Optimal Mortgage Refinancing Based on Monte Carlo Simulation

Jin Zheng, Siwei Gan, Xiaoxia Feng, and Dejun Xie

Abstract—**The pricing of mortgages in the context of stochastic interest rate plays an important role for financial management. The contributing factors impacting the mortgage contract value have been explored by abundant literatures. However, the market players anticipate a systematic but low-cost approach to minimize the net present value of the payment streams by taking advantage of refinancing, for instance. This paper focuses on finding a desirable refinancing time for mortgage borrowers to minimize the total payment in a stochastic interest rate environment. The underlying interest rate is assumed to follow a stochastic process with mean-reverting property, the setting of which is broad enough to accommodate a large spectrum of market realities. Two types of commonly adopted mortgage balance settlement schemes are analyzed and compared to ensure the applicability of our study. Our numerical algorithm is validated with with varying samplings, leading to several interesting characteristics pertaining to the optimal mortgage refinancing period. As one of the applications, we obtain the optimal boundary conditions for the value of the mortgage contract for all time before the expiry of the contract. Our approach and algorithm provide cost effective and easy to use financial tools for both institutional and individual property investors.**

Index Terms—**mortgage refinancing, loan valuation, financial optimization, Monte-Carlo simulation, stochastic interest rate model**

I. Introduction

As one of the most influential financial instruments in both the primary and secondary market, residential mortgage contract typically grants the borrower several options to facilitate his reacting to the market movement, among which the options of prepayment and refinancing are of pivotal importance. Prepayment refers to the behavior that the borrower chooses to settle all or in part the loan balances even though the lender's preference may be to keep receiving the contracted continuous or periodical instalments. The main financial reason leading to prepayment is typically the low investment return that the borrower may earn using the money at hand. The studies on this aspect have seen important development recently, especially those contained in [0], [0], [0], for instance, where the combination of advanced mathematical analysis with novelty numerical methods has made it possible to find very fast and cost effective solutions to the problem when the underlying interest rate is assumed as a specific but commonly adopted mean reverting model.

On the other hand, not all borrowers have sufficient fund to make alternative investment. In fact, a rather more common scenario in China's market is that majority mortgage borrowers make periodical mortgage payment using their fixed income inflow from other sources, typically in the form of salary, for instance. This economy reality underscores the importance of the option of refinancing in mortgage contract. The main reason for debtors to refinance is to improve the financial leverage efficiency by obtaining an alternative mortgage loan with a lower interest rate. Most of the previous literatures in this topic are empirical in natural from the perspective of optimal refinancing differentials, where the optimal differential is defined when the net present value of the interest payment saved reaches the sum of refinancing costs (see [0] and relevant references contained therein). In this work, we intend to address the problem by simulating the alternative interest rate that the market may offer with a rather simple assumption on the structure of the interest rate process. We exhibit the procedure with the Vasicek ([0]) model for its tractability and more importantly, for comparison with available results in related literatures. The vasicek Model has been widely used in financial modeling and financial products valuation, including characterizing the price of discount bond (see [0]) and residential mortgages (see [0], [0], [0], for instance). Another reason for using Vasicek model to implement our algorithm is the existence of convenient parameter estimation procedures for the model, including maximum likelihood method or Bayesian based method. References of such estimations can be found in [0], for instance. Matlab algorithms are proved to be helpful in solving the pertaining financial optimization problems. We remark that although the algorithm is exhibited with Vasicek model, the implementation of our approach does not restrict the choice of any stochastic model, as long as such a model explains the market trend with acceptable significance.

Our current work considers two types of loan payment in the financial market, one is matching the principal payment method and the other is matching the payment of principal and interest. Both of the two payment schemes are adopted to generate and compute the monthly installments within the whole contractual duration. To compare, we also consider two scenarios with respect to the present value of the future payment flows, one is for the zero discounting rate and the other is for a positive discounting rate. In addition, we would like to emphasize the following two commonly adopted practices in mortgage industry. First, the debtor is allowed to refinance only once after the contract is signed but before the expiry date. Under this assumption, the debtor should

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grasp the best opportunity of the a lower enough interest rate to minimize the total payment. Second, if a rational debtor chooses to refinance, he is no longer responsible for the subsequent residuary interest upon refinancing and the new debt is the outstanding loan balance inherited from the old debt. This assumption is essential since otherwise the debtor will have no motivation to refinance. However, in reality, the market is not completely efficient and liquid in the sense that the debtor is often required to pay additional refinancing fee. The impact of transaction fee is not considered in the current work.

The rest of this paper is organized as follows. We choose a model to simulate alternative mortgage rate, then derive the cash flow schemes for the two types of loan settlement in Section 2. In Section 3, we formulate and present our algorithm for numerical simulation. Numerical experimentation and discussion for finding the desirable refinancing time for debtors are provided in Section 4. The scenario of positive discounting factor is included and implemented in Section 5. In Section 6, we provide numerical experiments for model calibration with varying samplings. The optimal refinancing curve as a function in time is defined and presented in Section 7. We summarize in Section 8 with concluding remarks and possible applications in related fields.

II. Model derivation and interest rate simulation

A. Matching The Principal Repayment Method

Suppose the debtor borrows P_0 with monthly interest rate r_0 during the time period $[0\ T]$ and repays m_t at the beginning of each month, where t denotes the t^{th} month. According to matching the principal repayment method, m_t equals to a certain portion of principal plus a decreasing value of interest.

$$m_t = \frac{P_0}{n} + (1 - \frac{t-1}{n})P_0 r_0 \tag{1}$$

where n is the total number of payment times.

The term $\frac{P_0}{n}$ could be explained as a fixed portion of principal, and $(1 - \frac{t-1}{n})P_0 r_0$ is an amount of decreasing interest due to the reduction of principal every month.

