Riesz Potentials, Riesz Transforms on Lipschitz Spaces in Compact Lie Groups

Daning Chen, Jiecheng Chen & Dashan Fan

Abstract—Using the heat kernel characterization, we establish some boundedness properties for Riesz potentials and Riesz transforms on Lipschitz spaces in a compact Lie group.

Key words: Riesz transforms, Riesz potentials, Lipschitz spaces, Besov spaces, Heat kernel. 2000 AMS classification: 43A22, 43A32, 43B25.

I. INTRODUCTION

Let G be a connected, simply connected, compact semisimple Lie group of dimension n. The main purpose of this paper is to establish boundedness of Riesz transforms R_j $(j=1,2,\cdots,n)$ and Riesz potentials I_z on the Lipschitz spaces $\Lambda_\alpha^{p,q}$ $(\alpha>0, 1\le p,q\le\infty)$ on G. These operators were studied by E.M. Stein in [8], by using the heat kernel W_t on G. It was proved by Cowling, Mantero and Ricci in [6] that the Riesz transforms R_j are Calderón-Zygmund operators. Thus a standard argument shows that R_j are bounded operators on the Lebesgue space $L^p(G)$ for any $1<p<\infty$. Also, it is well-known that the Riesz transforms R_j are not bounded on the Lebesgue spaces $L^1(G)$ and $L^\infty(G)$. Our results (see Theorem 2) in this paper, however, seem little surprising, since we will show that on a compact Lie group G, the Riesz transforms R_j are bounded on the Lipschitz spaces $\Lambda_\alpha^{p,q}(G)$ for any $1\le p\le\infty$ if $\alpha>0$. Our main results are stated in the following theorems.

Theorem 1 Let $\mathrm{Re}(z) = \beta/2 > 0$. Then the Riesz potential $I_z = (-\Delta)^{-z}$ is a linear operator that maps $\Lambda_\alpha^{p,q}$ to $\Lambda_{\alpha+\beta}^{p,q}$ boundedly.

Theorem 2 The Riesz transforms are bounded on $\Lambda_\alpha^{p,q}$.

Theorem 3 Suppose that $\mathrm{Re}(z) = \beta/2 < 0$ and $\alpha + \beta > 0$. Then I_z is a linear bounded operator from $\Lambda_\alpha^{p,q}$ to $\Lambda_{\alpha+\beta}^{p,q}$.

Our proof uses a heat kernel characterization of the Lipschitz space obtained by Meda and Pini in [7], the semi-group property of the heat kernel and an estimate on the heat kernel in [6]. This proof might be a new one even in the classical case.

For the historic development of the Riesz potential on both classical case and Lie groups, one can refer [8],[9],[10] and the references therein. We also want to mention some recent articles [2],[3],[4] about harmonic analysis on compact Lie groups.

This paper is organized as following. In the second section, we will present some necessary notations on Lie group and definitions of operators and spaces that will be studied in the paper. We will introduce some lemmas in the third section and present the proofs of the theorems in Section 4. In this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \le CB$. We use the notation $A \approx B$ to mean that there are two positive constants c_1 and c_2 independent of all essential variables such that $c_1 A \le B \le c_2 A$.

II. NOTATIONS AND DEFINITIONS.

Let G be a connected, simply connected, compact semisimple Lie group of dimension n. Let g be the Lie algebra of G and τ the Lie algebra of a fixed maximal torus T in G of dimension m. Let A be a system of positive roots for (g, τ), so that Card(A) = (n-m)/2 and let $\delta = \sum_{\alpha\in A}\alpha$.

Let $|\cdot|$ be the norm of g induced by the negative of the Killing form B on $g^{\mathbb{C}}$, the complexification of g, then $|\cdot|$ induces a bi-invariant metric d on G. Furthermore, since $B|_{\tau^{\mathbb{C}}\times\tau^{\mathbb{C}}}$ is nondegenerate, given $\lambda\in\hom_{\mathbb{C}}(\tau^{\mathbb{C}},\mathbb{C})$, there is a unique H_λ in $\tau^{\mathbb{C}}$ such that $\lambda(H) = B(H, H_\lambda)$ for each $H\in\tau^{\mathbb{C}}$. We let <,> and

Manuscript received October 18, 2010, revised August 8, 2012. This work is partially supported by the NSF of China, (Grant No. 10931001, 10871173).

