Green's Relations in Ordered Gamma-Semigroups in Terms of Fuzzy Subsets

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***Abstract*—In this paper, we characterize the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ defined in terms of ordered filters on ordered Γ-semigroups in terms of fuzzy subsets. For an ordered Γ-semigroup we define relations $\mathcal{R}^F, \mathcal{L}^F, \mathcal{I}^F, \mathcal{N}^F$ in terms of fuzzy subsets and we prove that $\mathcal{R}^F = \mathcal{R}, \mathcal{L}^F = \mathcal{L}, \mathcal{I}^F = \mathcal{I}, \mathcal{N}^F = \mathcal{N}$.**

***Index Terms*—ordered Γ-semigroup, Green's relations, fuzzy subset, ordered filter.**

I. Introduction and Prerequisites

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [20] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [20], several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: In 1971, Rosenfeld [15] was the first who studied fuzzy sets in the structure of groups. Fuzzy semigroups have been first considered by Kuroki [7], [8], [9], [10], and fuzzy ordered groupoids and ordered semigroups, by Kehayopulu and Tsingelis [4], [5]. In 2007, Kehayopulu and Tsingelis [6] characterized the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ of ordered groupoids in terms of fuzzy subsets. In 2008, Shabir and Khan [18] defined fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterized ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. In 2009, Majumder and Sardar [12] studied fuzzy ideals and fuzzy ideal extensions in ordered semigroups. In 2010, Chinram and Malee [1] investigated some properties of L-fuzzy ternary subsemiring and L-fuzzy ideals in ternary semirings. In 2010, Chinram and Saelee [2] studied fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups. In 2010, Shah and Rehman [19] introduced Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids which are in fact a generalization of ideals and bi-ideals of AG-groupoids and studied some characteristics of Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids. In 2010, Iampan [3] characterized the relationship between the fuzzy ordered ideals (fuzzy ordered filters) and the characteristic mappings of fuzzy ordered ideals (fuzzy ordered filters) in ordered Γ-semigroups. In 2011, Saelee and Chinram [13] studied rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups.

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The Green's relations defined in terms of ordered right ideals, ordered left ideals, ordered ideals and the concept of fuzzy sets play an important role in studying the structure of ordered semigroups. Now, we characterize the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ defined in terms of ordered filters on ordered Γ-semigroups in terms of fuzzy subsets. To present the main theorems we discuss some elementary definitions that we use later.

Definition I.1. Let M be a set. A *fuzzy subset* of M is an arbitrary mapping $f\colon M \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line.

Definition I.2. Let M be a set and $A \subseteq M$. The *characteristic mapping* $f_A\colon M \to [0, 1]$ defined via

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic mapping, f_A is a mapping of M into $\{0, 1\} \subset [0, 1]$. Hence f_A is a fuzzy subset of M.

Definition I.3. Let M and Γ be any two nonempty sets. Then (M, Γ) is called a Γ*-semigroup* [17] if there exists a mappings $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$. A nonempty subset K of M is called a Γ*-subsemigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Definition I.4. A partially ordered Γ-semigroup $(M, \Gamma, \leq)$ is called an *ordered* Γ*-semigroup* [16] if for any $a, b, c \in M$ and $\gamma \in \Gamma, a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Definition I.5. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. For $H \subseteq M$, we define

$$(H] = \{t \in M \mid t \leq h \text{ for some } h \in H\}.$$

Definition I.6. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. An equivalence relation σ on M is called a *congruence* on M if $(a, b) \in \sigma$ implies $(a\gamma c, b\gamma c) \in \sigma$ and $(c\gamma a, c\gamma b) \in \sigma$ for all $a, b, c \in M$ and $\gamma \in \Gamma$. A congruence σ on M is called a *semilattice congruence* on M if $(a\gamma a, a) \in \sigma$ and $(a\gamma b, b\gamma a) \in \sigma$ for all $a, b \in M$ and $\gamma \in \Gamma$. A semilattice congruence σ on M is called *complete* if $a \leq b$ implies $(a, a\gamma b) \in \sigma$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition I.7. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset A of M is called an *ordered left ideal* of M if