At time k, the debtor prefers to refinance the debt with another lender when a lower interest rate r_k is offered. On the k^{th} month, he owes the previous bank P_k and has paid A_k.

$$\begin{aligned} P_k &= (1 - \frac{k-1}{n})P_0 \\ A_k &= \sum_{i=1}^{k-1} m_i = P_0(k-1)(r_0 + \frac{1}{n} - \frac{k-2}{2n} r_0) \end{aligned} \tag{2}$$

The amount of money $P(t)$ is the new principal the debtor borrows from another bank with the interest rate $r(t)$. This transaction will last from time k to time T. The total payment over time $[0\ T]$ could be described as follows:

$$\begin{aligned} P(T) &= A_k + \sum_{i=k}^{n} m_i \\ &= P_0(k-1)(r_0 + \frac{1}{n} - \frac{k-2}{2n} r_0) \\ &\quad + P_k[1 + \frac{(n^*+1)r_k}{2}] \end{aligned} \tag{3}$$

where $n^* = n - k + 1$

B. Matching the Repayment of Principal and Interest Method

The second method to repay loan is to match the repayment of principal and interest. Assume the debtor borrows P_0 with interest rate r_0 over time $[0\ T]$ and the amount of monthly payment is kept the same. In the beginning of the contract, the interest accounts for most of payment due to a large amount of loan while principal is small. Let $P(t)$ denote the amount of money owed at time t and m is the monthly payment.

$$\begin{cases} dP(t) = -m dt + r_0 P(t) dt \\ P(0) = P_0 \end{cases} \tag{4}$$

The monthly payment m, should be:

$$m = \frac{P_0 r_0 (1+r_0)^n}{(1+r_0)^n - 1} \tag{5}$$

where n is the number of total repayment times.

At time k, the debtor owes the $P(k)$ to the previous bank. Again, due to the lower interest rate r_k , the debtor would borrow $P(k)$ from another bank to repay the remaining debts $P(k)$. The total payment over time $[0\ T]$ could be described as follows:

$$P(T) = m_1 * (k-1) + m_2 * (n-k+1) \tag{6}$$

where

$$\begin{cases} m_1 = \dfrac{P_0 r_0 (1+r_0)^k}{(1+r_0)^k - 1} \\ m_2 = \dfrac{P_k r_k (1+r_k)^{n-k}}{(1+r_k)^{n-k} - 1} \\ P_k = \dfrac{m}{r_0}[1 - e^{r_0(k-T)}] \end{cases} \tag{7}$$

To carry out numerical simulations for both payment schemes, we assume that the principal P_0 is 100,000, the initial monthly lending rate r_0 is $\frac{5}{12}\%$, and the total payment period, counted in number of months, is $T = 240$.

III. Numerical Simulation

The Vasicek short term interest rate process is a mathematical model describing the evolution of interest rate (see [0]). The model specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \tag{8}$$

where k is the reversion rate, θ is long−term mean interest rate and σ is the standard deviation, all of which are positive constants. We let r_t denote the instantaneous spot rate at time t, and W_t is the standard Brownian Motion. which yields the explicit solution for equation (8)

$$r_t = e^{-kt} r_0 + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dW_s \qquad (9)$$

Under the Euler approximation, equation (1) can be rewritten as:

$$\Delta r = k(\theta - r_t)\Delta t + \sigma \Delta W_t \qquad (10)$$

Both equation (8) and equation (9) can be used equivalently to describe the alternative mortgage rate that a loan borrower may choose from the open market. But equation (10) is often more useful for simulation purposes. We would like to remark that although Vasicek model is considered in the current paper, our method is equally applicable to many other classes of stochastic models.

Then we use simulated data to carry out the experiment. The aim of our model is to obtain the best period to refinance. The best period in our experiment means the month during which to refinance yields a lowest total payment. We simulate both repayment methods to obtain the frequency distributions.

The major steps for the algorithms are as follows:

1. Initialize r_0 and generate interest rate r_t for each month by Monte Carlo simulation. By Euler's Approximation

$$\begin{aligned} r_j &= r_{j-1} + \Delta r_{j-1} \\ &= r_{j-1} + k(\theta - r_{j-1}) + \sigma dW_{j-1} \end{aligned}$$

2. Update total payment by the simulated interest rate. Both of the two payment schemes are experimented in this study.
3. Find the period where the total payment achieves the lowest. The followings Figures 1-6 are the example of our simulations by both methods. Figures 1-3 are simulated by matching the principal repayment method, and rest Figures 4-6 are simulated by matching the repayment of principal and interest.

We observe that for all 6 scenarios, the trend of the plot becomes flat after, say, 13-15 years, which means the total present value of payment roughly keeps constant after certain years. This is a strong hint that early stage of the contract is critical for prudent financial decisions. In the following Section 6, one will see that as $\sigma \to 0$, the solution curve defining the normalized net present value of total cash payments goes fast to an asymptotic value beyond certain time. Under this scenario, there is no difference to refinance or not after such a asymptotic time period. However for any $\sigma > 0$, the stochastic nature of the underlying interest rate can lead to a very volatile optimal decision making process before, say, $t < 13 \sim 15$, which means adjacent times for refinancing may result in quite divergent net present values of future payments.

IV. NUMERICAL EXPERIMENTATION AND DISCUSSION

In this section, we use simulated data to carry out the experiment. The aim of our model is to obtain the best period to refinance. The 'best period' in our experiment means the month during which to refinance yields a lowest total payment. We simulate both methods to obtain the frequency distributions.

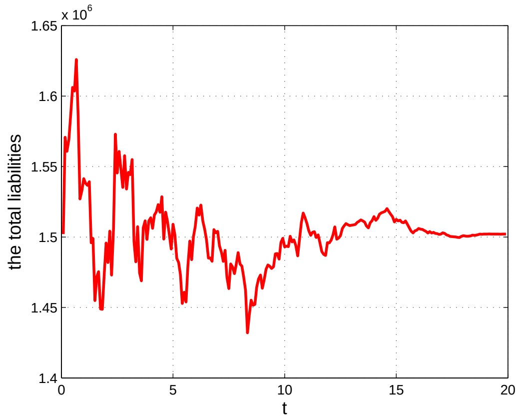

Fig. 1.

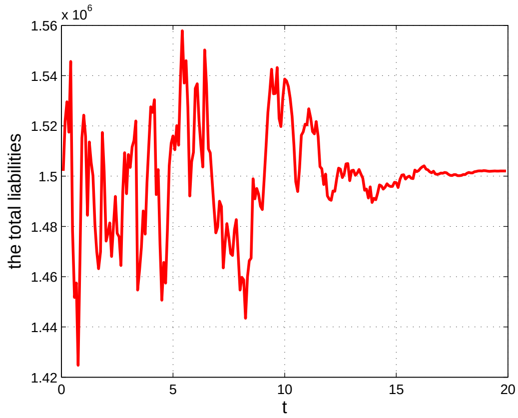

Fig. 2.