F. Daning Chen is with the Department of Mathematics at Jackson State University, Jackson, Mississippi 39217,USA. (e-mail: chendn@yahoo.com)

S. Jiecheng Chen is with the Department of Mathematics at Zhejiang University, Hangzhou , China. (e-mail: jcchen@zju.edu.cn)

T. Dashan Fan is with Department of Mathematics at University of Wisconsin-Milwaukee, Milwaukee, WI 53217, USA. (e-mail: fan@uwm.edu)

(Advance online publication: 27 August 2012)

$\|\cdot\|$ denote the inner product and norm transferred from τ to $\hom_{\mathbb{C}}(\tau, i\mathbb{R})$ by means of this canonical isomorphism.

Let $\mathbb{N} = \{H \in \tau, \exp H = I\}$, I being the identity in G. The weight lattice P is defined by $P = \{\lambda \in \tau : \langle \lambda, n \rangle \in 2\pi\mathbb{Z} \text{ for any } n \in \mathbb{N}\}$ with dominant weights defined by $\Lambda = \{\lambda \in P : \langle \lambda, \alpha \rangle \geq 0 \text{ for any } \alpha \in A\}$. Λ provides a full set of parameters for the equivalent classes of unitary irreducible representation of G: for λ∈Λ, the representation U_λ has dimension

$$d_\lambda = \prod_{\alpha \in A} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}$$

and its associated character is

$$\chi_\lambda(\xi) = \frac{\sum_{w \in W} 7(w) e^{i\langle w(\lambda+\delta), \xi \rangle}}{\sum_{w \in W} e^{i\langle w\delta, \xi \rangle}}$$

where $\xi \in \tau$, W is the Weyl group and $\varepsilon(w)$ is the signature of $w \in W$. Let $X_1, X_2, \ldots, X_n$ be an orthonormal basis of g. Form the Casimer operator

$$\Delta = \sum_{i=1}^{n} X_i^2 .$$

This is an elliptic bi-invariant operator on G which is independent of the choice of orthonormal basis of g. The solution of the heat equation on $G \times \mathbb{R}^+$

$$\Delta\Phi(x,t) = \frac{d\Phi}{dt}(x,t), \quad \Phi(x,0) = f(x)$$

for $f \in L^1(G)$ is given by $\Phi(x,t) = W_t * f(x)$, where W_t is the Gauss-Weierstrass kernel (heat kernel). It is well known that W_t is a central function and one can write it as, for $\xi \in \tau$ and t > 0,

$$W_t(\xi) = \sum_{\lambda \in \Lambda} e^{-t\left(\|\lambda+\delta\|^2 - \|\delta\|^2\right)} d_\lambda \chi_\lambda(\xi).$$

It is easy to see that W_t satisfies the semi-group property $W_{t+s} = W_t * W_s$ for any s, t > 0.

Using the heat kernel, one can defines various useful function spaces on G. One of such spaces is the Besov space $B_\alpha^{p,q}(G)$ defined in the following.

For any k > α ≥ 0, and 1 ≤ p ≤ ∞, we say that a function f is in the homogeneous Besov space $hB_\alpha^{p,q}(G)$ if the $hB_\alpha^{p,q}(G)$ norm of f

$$\|f\|_{hB_\alpha^{p,q}(G)} = \sum_{|I|=k} \left(\int_0^{2^{-n_0}} \left(t^{\frac{k-\alpha}{2}} \left\| X^I W_t * f \right\|_{L^p(G)} \right)^q t^{-1} dt \right)^{\frac{1}{q}} < \infty$$

when q ≠ ∞; and

$$\|f\|_{hB_\alpha^{p,q}(G)} = \sum_{|I|=k} \sup_{0<t\leq 2^{-n_0}} t^{k-\alpha} \left\| X^I W_t * f \right\|_{L^p(G)} < \infty$$

when q = ∞, where $X^I = X_1^{j_1} X_2^{j_2} \ldots X_n^{j_n}$ with the multi-index $I = (j_1, j_2, \ldots j_n)$.

We say that an L^p function f is in the Besov space $B_\alpha^{p,q}(G)$ if

$$\|f\|_{B_\alpha^{p,q}(G)} = \|f\|_{L^p(G)} + \|f\|_{hB_\alpha^{p,q}(G)}, \ (1 \leq p \leq \infty)$$

is finite.

