(t), \\
&S^1(P, q, 0) = 0,
\end{aligned} \tag{25}$$

the solution of which is assumed to be of the form

$$\begin{aligned}
S^1(P, q, t) &= \sum_{r=1}^{m} G_r(P, q)I_r + G_0(P, q)I_0 \\
&\quad + \sum_{v=1}^{m}\sum_{r=1}^{m} G_{v,r}(P, q)I_{v,r} \\
&\quad + \sum_{i=1}^{m}\sum_{v=1}^{m}\sum_{r=1}^{m} G_{i,v,r}(P, q)I_{i,v,r} \\
&\quad + \sum_{r=1}^{m} G_{0,r}(P, q)I_{0,r} + \sum_{r=1}^{m} G_{r,0}(P, q)I_{r,0} \\
&\quad + G_{0,0}(P, q)I_{0,0} \\
&\quad + \cdots,
\end{aligned} \tag{26}$$

where

$$I_{i_1,\cdots,i_j} = \int_0^t \int_0^{u_1} \cdots \int_0^{u_{j-1}} \circ dW_{i_1}(u_j) \cdots \circ dW_{i_j}(u_1), \tag{27}$$

in which $j \geq 1$, $j \in \mathcal{Z}_+$, i_k $(k = 1, \cdots j)$ takes value from the set $\{0, 1, \cdots, m\}$, and if $i_k = 0$, $dW_{i_k}(u_s) := du_s$. The functions containing the character G are to be determined by substituting the series (26) into the equation (25). The series S^1 in (6) is just obtained in this way.

More details about the stochastic generating function theory were given in [5] and [12].

Proposition 3. The second moment of the solution of (1) grows linearly with respect to time t, that is,

$$\mathbf{E}(p(t)^2 + q(t)^2) = \mathbf{E}(p_0^2 + q_0^2) + (\sigma^2 + \gamma^2)t. \tag{28}$$

Proof. A straightforward calculation of the second moment on the true solution (10) yields

$$\mathbf{E}(P^2 + Q^2) = \mathbf{E}(p^2 + q^2) + (\sigma^2 + \gamma^2)h, \tag{29}$$

which is equivalent to (28) by assigning $t = 0$ and substituting the notation h by t in (10). □

In the next section, we use the linear growth property (28) as a criterion of evaluating the numerical methods.

III. Numerical Tests

To compare the symplectic methods with non-symplectic ones, we take the non-symplectic Euler-Maruyama method applied to (1) as an example, which reads

$$\begin{aligned} p_{n+1} &= p_n + \gamma \Delta_n W_2 - hq_n \\ q_{n+1} &= q_n + \sigma \Delta_n W_1 + hp_n. \end{aligned} \tag{30}$$

For the implementation methods of $\Delta_n W_i$ $(i = 1, 2)$ and the stochastic integrals in (8) and (9), refer to e.g. [4], [6] and [7].

The numerical tests examine the behavior of the numerical methods from three aspects: first, closeness between the oscillation curves produced by the numerical (q_n) and the true solution $(q(t_n))$, to which Fig. 1, 2, and 3 are contributed; second, ability of preserving the linear growth property (28), as shown by Fig. 4, 5, and 6; and third, the empirical mean-square order of the methods illustrated by Fig. 7 and 8.

Both Fig. 1 and 2, produced by the methods (8) and (9) respectively, exhibit good coincidence between the numerical (blue dotted) and the true solution (red solid) curves, while obviously larger and larger deviation of the numerical curve created by the Euler-Maruyama method (30) from the true solution is observed in Fig. 3, which indicates the effectiveness of the symplectic methods (8) and (9), as well as the invalidity of the non-symplectic Euler-Maruyama method in solving the stochastic oscillator.

Fig. 4, 5, and 6 show the evolution of the numerical second moment $\mathbf{E}(p_n^2 + q_n^2)$ (blue solid) by the methods (8), (9) and (30), respectively, compared with the reference line (red dotted) indicating the theoretical path of the linear growth, from which it can be seen that the method (9) preserves the linear growth property (28) more accurately than (8), though both of them behave fairly well in this aspect. The Euler-Maruyama method, however, fails to reproduce the linear growth of the second moment. The expectation $\mathbf{E}$ in these tests is approximated by taking average over 500 sample solutions.