A. Matching The Principal Payment Method

The following Figure 7 provides the information of the frequency distribution of the best period throughout the contracted duration. The frequency space is 6 months. It can be seen that the frequency arrives the peak at the second half of the first year. The frequency of following months declines over time. From the results reported in Table 1, we find that until the $5th$ year, the total times to refinance is up to 9252 (the frequency rate is 92.52 %), which implies it is better to

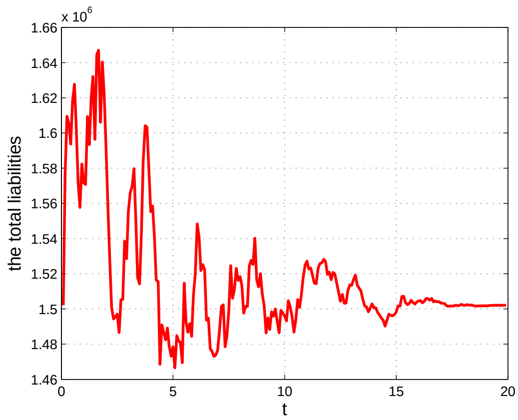

Fig. 3.

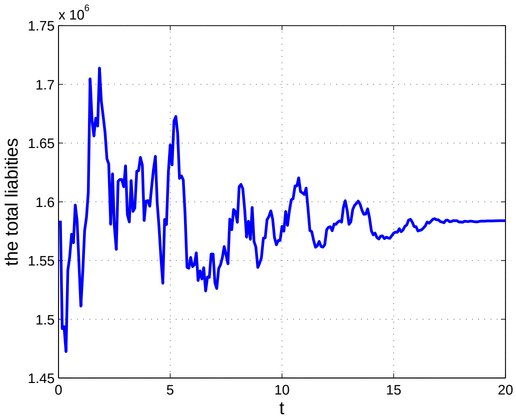

Fig. 4.

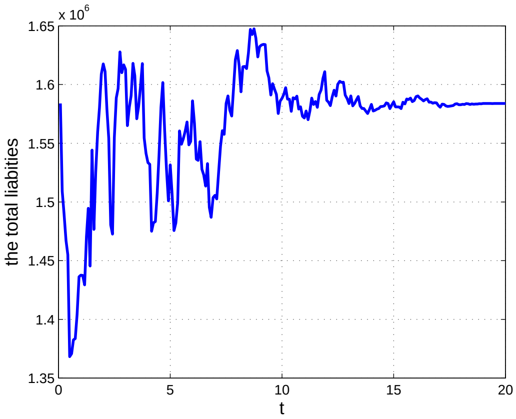

Fig. 5.

refinance early.

We include the interest rate factor into our implementation and discussion. As we might reasonably assume, the best opportunity to refinance probably arise when the loan interest is comparatively low. We define a new variable 'count' to record the times that the best month to refinance (m_f) coincide with the month where the lowest interest rate (m_r) occurs. In each simulation, if the difference between m_f and m_r is less than 3 months, we regard them to be a coincidence

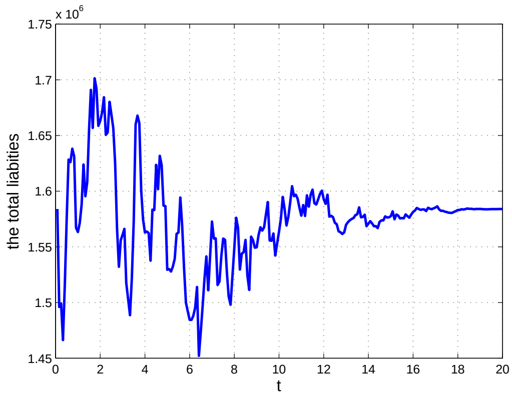

Fig. 6.

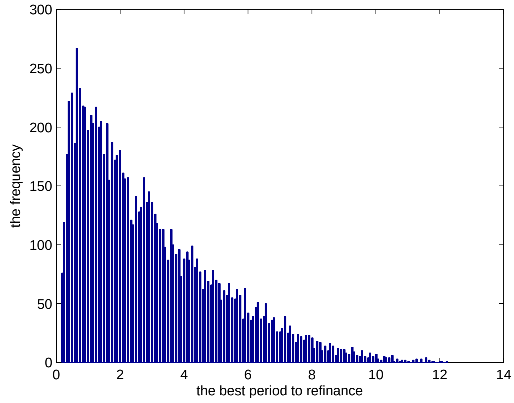

Fig. 7. The frequency distribution over 240 months' duration by 10000 times of simulations with matching the principal payment method.

TABLE I
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Months	Frequency	Cumulative Frequency
1-6	1037	1037
7-12	1649	2956
13-18	1424	4380
19-24	1174	5554
25-30	971	6525
31-36	791	7316
37-42	701	8017
43-48	527	8544
49-54	395	8939
55-60	313	9252
61-66	258	9510
67-72	193	9703
73-78	105	9808
79-84	78	9886
85-90	50	9936
91-96	30	9966
97-102	21	9987
103-108	7	9994
109-114	3	9997
115-120	2	9999
121-126	0	9999
127-132	1	10000
133-240	0	10000

and the value of count increases by 1.

$$\begin{aligned} for \quad & k = 1 : 10000 \\ if \quad & |m_f - m_r| \leq 3 \\ count \quad = \quad & count + 1 \end{aligned}$$

The above procedure is circulated 10 times and we choose (m_r) in different time intervals. The simulated results of the coincidence as measured by the variable 'count' are shown in Table II. The second column $1-36$ represents the time interval from the $1st$ month to the $36th$ month of the contract. Similarly, $1-60$, $1-90$ and $1-240$ represent the corresponding month intervals. For instance, the times that the optimal refinance period locates in the interval from the first month to the $90th$ month is 6295 in the first simulation.