Remark 4 (i) In the definition of the Besov spaces, the sum is taken over all the differential monomials of order k, and n_0 is a fixed big number. It is easy to check that in the definition of the Besov spaces, one can use the number n_0 - 1 to replace n_0.

(ii) One can pick any k > α in the definition. The Besov norms obtained from different k are equivalent.

Another important function space is the Lipschitz space $\Lambda_\alpha^{p,q}$ on G. Let k be a positive integer. For $f \in L^p(G)$ and for every element V ∈ g, with |V| =1, we define the k-th order difference operator centered at x with direction V by

$$\Delta_{sV}^k f(x) = \sum_{j=1}^{k} (-1)^k \binom{k}{j} f(x \exp(jsV)), \ s > 0.$$

The k-th order L^p-modulus of smoothness of f is the function

$$\varpi_k(t, f, L^p) = \sup_{|V|=1, 0<s<t} \left\| \Delta_{sV}^k f \right\|_{L^p}.$$

Let α ≥ 0. We say that a function $f \in L^p(G)$, 1 ≤ p ≤ ∞, is in the Lipschitz space $\Lambda_\alpha^{p,q}$, if the norm

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|f\|_{L^p(G)} + \left\{ \int_0^1 \left(t^{-\alpha} \varpi_k(t, f, L^p) \right)^q t^{-1} dt \right\}^{\frac{1}{q}}$$

if 1 ≤ q < ∞, and

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|f\|_{L^p(G)} + \sup_{0<t\leq 1} t^{-\alpha} \varpi_k(t, f, L^p) \ (\text{if } q = \infty)$$

is finite.

The Lipschitz space plays an important role in studying harmonic analysis on Lie groups. It is known in [1] that one can use the Lipschitz space to characterize the Hardy space H^p on a compact Lie group. Also, Meda and Pini proved that the Besov norm and the Lipschitz norm are equivalent (see [7]), therefore $\Lambda_\alpha^{p,q}(G) = B_\alpha^{p,q}(G)$.

Next, we recall that one can use the heat kernel to define Riesz transforms and Riesz potentials on G. The following definitions can be found in Stein [8]. The Riesz potential $I_z = (-\Delta)^{-z}$, $z \in \mathbb{C}, \operatorname{Re}(z) > 0$, is defined by

$$(-\Delta)^{-z} f = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1} W_t' * f dt.$$

where

$$W_t'(\xi) = \sum_{\lambda \in \Lambda \setminus \{0\}} e^{-t\left(\|\lambda+\delta\|^2 - \|\delta\|^2\right)} d_\lambda \chi_\lambda(\xi)$$

Thus, it is easy to see that

$$\int_G W_t'(x)\,dx = 0.$$

We can extend the definition of I_z to the complex plane by using the formula

$$(-\Delta)^{-z} = -\Delta(-\Delta)^{-z-1}.$$

We are interested in the particular case $z = 1/2$ and define the Riesz transforms $R_j\ (j = 1, 2, \ldots, n)$ by

$$R_j f = \pi^{-\frac{1}{2}}\int_0^\infty t^{-\frac{1}{2}} X_j W_t * f dt.$$

When in the case $z = i\lambda,\ \lambda \in \mathbb{R}\setminus\{0\}$,

$$(-\Delta)^{-i\gamma} f = \frac{-1}{\Gamma(1+i\lambda)}\int_0^\infty t^{i\gamma}\Delta W_t * f dt.$$

In [6], both operators $(-\Delta)^{-i\gamma}$ and R_j are shown to be Calderón-Zygmund operators. Combining that with the L^2 boundedness proved by E.M. Stein [8], it follows by a standard method of Calderón-Zygmund decomposition that both operators $(-\Delta)^{-i\gamma}$ and R_j are strong type (p, p), $1 < p < \infty$, and of weak type (1, 1). In this paper, we are also interested in the Riesz potentials $(-\Delta)^{-i\gamma+\frac{\alpha}{2}}$, $\alpha > 0$.

Some Lemmas

Let $H^{2,s}$ be the Sobolev space of functions f on G for which any $X_{j_1}, X_{j_2}, \ldots X_{j_s} \in g$, $\prod_{k=1}^s X_{j_k} f \in L^2(G)$. A norm on the subspace of central functions in $H^{2,s}$ is

$$\|f\|_{H^{2,s}} = \left\{\sum_{\lambda\in\Lambda} |f_\lambda|^2 \|\lambda+\delta\|^{2s} d_\lambda\right\}^{\frac{1}{2}}$$

where f_λ are the Fourier coefficients of f. Since the heat kernel W_t is a central function, we have the following estimate of W_t.