L1. $M\Gamma A \subseteq A$.
L2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

A nonempty subset A of M is called an *ordered right ideal* of M if

R1. $A\Gamma M \subseteq A$.
R2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

A nonempty subset A of M is called an *ordered ideal* of M if it is both an ordered left and an ordered right ideal of M. That is,

I1. $M\Gamma A \subseteq A$ and $A\Gamma M \subseteq A$.
I2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

We denote by $R(x), L(x), I(x)$ the ordered right ideal, ordered left ideal, ordered ideal of M, respectively, generated by x. For each $x \in M$, we have

$$R(x) = (x \cup x\Gamma M], L(x) = (x \cup M\Gamma x], I(x) = (x \cup x\Gamma M \cup M\Gamma x \cup M\Gamma x\Gamma M] \text{ [11]}.$$

Definition I.8. The Greens relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ are the equivalence relations on ordered Γ-semigroup M, defined as follows:

$$\begin{aligned} \mathcal{R} &:= \{(x, y) \mid R(x) = R(y)\}, \\ \mathcal{L} &:= \{(x, y) \mid L(x) = L(y)\}, \\ \mathcal{I} &:= \{(x, y) \mid I(x) = I(y)\}. \end{aligned}$$

Definition I.9. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A Γ-subsemigroup F of M is called an *ordered filter* of M if

F1. For any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $a, b \in F$.
F2. For any $b \in M$ and $a \in F, a \leq b$ implies $b \in F$.

For an element $x \in M$, we denote by $N(x)$ the ordered filter of M generated by x, and by $\mathcal{N}$ the relation on M defined by

$$\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}.$$

It has been proved in [14] that $\mathcal{N}$ is a semilattice congruence on M, in particular, $\mathcal{N}$ is an ordered semilattice congruence [14], and it is the least ordered semilattice congruence on M [14].

We can easily prove that for two subsets A, B of ordered Γ-semigroup M, we have $A \subseteq B$ if and only if $f_A \subseteq f_B$ and $A = B$ if and only if $f_A = f_B$. We denote by $F(M)$ the set of all fuzzy subsets of M. We define an order relation "$\subseteq$" on $F(M)$ as follows: For $f, g \in F(M)$, we define $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in M$. Clearly, $(F(M), \subseteq)$ is an ordered set. Let $0_F, 1_F$ be the fuzzy subsets of M defined by: $0_F \colon M \to [0,1] \mid x \mapsto 0_F(x) := 0$ and $1_F \colon M \to [0,1] \mid x \mapsto 1_F(x) := 1$. It is clear that the mapping 0_F is the least element of $F(M)$ and the mapping 1_F is the greatest element $F(M)$.

Definition I.10. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of M is called a *fuzzy ordered left ideal* of M if

FL1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.
FL2. $f(a\gamma b) \geq f(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered right ideal* of M if

FR1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.
FR2. $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered ideal* of M if it is both a fuzzy ordered left and a fuzzy ordered right ideal of M. That is,

FI1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.
FI2. $f(a\gamma b) \geq f(b)$ and $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition I.11. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of M is called a *fuzzy Γ-subsemigroup* of M if $f(a\gamma b) \geq \min\{f(x), f(y)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition I.12. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy subset f of M is called a *fuzzy ordered filter* of M if

FF1. For any $a, b \in M, a \leq b$ implies $f(a) \leq f(b)$.
FF2. $f(a\gamma b) = \min\{f(x), f(y)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

II. Main Results

Definition II.1. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. We define

$$\bigwedge_{i \in I} f_i \colon M \to [0,1] \mid x \mapsto (\bigwedge_{i \in I} f_i)(x) := \inf\{f_i(x) \mid i \in I\}$$

and

$$\bigvee_{i \in I} f_i \colon M \to [0,1] \mid x \mapsto (\bigvee_{i \in I} f_i)(x) := \sup\{f_i(x) \mid i \in I\}.$$