The bottom row in Table II displays the average value

TABLE II
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Times	1-36	1-60	1-90	1-240
1	5612	6765	6295	2721
2	5584	6823	6446	2820
3	5721	6838	6321	2761
4	5666	6973	6393	2749
5	5660	6890	6442	2760
6	5731	6853	6348	2708
7	5607	6710	6260	2671
8	5626	6770	6326	2743
9	5679	6829	6351	2807
10	5714	6767	6335	2713
Average	5660	6821.8	6447	2745.3

of 'count'. It is observed that the average percentage value of 'count' during $1st - 240th$ is only 27.45%, which is the lowest compared to others. This result is not surprising since it has been shown in above that the possibility of refinance is up to 92.52% in the first five years. As the time interval is shortened to, say, the first three years or the first five years, the percentage of coincidence substantially increases.

From Table I, we have seen that the best refinancing month is considerably more possibly located in the earlier time. But how early is still a problem deserving careful analysis. The duration of the first 90 months apparently shows the highest possibility 99.36%. However, the interval is so long that it may not be an operative suggestion to debtors. In fact, the frequency rate steadily increases after the $60th$ month. On the other hand, when we inspect the first 36 months' duration, it is noted that although the range becomes small, the possibility that the best period to refinance locates in this range is still as high as 73.16%. As for the duration of the first 60 months, the frequency rate is 92.52%, and the corresponding average percentage of coincidence is the highest among all these three cases. This comparison provides a useful hint on the distributional pattern of the best refinance period, which, taken in conjunction with the observations of the real market interest rate, will facilitate the borrower's financial decisions.

We focus on a period of 60 months. In the above discussion, we define the 'coincidence' as the difference between the smallest interest rate month in a specific period and the best refinance month less than three months. As a matter of fact, when we strictly define the 'coincidence' means the best refinance month equal to the local smallest interest rate month, the consequence reveals the value of 'count' just reduces by around 3.3%. In addition, there is more likely to refinance after the smallest interest rate happens than refinance before it.

B. Matching The Payment of Principal and Interest

Figure 8 is the frequency distribution generated by simulating 10000 times of matching payment of principal and interest method. It has the similar but not identical properties compared to Figure 7. In this payment scheme, the principal balance decreases rather slowly at early stage while in the first payment scheme (matching the payment of principle) that the principle decreases by an equal amount each month.

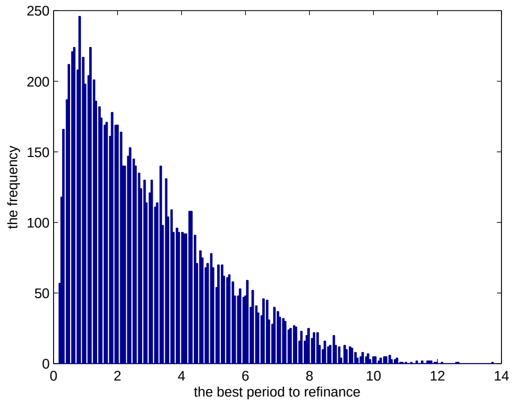

Fig. 8. The frequency distribution over 240 months' duration by 10000 times of simulations with matching the payment of principal and interest method.

TABLE III
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Months	Frequency	Cumulative Frequency
1-6	727	727
7-12	1038	1765
13-18	1259	3024
19-24	1162	4186
25-30	961	5047
31-36	766	5913
37-42	670	6583
43-48	588	7171
49-54	553	7704
55-60	468	8172
61-66	381	8553
67-72	334	8887
73-78	274	9161
79-84	233	9494
85-90	161	9625
91-96	136	9791
97-102	106	9797
103-108	58	9855
109-114	45	9900
115-120	38	9938
121-126	34	9972
127-132	15	9987
133-138	11	9999
139-142	1	10000
143-240	0	10000

Thus, the less indifference of change of principle leads to the more divergent distribution.

Again, the interest rate factor should be involved in our discussion. As mentioned above, we use 'count' to record the times that the best month to refinance (m_t) coincides with the month when the smallest interest rate (m_r) occurs.

Figure 8 and Table III illustrates the frequency distribution for a contract of 20 years. The results we have obtained are similar to the previous method. The frequency first increases, reaching the peak during the $13th$ month to the $18th$ month. Afterwards it decreases gradually, down to 0 after 11 years. Until the $7th$ year, the cumulative frequency is 9494 in total,

TABLE IV
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Times	1-36	1-60	1-90	1-240
1	5579	6862	6465	2777
2	5622	6833	6460	2824
3	5572	6821	6549	2877
4	5531	6769	6456	2757
5	5587	6764	6528	2800
6	5647	6856	6485	2802
7	5574	6798	6450	2761
8	5624	6847	6478	2903
9	5633	6821	6435	2739
10	5540	6914	6510	2809
Average	5590.9	6828.5	6481.6	2804.9

which provides a strong evidence for early refinance. As for coincidence, again, the duration of 90 months has the highest value in these three periods.

C. Comments on the Results

In our paper, we wish to determine which period is a better choice for debtors to refinance. The study has found some important properties for refinancing. First, the possibility of refinancing in the early stage may surpass 90%, which implies that debtors should refinance early. Second, the frequency curve arrives its peak at the last half of the first year. After that, the frequency of refinancing will drop and the coincidence increases at first and decreases after its peak value. Finally, a duration neither relatively too long nor too short is regarded as a perfect solution, i.e., a duration of 90 months (7.5 years) is relatively too long to the whole duration of 20 years. In consideration of these four properties, the debtors should refinance in the period of the $1st$ to the $60th$ month when the interest rate is locally low, for contract conditions and market rate movement specified in this paper. That means in certain month when the interest rate will be expected to fall down to certain lower enough level, it is probably the best time to refinance.