Lemma 5 For any multi-index I and any $1 \le p \le \infty$, $\|X^I W_t\|_{L^p(G)} \preceq t^{-N}$ for any $N > 0$ uniformly for $t > \sigma$.

Proof. Use the Hölder's inequality, semi group property of W_t and the left invariance of X, one has

$$\begin{aligned}\|X^I W_t\|_{L^p(G)} &\preceq \|X^I W_t\|_{L^\infty(G)} \\ &\preceq \|X^I W_{t/2} * W_{t/2}\|_{L^\infty(G)} \preceq \|X^I W_{t/2}\|_{L^2(G)} \|W_{t/2}\|_{L^2(G)} \\ &\approx \|W_{t/2}\|_{H^{2,s}(G)} \|W_{t/2}\|_{L^2(G)},\end{aligned}$$

with $s = |I|$.

Thus, the lemma follows easily from the definition of W_t.

By the Poisson summation formula (see [5], or [6]), we know that

$$W_t(\xi) = \frac{e^{t\|\rho\|^2} t^{-\frac{m}{2}}}{D(\xi)} \sum_{\lambda\in\mathbb{N}}\left(\prod_{\alpha\in A}\langle \xi+\lambda, \alpha\rangle e^{-\frac{\|\lambda+\xi\|^2}{4t}}\right),$$

where

$$D(\xi) = \sum_{w\in W} e^{i\langle w\delta, \xi\rangle}.$$

Using this expression of the heat kernel, we can obtain the following estimate.

Lemma 6 For any multi-index I with $|I| = k$, $\left\|t^{\frac{k}{2}} X^I W_t\right\|_{L^1(G)} \preceq 1$ uniformly for $t > 0$.

Proof: By Lemma 5, we may assume that t is small. Let

U be a neighborhood of 0 in τ such that it translates by elements of Λ are all disjoint, and let η(x) be a C^∞ function supported on U, radial and identically one on a neighborhood of 0. One defines two modified kernels K_t and V_t by

$$V_t(\xi) = e^{2t\|\rho\|^2} t^{-\frac{n}{2}} \sum_{\lambda \in \mathbb{N}} e^{-\frac{\|\lambda+\xi\|^2}{4t}},$$

$$K_t(\xi) = e^{2t\|\rho\|^2} t^{-\frac{n}{2}} \sum_{\lambda \in \mathbb{N}} \eta(\xi+\lambda) e^{-\frac{\|\lambda+\xi\|^2}{4t}}.$$

By Theorem 4 of [6], it is known that for any pair of integers s and N,

$$\|V_t - K_t\|_{H^{2,s}(G)} = O(t^N), \quad t \to 0.$$

Also, by Theorem 2 of [6], we know that given any pair of integers s and N, there is an integer L such that

$$\|\Delta_{L,t} V_t - W_t\|_{H^{2,s}(G)} = O(t^N), \quad t \to 0,$$

where

$$\Delta_{L,t} = \sum_{j=0}^{M} t^j D_{j,L}, M = \frac{L(n-m)}{2}$$

and $D_{j,L}$, j = 0,1,...,M, are differential operators of order j, which are invariant under both left and right translations. Thus, for $t \le \sigma$, we have

$$\|X^I W_t\|_{L^1(G)} \preceq \|X^I (W_t - \Delta_{L,t} V_t)\|_{L^2(G)}$$

$$\preceq \|X^I \Delta_{L,t}(V_t - K_t)\|_{L^2(G)} + \|X^I \Delta_{L,t} K_t\|_{L^1(G)}$$

$$\preceq \|V_t - K_t\|_{H^{2,s}(G)} + \|\Delta_{L,t} V_t - W_t\|_{H^{2,r}(G)} + \|X^I \Delta_{L,t} K_t\|_{L^1(G)}$$

for some suitable integers s and r. Therefore, for $0 < t \le \sigma$,

$$\|X^I W_t\|_{L^1(G)} \preceq 1 + \|X^I \Delta_{L,t} K_t\|_{L^1(G)}.$$

Recalling that the function K_t, considered as a function on G, is supported on a small neighborhood of I of G, one introduces on this neighborhood the regular coordinates $(\xi_1, \ldots, \xi_n)$, where $(\xi_1, \ldots, \xi_n) \to \exp\left(\sum_{j=1}^n \xi_j X_j\right)$.