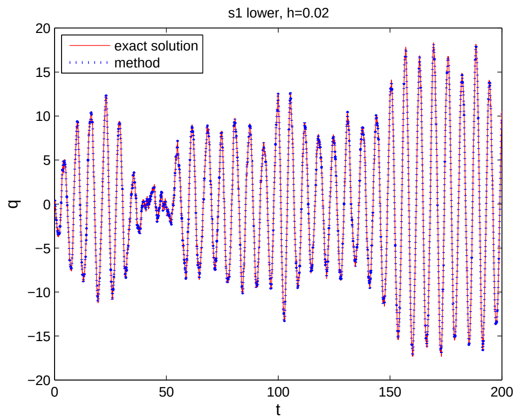

Fig. 1. A Sample Trajectory arising from the Numerical Method (8) (blue dotted) and the True Solution (10) (red solid)

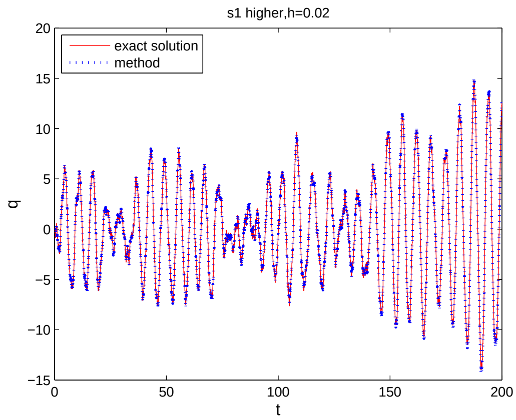

Fig. 2. A Sample Trajectory arising from the Numerical Method (9) (blue dotted) and the True Solution (10) (red solid)

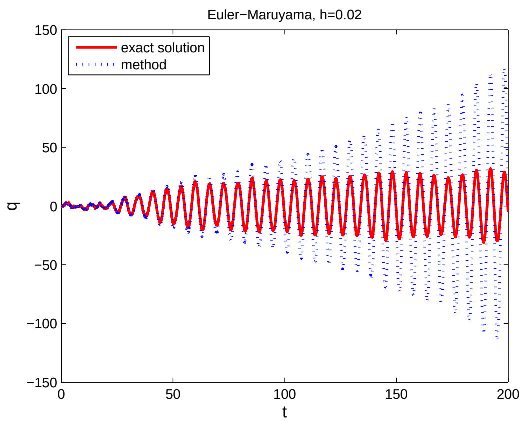

Fig. 3. A Sample Trajectory arising from the Euler-Maruyama Method (30) (blue dotted) and the True Solution (10) (red solid)

The data for the tests are: $(p_0, q_0) = (0, 0)$, $\sigma = \gamma = 1$, $t \in [0, 200]$, and the time step-size $h = 0.02$.

The log-log plot between the step-sizes h and the corresponding mean-square error at $t = 200$, i.e. $\left[\mathbf{E}[(p_N - p(200))^2 + (q_N - q(200))^2]\right]^{\frac{1}{2}}$, where $N = 200/h$, arising from the numerical schemes (8) and (9) are exhibited in Fig. 7 and 8 respectively. Five different values of h, i.e. 0.01, 0.02, 0.05, 0.1, 0.2 are chosen for the test, corresponding to the five circle markers on the blue solid lines. The red dotted straight lines are of slope 1 in Fig. 7 and 2 in Fig. 8. It is indicated by the parallelism between

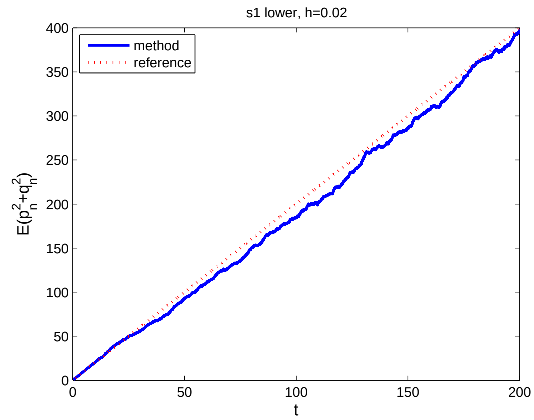

Fig. 4. Evolution of the Sample Average (over 500 samples) of $p_n^2 + q_n^2$ by the Numerical Method (8) (blue solid) and the Exact Second Moment (red dotted)