V. Optimal Refinancing with discounted payment

In real financial market, the consideration of discounted payment with matching the principal and interest rate method is more relevant and applicable to most of the industry practitioners. We assume the debtor borrows P_0 with monthly interest rate r_0 during the time period [0 T] and pays m for each month. At time t, where the market interest reaches r_t and the debtor has the choice of whether to refinance or not. The monthly payment corresponding to the originally contracted mortgage rate is

$$m = \frac{r_0 P_0}{1-(1+r_0^{-T})} \tag{11}$$

If the debtor does not want to refinance and holds the current contract, the present value of total payment from time 1 to

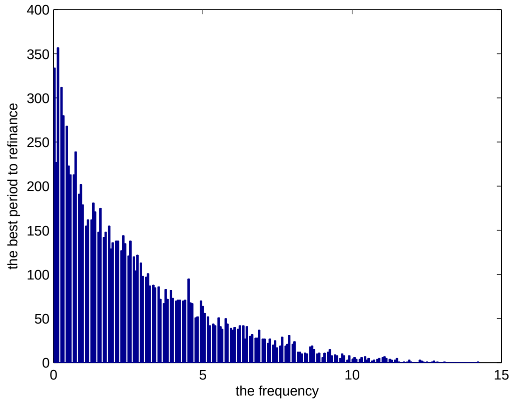

Fig. 9. The frequency distribution over 240 months' duration by 10000 times of simulations with discounted future payment.

T should be:

$$\begin{aligned} M_1(t) &= \frac{m}{1+r_1} + \frac{m}{(1+r_1)(1+r_2)} \\ &\quad + \frac{m}{(1+r_1)(1+r_2)(1+r_3)} + \cdots \\ &\quad + \frac{m}{(1+r_1)(1+r_2)(1+r_3)\cdots(1+r_T)} \\ &= \sum_{i=1}^{T} \frac{m}{\prod_{j=1}^{i}[1+r_j]} \end{aligned} \tag{12}$$

At time t, if the debtor chooses to refinance with a lower interest rate r_t. After refinance, the monthly payment becomes m_1 instead of m, and the discounted total payment is:

$$\begin{aligned} M_2(t) &= \frac{m}{1+r_1} + \frac{m}{(1+r_1)(1+r_2)} \\ &\quad + \frac{m}{(1+r_1)(1+r_2)(1+r_3)} + \cdots \\ &\quad + \frac{m}{(1+r_1)(1+r_2)(1+r_3)\cdots(1+r_t)} \\ &\quad + \frac{m_1}{1+r_{t+1}} + \frac{m_1}{(1+r_{t+1})(1+r_{t+2})} \\ &\quad + \frac{m_1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} + \cdots \\ &\quad + \frac{m_1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})\cdots(1+r_T)} \\ &= \sum_{i=1}^{t} \frac{m}{\prod_{j=1}^{i}[1+r_j]} + \sum_{i=t+1}^{T} \frac{m_1}{\prod_{j=1}^{i}[1+r_j]} \end{aligned} \tag{13}$$

The mathematical formulation and the simulation procedure are similar to those discussed in the previous section, where the discounting factor is zero. The only difference is that we now discount future cash payment in computing the present value. Similar numerical experiments are carried out with 10000 simulated interest trajectories. We use the same estimated parameters for Vasicek model and the same initial contract conditions.

Figure 9 describes the frequency distribution with discounted future payments before the expiration of the contract by matching the principal and interest rate method. The frequency reaches its peak in the first 6 months and decline exponentially over the time. Compared to Figure 8, the

TABLE V
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Months	Frequency	Cumulative Frequency
1-6	1531	1531
7-12	919	2450
13-18	788	3238
19-24	728	3966
25-30	662	4628
31-36	553	5181
37-42	521	5702
43-48	438	6140
49-54	426	6566
55-60	383	6949
61-66	336	7285
67-72	337	7622
73-78	278	7909
79-84	248	8157
85-90	234	8391
91-96	224	8615
97-102	178	8793
103-108	154	8947
109-114	164	9111
115-120	126	9237
121-126	96	9333
127-132	121	9454
133-138	113	9567
139-142	64	9631
143-148	80	9711
149-154	52	9763
153-158	49	9812
159-164	36	9848
165-170	40	9888
171-240	112	10000

frequency distribution is Figure 9 tends to have a 'long-tail', which means the minimum total payment also be achieved when the time is close to maturity. The comparison of Table III and Table V shows such a difference for the two different discounting scenarios. In Table III, the frequency is 0 after 143 months, which reflects the fact that after 143 months, it is not necessary to replace the existing debt obligation. Nevertheless, even near the expiration date, the discounted future payment scheme allows an optimal refinance to minimize the total payment. Thus the difference between able III and Table V verifies the time value of money conserved in the discounted payment scheme.

VI. APPLICATIONS

A. Analytical Solutions

The results in previous sections show that the debtors should refinance as earlier as possible when the lending rate is relatively low. In this section, we try to use analytical approach to find the closed form solutions for some special cases. On the other hand, the numerical results are also verified, in part, by the following theoretical analysis on the ration of $\frac{P(T)}{P_0}$. To illustrate the idea of our analytical idea, we assume that the contract adopts the matching the principal payment method for the borrower to pay back his debt. Let $t = k - 1$, then equation (9) yields

$$\begin{aligned}\frac{P(T)}{P_0} &= [r_0 - \frac{r_{t+1}}{2} + \frac{r_0}{2n} - \frac{(n+1)r_{t+1}}{2n}]t \\ &\quad + \frac{r_{t+1} - r_0}{2n}t^2 + \frac{n+1}{2}r_{t+1} + 1 \\ &= \frac{r_{t+1} - r_0}{2n}t^2 + (1 + \frac{1}{2n})(r_0 - r_{t+1})t \\ &\quad + \frac{n+1}{2}r_{t+1} + 1 \qquad (14)\end{aligned}$$

We proceed the analysis by identifying the following two scenarios.

1) $r_0 = \theta$ *:* When the initial borrowing rate equals to the long term mean rate, the stochastic process for the market interest rate becomes

$$\begin{aligned}r_t &= e^{-kt}r_0 + \theta(1 - e^{-kt}) + \sigma\int_0^t e^{-k(t-s)}dW_s \\ &= r_0 + \sigma\int_0^t e^{-k(t-s)}dW_s \qquad (15)\end{aligned}$$

It is intuitive and worthwhile to note that the debtor is likely to refinance only when the instantaneous spot rate is less than the initial borrowing rate, i.e., only when the stochastic integral term $\sigma\int_0^t e^{-k(t-s)}dW_s$ results in a negative value. But even with this in mind, the statistically measured minimizer t to the stochastic function $\frac{P(T)}{P_0}$ is not immediate since the equation (17), as a quadratic form in t with stochastic coefficients, is composed of terms with different signs in differentials in t. For instance, one might want t go to zero on the set of t where $r_0 > r_t$ if only the first order term of t is concerned, but this move may not grant enough time for r_t to achieve sufficiently lower level, which is desirable if the second order or zero order term of t is concerned. An equilibrium of the opposing factors in (17), as shown by our simulated results in Figure 7 and 8, says that the best refinance time is most likely located in the early stage of the contract for the usual conditions set in this paper. This is true despite that the expectation of $\frac{P(T)}{P_0}$ is independent of time t. The result is consistent with the numerical results contained in the previous section and offers a statistical explanation to the optimal strategy that a borrower should take to minimize his total financial cost.