In this coordinates,

$$K_t(\xi) = e^{2t\|\rho\|^2} t^{-\frac{n}{2}} \eta(\xi) e^{-\frac{\|\xi\|^2}{4t}}$$

By the proof of the Lemma 5 in [6], it is easy to see that

$$\left|X^I \Delta_{L,t} K_t(\xi)\right| \preceq t^{-\frac{n}{2}-|I|} \|\xi\|^{|I|} e^{-\frac{\|\xi\|^2}{4t}}.$$

Thus

$$\left| t^{\frac{k}{2}} X^I W_t(\xi) \right| \preceq t^{-\frac{n}{2}-\frac{|I|}{2}} \|\xi\|^{|I|} e^{-\frac{\|\xi\|^2}{4t}} + O(1)$$

is an integrable kernel on G. The lemma is proved.

By the proof of Lemma 6, it is easy to obtain the following estimate. We skip the proof of it.

Proposition 7 Let $0 < t \le 1$. Then

$$\left|X^J I_z W_t(\xi)\right| \preceq \left\{\|\xi\|^2 + t\right\}^{\frac{\beta-\eta-|J|}{2}}$$

for any multi-index J with $|J| \ge \beta = 2\mathrm{Re}(z)$.

Proofs of Theorems

Proof of Theorem 1

Since the norms of Besov space and Lipschitz space are equivalent, it suffices to show our theorems in the Besov norms. For any $1 \le p \le \infty$, by the Minkowski inequality and Hölder's inequality,

$$\|I_z(f)\|_{L^p}$$

$$\preceq \|f\|_{L^p} \left\{ \int_0^1 t^{\frac{\beta}{2}} \|W_t'\|_{L^1(G)} t^{-1} dt + \int_0^1 t^{\frac{\beta}{2}} \|W_t'\|_{L^2(G)} t^{-1} dt \right\}.$$

Thus, by Lemma 5 and Lemma 6, we have

$$\|I_z(f)\|_{L^p} \preceq \|f\|_{L^p}.$$

Now, by the definition of the Besov norm, it suffices to show

$$\|I_z(f)\|_{hB^{p,q}_{\alpha+\beta}} \preceq \|f\|_{B^{p,q}_\alpha}.$$

First, we study the case $q = \infty$. It is easy to see that we only need to prove that if $k > \alpha + \beta$, then for any multi-index I with $|I| = k$, one has

$$(1) \quad \sup_{0<t\le 2^{-n_0}} t^{\frac{k-\alpha-\beta}{2}} \|X^I W_t * I_Z f\|_{L^p(G)}$$

$$\preceq \sum_{|I|=k} \sup_{0<t\le 2^{-n_0}} t^{\frac{k-\alpha}{2}} \|X^I W_t * f\|_{L^p(G)} + \|f\|_{L^p(G).}$$

Denote $2^{-n_0} = \sigma$. Using the semi-group property of W_t and the left invariance of X^I, we have

$$\left\| X^I W_t * I_Z f \right\|_{L^p(G)}$$

$$\preceq \left\{ \int_G \left| \int_\sigma^\infty s^{z-1} X^I W_{t+s} * f(x) ds \right|^p dx \right\}^{\frac{1}{p}}$$

$$+ \left\{ \int_G \left| \int_t^\sigma s^{z-1} X^I W_{t+s} * f(x) ds \right|^p dx \right\}^{\frac{1}{p}}$$

$$+ \left\{ \int_G \left| \int_0^t s^{z-1} X^I W_{t+s} * f(x) ds \right|^p dx \right\}^{\frac{1}{p}} = J_1 + J_2 + J_3.$$

For any $0 < t \le \sigma$, when $k > \alpha + \beta$, by the Minkowski inequality we see that $t^{\frac{k-\alpha-\beta}{2}} J_1$ is dominated by

$$\preceq \left\{ \int_G \left| \int_\sigma^\infty s^{-1-\frac{\beta}{2}} \left\| X^I W_{t+s} \right\|_\infty \left\| f \right\|_{L^1(G).} \right|^p ds dx \right\}^{\frac{1}{p}}.$$