Proposition II.2. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. Then*

$$\bigwedge_{i \in I} f_i \in F(M) \text{ and } \bigvee_{i \in I} f_i \in F(M).$$

Proof: Let $x \in M$. The set $\{f_i(x) \mid i \in I\}$ is a nonempty lower bounded subset of $\mathbb{R}$, so there exists the $\inf\{f_i(x) \mid i \in I\}$ in $\mathbb{R}$. Since $0 \leq f_i(x) \leq 1$ for each $i \in I$, we have $0 \leq \inf\{f_i(x) \mid i \in I\} \leq 1$. If $x, y \in M$ is such that $x = y$, then clearly $(\bigwedge_{i \in I} f_i)(x) = (\bigwedge_{i \in I} f_i)(y)$. Hence $\bigwedge_{i \in I} f_i \in F(M)$. Similarly, $\bigvee_{i \in I} f_i \in F(M)$. ■

Remark II.3. By Proposition II.2, we have $(F(M), \subseteq)$ is a complete lattice.

Remark II.4. The following two statements hold true:

(i) $\bigwedge_{i \in I} f_i \subseteq f_j$ for all $j \in I$,
(ii) $f_j \subseteq \bigvee_{i \in I} f_i$ for all $j \in I$.

Proposition II.5. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. Then we have the following:*

(i) $\bigwedge_{i \in I} f_i = \inf\{f_i \mid i \in I\}$,
(ii) $\bigvee_{i \in I} f_i = \sup\{f_i \mid i \in I\}$.

Proof: (i) By Proposition II.2, $\bigwedge_{i \in I} f_i \in F(M)$ and by Remark II.4, $\bigwedge_{i \in I} f_i \subseteq f_i$ for all $i \in I$. Thus $\bigwedge_{i \in I} f_i$ is a lower bound of $\{f_i \mid i \in I\}$. Let $g \in F(M)$ be such that $g \subseteq f_i$ for all $i \in I$. Then $g(x) \leq f_i(x)$ for all $x \in M$ and $i \in I$, so $g(x)$ is a lower bound of $\{f_i(x) \mid i \in I\}$ for all $x \in M$. By the proof of Proposition II.2, we have

$$g(x) \leq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(x) \text{ for all } x \in M.$$

Thus $g \subseteq \bigwedge_{i\in I} f_i$, so $\bigwedge_{i\in I} f_i = \inf\{f_i \mid i \in I\}$. The proof of (ii) is similar. ■

Proposition II.6. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. Then we have the following:*

(i) *If f_i is a fuzzy ordered right ideal of M for all $i \in I$, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy ordered right ideals of M.*
(ii) *If f_i is a fuzzy ordered left ideal of M for all $i \in I$, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy ordered left ideals of M.*
(iii) *If f_i is a fuzzy ordered ideal of M for all $i \in I$, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy ordered ideals of M.*
(iv) *If f_i is a fuzzy Γ-subsemigroup of M for all $i \in I$, then $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M.*
(v) *If f_i is a fuzzy ordered filter of M for all $i \in I$, then $\bigwedge_{i\in I} f_i$ is a fuzzy ordered filter of M.*

Proof: (i) By Proposition II.2, $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Let $x, y \in M$ and $\gamma \in \Gamma$. Since f_i is a fuzzy ordered right ideal of M, we have $f_i(x\gamma y) \geq f_i(x)$ for all $i \in I$. Then

$$(\bigwedge_{i\in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \geq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(x).$$

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy ordered right ideal of M, we have $f_i(x) \geq f_i(y)$ for all $i \in I$. Then