2) $r_0 > \theta$*:* When the initial borrowing rate is higher than the long term mean rate, note that the stochastic process for the market interest rate can be written as

$$r_t = (r_0 - \theta)e^{-kt} + \theta + \sigma\int_0^t e^{-k(t-s)}dW_s \qquad (16)$$

Figure 10 reveals that when the value of σ is small (i.e. 0.001 or less), the simulated interest rates are fluctuating around the 'drift' with very small deviations. In this scenario, the general trend of interest rate drops exponentially to the mean level. With the parameters we choose for the model, and with the current simulation specifications, such as the time step for the Euler approximation and the maximum number of simulated trajectories, contained in this paper, we find that the stochastic integral term $\sigma\int_0^t e^{-k(t-s)}dW_s$ is negligible in statistical sense for understanding the refinancing strategy.

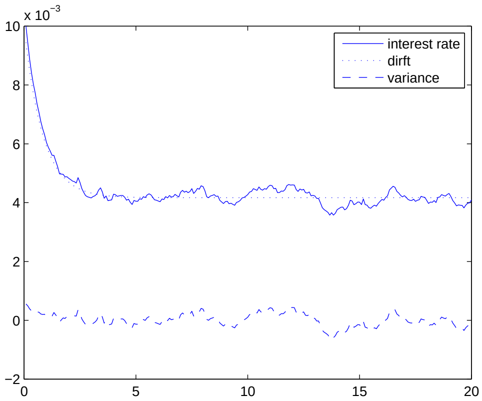

Fig. 10. 'drift' represents the term $(r_0-\theta)e^{-kt}+\theta$, 'variance' represents the term $\sigma\int_0^t e^{-k(t-s)}dW_s$ and 'interest rates' are the simulated spot instantaneous rates, where r_0=0.12, k=0.1 and σ=0.001.

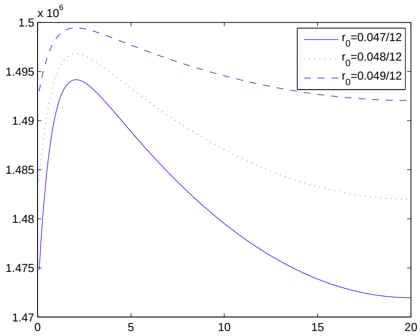

Fig. 11. The distribution of total payment for $r_0<\theta$, when θ =0.05/12, σ =0.003

3) $\sigma=0$: As present in the previous section, the stochastic integral term is negligible in statistical sense. In this section we ignore the effect of the stochastic term and consider the impact of r_0 and θ on total payment. From equation (14) and (16), we find that

$$\begin{aligned}\frac{P(T)}{P_0} &= \frac{(r_0-\theta)e^{-kt}-r_0}{2n}t^2 \\ &\quad+(1+\frac{1}{2n})(r_0-(r_0-\theta)e^{-kt})t \\ &\quad+\frac{n+1}{2}(r_0-\theta)e^{-kt}+1 \qquad (17)\end{aligned}$$

Consider when $r_0=\theta$, the equation(17) becomes $\frac{P(T)}{P(0)}=\frac{n+1}{2}r_0+1$ and the total payment is constant over the contractual duration. It is not essential for the debtor to refinance because $r_t=r_0$ at any time, implying the market interest rate is fixed. Figure 11 and Figure 12 describe the distribution of total payment when $r_t\neq r_0$. In Figure 11, when $r_0<\theta$, due to the mean-reverting property of the Vasicek model, the interest rate generated will increase until asymptotically reaching θ. In this case, the debtor will not choose to refinance for the sake of taking advantage of lower interest rate r_0. However, when $r_t\geq r_0$, the interest monotonically decreases to the asymptote. In this case, there exists a unique optimal refinance time between $t=0$ and $t=T$, as shown in Figure 12.

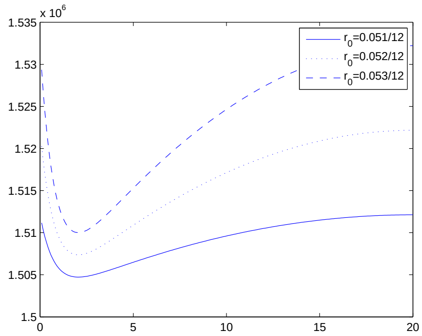

Fig. 12. The distribution of total payment for $r_0>\theta$, when θ =0.05/12, σ =0.003

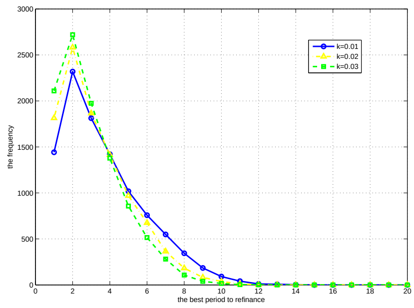

Fig. 13. The frequency distribution at different values of k, when θ =0.05/12, σ =0.003 and r_0=0.05/12.

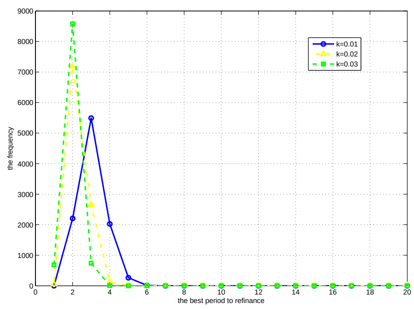

Fig. 14. The frequency distribution at different values of k, when θ =0.05/12, σ =0.003 and r_0=0.12/12.