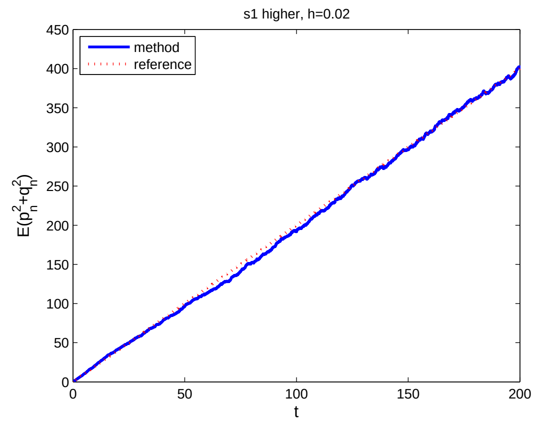

Fig. 5. Evolution of the Sample Average (over 500 samples) of $p_n^2 + q_n^2$ by the Numerical Method (9) (blue solid) and the Exact Second Moment (red dotted)

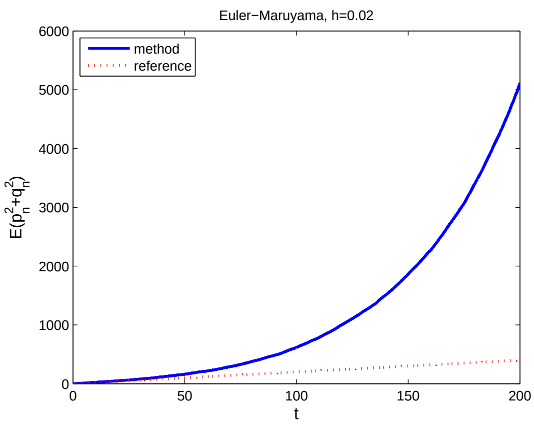

Fig. 6. Evolution of the Sample Average (over 500 samples) of $p_n^2 + q_n^2$ by the Euler-Maruyama Method (30) (blue solid) and the Exact Second Moment (red dotted)

the numerical plot and the corresponding reference lines that, the mean-square order of (8) is 1, and that of (9) is 2, which coincides with the theoretical results in Proposition 1.

IV. Conclusion

Symplectic methods are very important in the simulation of stochastic Hamiltonian systems, especially in long time simulation problems. The paper applies the stochastic generating function approach, which is a systematic way of constructing symplectic scheme for stochastic Hamiltonian systems, to a concrete stochastic Hamiltonian system, the linear stochastic oscillator (1), to build symplectic schemes for it, which might serve as a demonstration of the application of the stochastic generating function approach. Although only two schemes are given here, many others, in fact, can be produced, by different truncations of the same generating function series, or different choices of generating functions, such as S, S^2 or S^3 ([12]). Construction and implementation of symplectic schemes with even higher orders, however, are still subject to further investigation.

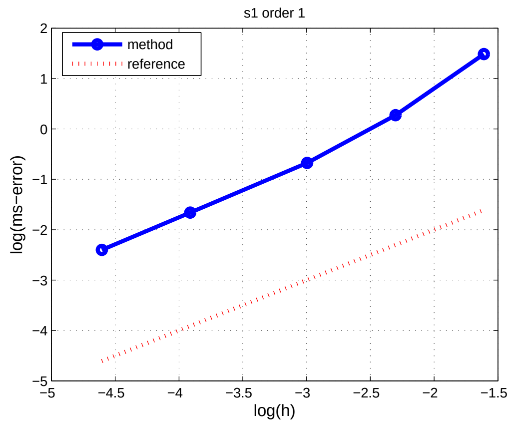

Fig. 7. Logarithm of the Mean-Square Error at Time $t = 200$ by the Numerical Method (8), versus the Logarithm of the time step-size h, for h=0.01, 0.02, 0.05, 0.1, 0.2 (blue solid), and the Reference Line of Slope 1 (red dotted)

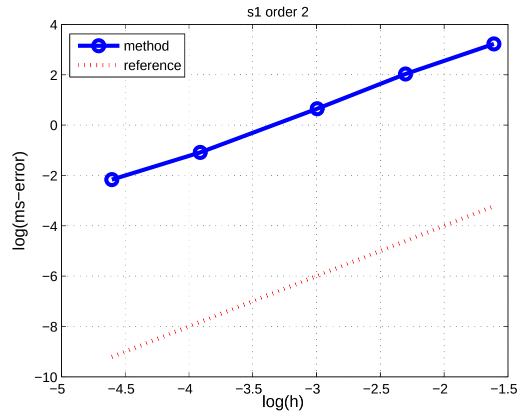

Fig. 8. Logarithm of the Mean-Square Error at Time $t = 200$ by the Numerical Method (9), versus the Logarithm of the time step-size h, for h=0.01, 0.02, 0.05, 0.1, 0.2 (blue solid), and the Reference Line of Slope 2 (red dotted)

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