$$(\bigwedge_{i\in I} f_i)(x) := \inf\{f_i(x) \mid i \in I\} \geq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(y).$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy ordered right ideals of M. Similarly, $\bigvee_{i\in I} f_i$ is a fuzzy ordered right ideals of M.
(ii) It is proved similarly.
(iii) It follows from (i) and (ii).
(iv) Let $x, y \in M$ and $\gamma \in \Gamma$. By Proposition II.2, $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Since f_i is a fuzzy Γ-subsemigroup of M, we have $f_i(x\gamma y) \geq \min\{f_i(x), f_i(y)\}$ for all $i \in I$. Then $f_i(x\gamma y) \geq f_i(x)$ or $f_i(x\gamma y) \geq f_i(y)$ for all $i \in I$. Since $f_i(x) \geq \inf\{f_i(x) \mid i \in I\}$ and $f_i(y) \geq \inf\{f_i(y) \mid i \in I\}$, we have $f_i(x\gamma y) \geq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(x)$ or $f_i(x\gamma y) \geq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(y)$ for all $i \in I$. Thus $f_i(x\gamma y) \geq \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}$ for all $i \in I$. Hence

$$(\bigwedge_{i\in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \geq \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}.$$

Therefore $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M.
(v) Let $x, y \in M$ and $\gamma \in \Gamma$. By Proposition II.2, $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Since f_i is a fuzzy ordered filter of M, we have $f_i(x\gamma y) = \min\{f_i(x), f_i(y)\}$ for all $i \in I$. Then $f_i(x\gamma y) = \min\{f_i(x), f_i(y)\} \leq f_i(x), f_i()y$ for all $i \in I$. Thus $(\bigwedge_{i\in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \leq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(x)$ and $(\bigwedge_{i\in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \leq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(y)$. Hence

$$(\bigwedge_{i\in I} f_i)(x\gamma y) \leq \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}.$$

On the other hand, by (iv), the mapping $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M, which means that $(\bigwedge_{i\in I} f_i)(x\gamma y) \geq \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}$. Thus

$$(\bigwedge_{i\in I} f_i)(x\gamma y) = \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}.$$

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy ordered filter of M, we have $f_i(x) \leq f_i(y)$ for all $i \in I$. Then

$$(\bigwedge_{i\in I} f_i)(x) := \inf\{f_i(x) \mid i \in I\} \leq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(y).$$

Therefore $\bigwedge_{i\in I} f_i$ is a fuzzy ordered filter of M. ■

Remark II.7. Clearly, $1_F \in R_f, 1_F \in L_f, 1_F \in I_f$ and $1_F \in N_f$. Since R_f, L_f, I_f and N_f are nonempty subsets of $F(M)$, by Proposition II.2, the elements $\bigwedge_{g\in R_f} g$, $\bigwedge_{g\in L_f} g$, $\bigwedge_{g\in I_f} g$ and $\bigwedge_{g\in N_f} g$ exist in $F(M)$. Moreover, by Proposition II.6, we have the following:

(i) $\bigwedge_{g\in R_f} g$ is a fuzzy ordered right ideal of M,
(ii) $\bigwedge_{g\in L_f} g$ is a fuzzy ordered left ideal of M,
(iii) $\bigwedge_{g\in I_f} g$ is a fuzzy ordered ideal of M,
(iv) $\bigwedge_{g\in N_f} g$ is a fuzzy ordered filter of M.

Proposition II.8. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and f a fuzzy subsets of M. Then we have the following:*

(i) $\bigwedge_{g\in R_f} g \in R_f$ *(resp.* $\bigwedge_{g\in L_f} g \in L_f$, $\bigwedge_{g\in I_f} g \in I_f$ *and* $\bigwedge_{g\in N_f} g \in N_f$*),*
(ii) $\bigwedge_{g\in R_f} g \subseteq h$ *(resp.* $\bigwedge_{g\in L_f} g \subseteq h$, $\bigwedge_{g\in I_f} g \subseteq h$ *and* $\bigwedge_{g\in N_f} g \subseteq h$*) for all* $h \in R_f$ *(resp.* $h \in L_f, I_f, N_f$*).*

Proof: (i) By Remark II.7, the element $\bigwedge_{g\in R_f} g$ is a fuzzy ordered right ideal of M. Let $x \in M$ and $g \in R_f$. Since $g \supseteq f$, we have $f(x) \le g(x)$. Thus $f(x) \le g(x)$ for all $g \in R_f$, so

$$f(x) \le \inf\{g(x) \mid g \in R_f\} := (\bigwedge_{g \in R_f} g)(x).$$

Hence $\bigwedge_{g \in R_f} g \supseteq f$, so $\bigwedge_{g \in R_f} g \in R_f$.