B. Variation Analysis

The previous section numerically displays similar results of these two payment schemes. The factors leading to such consequence include, say, the parameter value and the trend of interest rate in the context of Vasicek model. Here we provide more numerical experiments to show how the optimal refinancing frequency distribution will change as the the parameter value changes. Due to the mean-reverting property, the simulated interest rate, r_t, is expected to be reverting to the mean value θ in the long run. We simulate the process for 10000 times for different values of parameters and adopt the matching the principal repayment method in all simulations. We also include two conditions into simulation. One is that the initial interest rate equals to long-term mean interest rate. The other is an extreme condition that the initial interest rate is greater than the long-term mean interest rate.

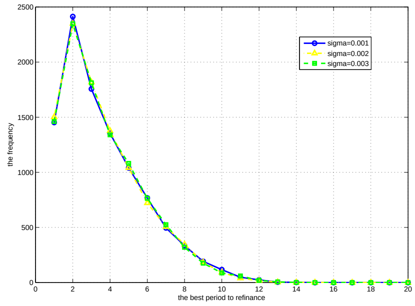

Fig. 15. The frequency distribution at different values of σ, when θ=0.05/12, k=0.1 and r_0=0.05/12.

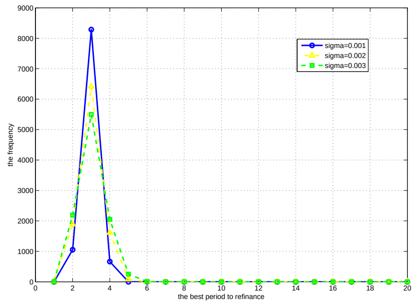

Fig. 16. The frequency distribution at different values of σ, when θ=0.05/12, k=0.1 and r_0=0.12/12.

1) Parameter k: The parameter k measures how fast the interest rate process will be driven back to the long term mean under Vasicek model. Figure 13 and 14 show the fact that, as the value of k rises, the likelihood of refinancing in the second half of the first year sees a growth when the initial lending rate equals to the mean lending rate. In the extreme condition that the initial interest rate is greater than the initial lending rate, the increase of the reversion speed leading to relatively early optimal refinancing.

2) Parameter σ: To observe the effect of market rate volatility on the refinance frequency distribution, we change the value of σ while keeping other parameters fixed. Figure 6 provides the numerical outputs when r_0 equals to the long-term mean. In this example, changes in the value of σ do not lead to significant changes in the frequency distribution. When r_0 is relatively higher than the long-term mean interest rate θ, the consequence is more apparent. Figure 15 shows the numerical plots for this scenario. An apparent convergent pattern can be drawn from Figure 15, where the best refinance period converges to around the $25th$ month as σ decreases.

VII. Optimal Refinancing Boundary

In this section, we review the problem and our solutions from stochastic control perspective. Consider a mortgage contract model with matching the repayment of principal and interest method. We assume the debtor pays m each month with initial interest rate r_0. Again, the debtor is only allowed to refinance once when he or she thinks the time is reasonable. The outstanding balance M(t), can be determined by the ODE:

$$dM(t) = -mdt + r_0 M(t)dt \tag{18}$$

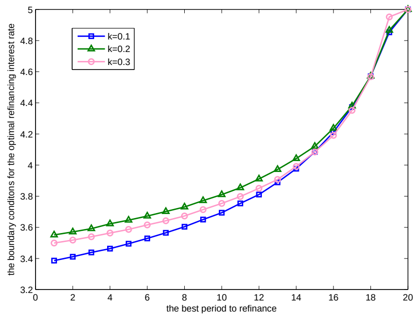

Fig. 17. The optimal refinancing interest rate boundary at different values of k.

At the expiration time T, $M(T) = 0$, and the ODE offers a unique solution

$$M(t) = \frac{m}{r_0}(1 - e^{-r_0(T-t)}) \tag{19}$$

To study the optimal refinancing strategy in general, we set $V(r,t)$ as the present value, at time t, of a continuous future cashflow of m_s for $t \le s \le T$, subject to a stochastic interest rate process of r_s, for $t \le s \le T$, where $r_t = r$. Then

$$V(r,t) = E_r[\int_t^T m_s exp(-\int_t^s r_u du)ds] \tag{20}$$

According to the properties of mortgage contract, at time T, the expiration time, the value of the contract should be

$$V(r,T) = 0 \tag{21}$$

Also, one may impose another boundary condition for $r \to \infty$ as one financial market requirement:

$$V(\infty,t) = 0, \forall 0 \le t \le T \tag{22}$$

Consider the current study where one but only one refinancing is allowed at some stage throughout the duration of the contract. Suppose the current time is t and the corresponding market interest rate is r, to find the optimal market interest $r_{optimal}$, at a specified current time t, for the borrower to refinance, the problem becomes the following stochastic control problem. Find $r_{optimal}$ such that the rational debtor prefers to refinance when the market interest rate is less or equal to $r_{optimal}$. Since the contractual duration is 240 month, we should obtain $r_{optimal}$ for each month. Under the assumption that the contract is signed at time 0, we should obtain a curve of $r_{optimal}$ as a function of t. If the market interest rate level is equal to or below $r_{optimal}$, the debtor can grasp the opportunity to refinance.

Because of the challenge in solving such a stochastic control problem analytically, we appeal to the iterative Monte Carlo simulation method to find its numerical solution. For this purpose, we define the optimal refinancing interest rate, $r_{optimal}$, as the interest rate that the likelihood to refinance at time t is in the interval P=[0.902, 0.904]. The interval is selected for the following reasonable considerations. First, if the probability is too low for optimal refinancing, the results will not be meaningless for mortgage borrowers. A rational investor may not prefer to refinance if he or she will bear more risk. Second, if a higher or safer probability is chosen, say 99%, the optimal refinancing interest rate will drop drastically to 1.8% in the first year and 2% in the second

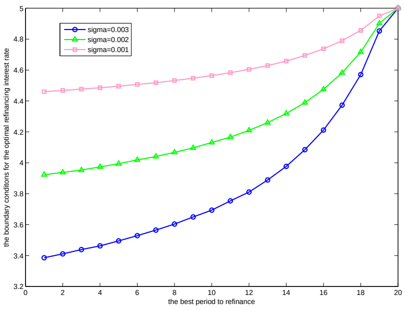

Fig. 18. The optimal refinancing interest rate boundary at different values of σ.

year, which is unrealistic in the real financial market. On balance, it is necessary to simulate optimal interest rate level which converges to a relatively stable value while the the randomness of the interest model is taken into consideration.