(ii) It follows from Remark II.4. ∎

Remark II.9. By Proposition II.8, we have the following:

(i) $\bigwedge_{g \in R_f} g$ is a fuzzy ordered right ideal of M generated by f,

(ii) $\bigwedge_{g \in L_f} g$ is a fuzzy ordered left ideal of M generated by f,

(iii) $\bigwedge_{g \in I_f} g$ is a fuzzy ordered ideal of M generated by f,

(iv) $\bigwedge_{g \in N_f} g$ is a fuzzy ordered filter of M generated by f.

Proposition II.10. *Let $(M, \Gamma, \le)$ be an ordered Γ-semigroup and f a fuzzy subsets of M. Let h be a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by f. Then $h = \bigwedge_{g \in R_f} g$ (resp. $h = \bigwedge_{g \in L_f} g, h = \bigwedge_{g \in I_f} g$ and $h = \bigwedge_{g \in N_f} g$).*

Proof: Since $h \in R_f$ and by Remark II.4, we have $\bigwedge_{g \in R_f} g \subseteq h$. By Proposition II.8, we have $\bigwedge_{g \in R_f} g \in R_f$. Since h is a fuzzy ordered right ideal of M generated by f, we have $h \subseteq \bigwedge_{g \in R_f} g$. Hence $h = \bigwedge_{g \in R_f} g$. The other cases are proved similarly. ∎

Notation II.11. If $(M, \Gamma, \le)$ be an ordered Γ-semigroup and f a fuzzy subsets of M, we denote by $R(f)$ (resp. $L(f), I(f)$ and $N(f)$) the fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by f. That is,

$$R(f) := \bigwedge_{g \in R_f} g \text{ (resp. } L(f) := \bigwedge_{g \in L_f} g, I(f) := \bigwedge_{g \in I_f} g \text{ and } N(f) := \bigwedge_{g \in N_f} g).$$

Remark II.12. Clearly, $1_F \in R_x, 1_F \in L_x, 1_F \in I_x$ and $1_F \in N_x$. Since R_x, L_x, I_x and N_x are nonempty subsets of $F(M)$, by Proposition II.2, the elements $\bigwedge_{g \in R_x} g$, $\bigwedge_{g \in L_x} g$, $\bigwedge_{g \in I_x} g$ and $\bigwedge_{g \in N_x} g$ exist in $F(M)$. Moreover, by Proposition II.6, we have the following:

(i) $\bigwedge_{g \in R_x} g$ is a fuzzy ordered right ideal of M,

(ii) $\bigwedge_{g \in L_x} g$ is a fuzzy ordered left ideal of M,

(iii) $\bigwedge_{g \in I_x} g$ is a fuzzy ordered ideal of M,

(iv) $\bigwedge_{g \in N_x} g$ is a fuzzy ordered filter of M.

Theorem II.13. *Let $(M, \Gamma, \le)$ be an ordered Γ-semigroup and $x \in M$. Then we have the following:*

(i) $R_x = R_{f_{\{x\}}}$,

(ii) $L_x = L_{f_{\{x\}}}$,

(iii) $I_x = I_{f_{\{x\}}}$,

(iv) $N_x = N_{f_{\{x\}}}$.

Proof: (i) Let $f \in R_x$. Then f is a fuzzy ordered right ideal of M and $f(x) = 1$. Let $y \in M$. If $y = x$, then $1 = f(x) = f(y)$ and $f_{\{x\}}(y) = f_{\{x\}}(x) = 1$, so $f_{\{x\}}(y) = f(y)$. If $y \ne x$, then $f_{\{x\}}(y) = 0 \le f(y)$. Since $f_{\{x\}}(y) \le f(y)$ for all $y \in M$, we have $f \supseteq f_{\{x\}}$. Thus $f \in R_{f_{\{x\}}}$. On the other hand, let $g \in R_{f_{\{x\}}}$. Then g is a fuzzy ordered right ideal of M and $g \supseteq f_{\{x\}}$. Thus $1 = f_{\{x\}}(x) \le g(x)$, so $g(x) = 1$. Hence $g \in R_x$. Therefore $R_x = R_{f_{\{x\}}}$. The other cases can be proved similarly. ∎

Proposition II.14. *Let $(M, \Gamma, \le)$ be an ordered Γ-semigroup and f a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M. Then we have the following:*

(i) *$f^{-1}(1) = \emptyset$ or $f^{-1}(1)$ is an ordered right ideal (resp. ordered left ideal, ordered ideal and ordered filter) of M,*

(ii) *$f_{h^{-1}(1)} \subseteq h$ for all $h \in R_x$ (resp. for all $h \in L_x, h \in I_x$ and $h \in N_x$).*

Proof: (i) Assume that $f^{-1}(1) \ne \emptyset$. Let $a \in f^{-1}(1), b \in M$ and $\gamma \in \Gamma$. Then $f(a) = 1$. Since f is a fuzzy ordered right ideal of M, we have $f(a\gamma b) \ge f(a) = 1$. Thus $f(a\gamma b) = 1$, so $a\gamma b \in f^{-1}(1)$. Let $a \in M$ and $b \in f^{-1}(1)$ be such that $a \le b$. Then $f(b) = 1$. Since f is a fuzzy ordered right ideal of M and $a \le b$, we have $f(a) \ge f(b) = 1$. Thus $f(a) = 1$, so $a \in f^{-1}(1)$. Hence $f^{-1}(1)$ is an ordered right ideal of M. The rest of the proof is similar.

(ii) Let $h \in R_x$ and let $x \in M$. If $x \in h^{-1}(1)$, then $f_{h^{-1}(1)}(x) = 1$ and $h(x) = 1$, so $f_{h^{-1}(1)}(x) = h(x)$. If $x \notin h^{-1}(1)$, then $f_{h^{-1}(1)}(x) = 0 \le h(x)$. Since $f_{h^{-1}(1)}(x) \le h(x)$ for all $x \in M$, we have $f_{h^{-1}(1)} \subseteq h$. ∎

Lemma II.15. *[3] Let $(M, \Gamma, \le)$ be an ordered Γ-semigroup. A nonempty subset R (resp. L) of M is an ordered right ideal (resp. ordered left ideal) of M if and only if the characteristic function f_R (resp. f_L) is a fuzzy ordered right ideal (resp. fuzzy ordered left ideal) of M. A nonempty subset I of M is an ordered ideal (resp. ordered filter) of M if and only if the characteristic function f_I is a fuzzy ordered ideal (resp. fuzzy ordered filter) of M.*

Remark II.16. If $(M, \Gamma, \le)$ is an ordered Γ-semigroup and A an ordered right ideal (resp. ordered left ideal, ordered ideal and ordered filter) of M, then there exists a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) f of M such that $f^{-1}(1) = A$. Indeed: If A an ordered right ideal of M, then the characteristic function f_A is a fuzzy ordered right ideal of M. On the other hand, we have $x \in A$ if and only if $f_A(x) = 1$ if and only if $x \in f_A^{-1}(1)$. Hence $f_A^{-1}(1) = A$.

Theorem II.17. *Let $(M, \Gamma, \le)$ be an ordered Γ-semigroup and $x \in M$. Then we have the following:*

(i) $R(f_{\{x\}}) = f_{R(x)}$,

(ii) $L(f_{\{x\}}) = f_{L(x)}$,

(iii) $I(f_{\{x\}}) = f_{I(x)}$,
(iv) $N(f_{\{x\}}) = f_{N(x)}$.