Suppose that the optimal refinancing interest rates at the k_{th} and the $(k-1)_{th}$ year are computed as $r_k, (r_{k-1})$, respectively. Let $F(x)$ denote the probability to refinance when interest rate is x. The probability is obtained with 50,000 simulations. The following procedures are carried out to obtain the initial guess of $r_{optimal}$ for time at $(k-2)$ year.

1. Given the value of r_k and r_{k-1}, we set the upper and lower boundary of the optimal refinancing interest rate for r_{k-2}.

The upper bound: $u = r_{k-1}$

The lower bound: $l = 2r_{k-1} - r_k$

2. Let $j = (u + l)/2$

If $F(j)$ is located in the interval P

Then we set $r_{k-2} = j$.

3. If $F(j) < 90.2$

Set the new upper bound $u = j$

Else if $F(j) > 90.4$

Set the new lower bound $l = j$

Return to the Step 2 until the value of j converges to the optimal interest rate as defined by the interval P;

4. Repeat the step $1-3$ to obtain the optimal interest rate on specific date and boundary of the optimal refinancing interest rate. To simplify the problem, the interest rate in simulated in the unit of month.

Since $F(j)$ is computed by simulation, the optimal interest rates obtained by the previous procedures are unstable and the boundary line is rough. Therefore, it is necessary to modify and enhance the accuracy at each point by the following two steps.

1. Check whether the value of $F(r_k)$ is located in the interval P for continuously 3 times. If it does, the interest rate r_k is considered as the eligible one, else go the next step.

2. If $F(r_k) > 90.4$

Then $r_k = r_k - 0.001$

Else if $F(r_k) < 90.2$

Then $r_k = r_k + 0.001$

Repeat the Step 1-2 to get an enough accurate value of r_k. Figure 17 and 18 provides numerical plots of $r_{optimal}$ using typical parameters pertaining to the usual mortgage contract under study. From these plots, one may tend to conjecture some analytical features of $r_{optimal}$ as a function in time, including monotonicity and lower boundedness, for instance. It may be also worthwhile to compare such an optimal refinance boundary with the optimal prepayment boundary contained in [0] and [0]. However, the connections between these two types of boundaries are not immediately clear. Theoretically, many stochastic control problems can be equivalently formulated as variational inequalities or partial differential equation with free boundaries (see [0]). The optimal point for an option holder to exercise the contracted right usually corresponds to the free boundary for the partial differential equation. To successfully formulate the problem into a partial differential equation system, solution conditions on the free boundary must be specified. For instance, in [0], the following two free boundary conditions are prescribed in the study of the prepayment strategy under similar interest rate process:

$$\begin{aligned} \frac{\partial V}{\partial r}\Big|_{r=r_{optimal}} &= 0 \\ \frac{\partial^2 V}{\partial r^2}\Big|_{r=r_{optimal}} &\geq 0 \end{aligned} \tag{23}$$

This might presents useful hints to the optimal refinancing boundary for mortgage borrowers. However, further research needs to be carried out to identify the right boundary conditions for the problem discussed in this work.

TABLE VI
THE OPTIMAL REFINANCING INTEREST RATE FOR DIFFERENT k

Time	$k=0.1$	$k=0.2$	$k=0.3$
1	3.386	3.552	3.499
2	3.411	3.571	3.518
3	3.439	3.592	3.540
4	3.463	3.623	3.563
5	3.495	3.646	3.587
6	3.529	3.672	3.615
7	3.565	3.702	3.642
8	3.604	3.731	3.673
9	3.650	3.772	3.714
10	3.694	3.810	3.754
11	3.754	3.855	3.798
12	3.811	3.911	3.851
13	3.889	3.972	3.908
14	3.977	4.042	3.992
15	4.085	4.120	4.085
16	4.212	4.237	4.191
17	4.373	4.380	4.352
18	4.571	4.572	4.568
19	4.853	4.867	4.952
20	5.000	5.000	5.000

TABLE VII
THE OPTIMAL REFINANCING INTEREST RATE FOR DIFFERENT σ

Time	$\sigma = 0.001$	$\sigma = 0.002$	$\sigma = 0.003$
1	3.386	3.923	4.461
2	3.411	3.938	4.468
3	3.439	3.953	4.477
4	3.463	3.974	4.486
5	3.495	3.994	4.495
6	3.529	4.019	4.508
7	3.565	4.041	4.518
8	3.604	4.068	4.533
9	3.650	4.097	4.548
10	3.694	4.132	4.564
11	3.754	4.166	4.583
12	3.811	4.210	4.605
13	3.889	4.260	4.629
14	3.977	4.319	4.658
15	4.085	4.389	4.695
16	4.212	4.476	4.738
17	4.373	4.582	4.790
18	4.571	4.717	4.857
19	4.853	4.902	4.950
20	5.000	5.000	5.000

VIII. CONCLUDING REMARKS

This paper focuses on the numerical simulation approach for finding the best refinancing strategy for mortgage borrowers in a stochastic interest environment. Interesting properties of the optimal refinancing time, including its relative closeness to the origination of the contract and the statistically lowest point of the interest curve, are discovered. In this work, Vasicek Model is applied to simulate the monthly interest rate and both matching the principal payment method and matching the payment of principal and interest method are considered to generate the total payment. Results from these empirical experiments tend to suggest relatively early refinancing for both scenarios under the conditions of the mortgage contracts set in the paper, particulary when the initial borrowing rate is large compared to the long term mean rate. These findings shed lights on the very important financial queries for many property investors.

In addition, since mortgage contract is also a type of option, the usefulness of our approach is not limited to the problem at hand. Traditional analytical techniques for characterizing option contracts, if possible, usually require mathematically strong and sometimes parameter sensitive properties attached to the formulation of the problem, such as the convexity existed in the early exercise boundary of the classic American put option (see [0], [0], for instance). In comparison to such analytical methods, our approach is robust and easy to implement. The algorithms contained in this work can be readily applied to a broad class of problems arising from financial optimization and option pricing.

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