Proof: (i) According to Notation II.11, Theorem II.13 and Remark II.4, we have

$$R(f_{\{x\}}) := \bigwedge_{h \in R_{f_{\{x\}}}} h = \bigwedge_{h \in R_x} h \subseteq h \text{ for all } h \in R_x. \quad (\star)$$

Since $R(x)$ is an ordered right ideal of M, it follows from Lemma II.15 that $f_{R(x)}$ is a fuzzy ordered right ideal of M. Since $x \in R(x)$, we get $f_{R(x)}(x) = 1$. Thus $f_{R(x)} \in R_x$. Then, by $(\star)$, we have $R(f_{\{x\}}) \subseteq f_{R(x)}$. On the other hand, let $h \in R_x$. Since $h(x) = 1$, we have $x \in h^{-1}(1)$. By Proposition II.14(i), we have $h^{-1}(1)$ is an ordered right ideal of M containing x, so $R(x) \subseteq h^{-1}(1)$. By Proposition II.14(ii), we have $f_{R(x)} \subseteq f_{h^{-1}(1)} \subseteq h$. Thus

$$f_{R(x)} \subseteq h \text{ for all } h \in R_x. \quad (\star\star)$$

By $(\star\star)$, Proposition II.5 and $(\star)$, we have

$$f_{R(x)} \subseteq \inf\{h \mid h \in R_x\} = \bigwedge_{h \in R_x} h = R(f_{\{x\}}).$$

Hence $R(f_{\{x\}}) = f_{R(x)}$. The other cases can be proved similarly. ■

As we have already seen, $R(f_{\{a\}})$ is the fuzzy ordered right ideal of M generated by $f_{\{a\}}$, $L(f_{\{a\}})$ is the fuzzy ordered left ideal of M generated by $f_{\{a\}}$ and $I(f_{\{a\}})$ (resp. $N(f_{\{a\}})$) is the fuzzy ordered ideal (resp. fuzzy ordered filter) of M generated by $f_{\{a\}}$.

Notation II.18. For an element $a \in M$, we denote:

$$\begin{aligned} R^F(a) &= R(f_{\{a\}}), \\ L^F(a) &= L(f_{\{a\}}), \\ I^F(a) &= I(f_{\{a\}}), \\ N^F(a) &= N(f_{\{a\}}). \end{aligned}$$

Definition II.19. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup, f a fuzzy subset of M and $x \in M$. We say that f contains x if $f(x) = 1$.

Remark II.20. According to Notation II.18, Notation II.11, and Theorem II.13, for any $x \in M$, we have the following:
(i) $R^F(x) = R(f_{\{x\}}) = \bigwedge_{g \in R_{f_{\{x\}}}} g = \bigwedge_{g \in R_x} g$,
(ii) $L^F(x) = L(f_{\{x\}}) = \bigwedge_{g \in L_{f_{\{x\}}}} g = \bigwedge_{g \in L_x} g$,
(iii) $I^F(x) = I(f_{\{x\}}) = \bigwedge_{g \in I_{f_{\{x\}}}} g = \bigwedge_{g \in I_x} g$,
(iv) $N^F(x) = N(f_{\{x\}}) = \bigwedge_{g \in N_{f_{\{x\}}}} g = \bigwedge_{g \in N_x} g$.

Proposition II.21. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $x \in M$. Then we have the following:*
(i) *$R^F(x) \in R_x$ and $R^F(x) \subseteq h$ for all $h \in R_x$,*
(ii) *$L^F(x) \in L_x$ and $L^F(x) \subseteq h$ for all $h \in L_x$,*
(iii) *$I^F(x) \in I_x$ and $I^F(x) \subseteq h$ for all $h \in I_x$,*
(iv) *$N^F(x) \in N_x$ and $N^F(x) \subseteq h$ for all $h \in N_x$.*

Proof: (i) Since $R^F(x) = \bigwedge_{g \in R_x} g$, $R_x \neq \emptyset$ and each $g \in R_x$ is a fuzzy ordered right ideal of M and by Proposition II.6, we have $R^F(x)$ is a fuzzy ordered right ideal of M. On the other hand, $(\bigwedge_{g \in R_x} g)(x) := \inf\{g(x) \mid g \in R_x\} = 1$. According to Definition II.19, $R^F(x)$ is a fuzzy ordered right ideal of M containing x and so $R^F(x) \in R_x$. By Remark II.4, we have $R^F(x) = \bigwedge_{g \in R_x} g \subseteq h$ for all $h \in R_x$. The other cases can be proved similarly. ■

Definition II.22. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $x \in M$. Then $R^F(x)$ (resp. $L^F(x)$, $I^F(x)$ and $N^F(x)$) is called the *fuzzy ordered right ideal* (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) *of M generated by x.*

Definition II.23. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. We define relations $\mathcal{R}^F, \mathcal{L}^F, \mathcal{I}^F, \mathcal{N}^F$ and $\mathcal{N}^F$ on M as follows:

$$\begin{aligned} a\mathcal{R}^F b &\Leftrightarrow R^F(a) = R^F(b), \\ a\mathcal{L}^F b &\Leftrightarrow L^F(a) = L^F(b), \\ a\mathcal{I}^F b &\Leftrightarrow I^F(a) = I^F(b), \\ a\mathcal{N}^F b &\Leftrightarrow N^F(a) = N^F(b). \end{aligned}$$

Theorem II.24. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. Then we have the following:*
(i) $\mathcal{R}^F = \mathcal{R}$,
(ii) $\mathcal{L}^F = \mathcal{L}$,
(iii) $\mathcal{I}^F = \mathcal{I}$,
(iv) $\mathcal{N}^F = \mathcal{N}$.

Proof: (i) For all $a, b \in M$, we have

$$\begin{aligned} a\mathcal{R}^F b &\Leftrightarrow R^F(a) = R^F(b) \\ &\Leftrightarrow R(f_{\{a\}}) = R(f_{\{b\}}) \\ &\Leftrightarrow f_{R(a)} = f_{R(b)} \text{ (by Theorem II.17)} \\ &\Leftrightarrow R(a) = R(b) \\ &\Leftrightarrow a\mathcal{R}b. \end{aligned}$$

Hence $\mathcal{R}^F = \mathcal{R}$. The other cases can be proved similarly. ■

Proposition II.25. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{A_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. Then*

$$\bigwedge_{i \in I} f_{A_i} = f_{\bigcap_{i \in I} A_i}.$$

Proof: Since f_{A_i} is a fuzzy subset of M for all $i \in I$ and by Proposition II.2, we have $\bigwedge_{i \in I} f_{A_i}$ is a fuzzy subset of M. Let $x \in M$. Then $(\bigwedge_{i \in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i \in I\}$. If $x \in A_i$ for all $i \in I$, then $f_{A_i}(x) = 1$ for all $i \in I$ and $(f_{\bigcap_{i \in I} A_i})(x) = 1$. Thus

$$(\bigwedge_{i \in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i \in I\} = 1 = (f_{\bigcap_{i \in I} A_i})(x).$$

If $x \notin A_j$ for some $j \in I$, then $x \notin \bigcap_{i \in I} A_i$. Thus $(f_{\bigcap_{i \in I} A_i})(x) = 0$. Since $x \notin A_j$, we have $f_{A_j}(x) = 0$. Since $0 \leq f_{A_i}(x)$ for all $i \in I$, we have $0 \leq \inf\{f_{A_i}(x) \mid i \in I\} \leq f_{A_j}(x) = 0$. Thus

$$(\bigwedge_{i\in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i \in I\} = 0 = (f_{\bigcap_{i\in I} A_i})(x).$$

Hence $\bigwedge_{i\in I} f_{A_i} = f_{\bigcap_{i\in I} A_i}$. ■

In a similar way we prove the following.

Proposition II.26. *Let $(M,\Gamma,\leq)$ be an ordered Γ-semigroup and $\{A_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. Then*

$$\bigvee_{i\in I} f_{A_i} = f_{\bigcup_{i\in I} A_i}.$$

Remark II.27. For an ordered Γ-semigroup $(M,\Gamma,\leq)$, we have

$$R(a) = \bigcap\{R \mid R \text{ ordered right ideal of } M \text{ and } a \in R\}.$$

Thus, by Proposition II.25, we have

$$f_{R(a)} = \bigwedge\{f_R \mid R \text{ ordered right ideal of } M \text{ and } a \in R\}.$$

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