Thus, by Lemma 5, we have

$$t^{\frac{k-\alpha-\beta}{2}} J_1 \preceq \left\| f \right\|_{L^p(G).}.$$

Let p' be the conjugate index of p. By Hölder's inequality, we have

$$J_3 \preceq \left\{ \int_G \left| \int_0^t s^{\frac{\beta}{2}-1} \left| X^I W_{t+s} * f(x) \right|^p \right| ds dx \right\}^{\frac{1}{p}} \left(\int_0^t s^{\frac{\beta}{2}-1} ds \right)^{\frac{1}{p'}}$$

$$\preceq t^{-\frac{k-\alpha-\beta}{2}} \sup_{0<t\le 2\sigma} t^{\frac{k-\alpha}{2}} \left\| X^I W_t * f \right\|_{L^p(G)}.$$

Similarly, by Hölder's inequality, the second integral J_2 is dominated by, up to a positive constant,

$$t^{-1+\frac{1}{p'}} \left\{ \int_t^\sigma \left\| X^I W_{t+s} * f \right\|_{L^p(G)}^p s^{\frac{p(k-\alpha)}{2}} s^{\frac{p(\alpha+\beta-k)}{2}} ds \right\}^{\frac{1}{p}}$$

$$\preceq t^{-\frac{k-\alpha-\beta}{2}} \sup_{0<s\le 2\sigma} t^{\frac{(k-\alpha)}{2}} \left\| X^I W_t * f \right\|_{L^p(G)}.$$

Combining the estimates on J_1, J_2 and J_3, we obtain the estimate (1), which proves the case $q = \infty$.

Next, we show the case $1 \le q < \infty$. Checking the proof for the case $q = \infty$, we only need to show that both integrals

$$(2) \left\{ \int_0^\sigma \left[t^{\frac{k-\alpha-\beta}{2}} \left(\int_G \left| \int_0^t s^{z-1} X^I W_{t+s} * f(x) ds \right|^p dx \right)^{\frac{1}{p}} \right]^q t^{-1} dt \right\}^{\frac{1}{q}},$$

$$(3) \left\{ \int_0^\sigma \left[t^{\frac{k-\alpha-\beta}{2}} \left(\int_G \left| \int_t^\sigma s^{z-1} X^I W_{t+s} * f(x) ds \right|^p dx \right)^{\frac{1}{p}} \right]^q t^{-1} dt \right\}$$

are bounded by $\left\| f \right\|_{B_\alpha^{p,q}(G)}$, up to a constant multiple. Using the Hölder Inequality, we easily obtain that the inside integral in (2)

$$\left(\int_G \left| \int_0^t s^{z-1} X^I W_{t+s} * f(x) ds \right|^p dx \right)^{\frac{1}{p}}$$

$$\preceq t^{\frac{\beta}{2}} \left\| W_s \right\|_{L^1(G)} \left\| X^I W_t * f \right\|_{L^p(G)}$$

which implies that the integral (2) is bounded by $C \left\| f \right\|_{B_\alpha^{p,q}(G)}$.

Now for $k > 2\alpha + 2\beta + 1$, and any I with $|I| = k$, we write $X^I = X^{I_1} X^{I_2}$ such that $|I_1| = k_1 > \alpha + \beta$, $|I_2| = k_2 > \alpha + (\beta/2) + 1$. Then the integral (3) is bounded by

$$\left\{ \int_0^\sigma t^{\frac{(k-\alpha-\beta)q}{2}} \left(\int_G \left(\int_t^\sigma s^{-p'} ds \right)^{\frac{p}{p'}} \left(\int_t^\sigma s^{\frac{p\beta}{2}} \left| X^I W_{t+s} * f(x) \right|^p ds \right) dx \right)^{\frac{q}{p}} t^{-1} dt \right\}^{\frac{1}{q}}$$

$$\preceq \left\{ \int_0^\sigma t^{\frac{2(k-\alpha-\beta)q}{2}} t^{p\left(\frac{1}{p'}-1\right)} \left\| X^{I_2} W_t * f \right\|_p^q \left(\int_t^\sigma s^{\frac{p\beta}{2}} \left\| X^{I_1} W_s \right\|_{L^1}^p ds \right)^{\frac{p}{q}} t^{-1} dt \right\}^{\frac{1}{q}}$$

$$\preceq \left\{ \int_0^\sigma t^{\frac{2(k_2-\alpha)q}{2}} \left\| X^{I_2} W_t * f \right\|_p^q t^{-1} dt \right\}^{\frac{1}{q}} \preceq \left\| f \right\|_{B_\alpha^{p,q}(G)}.$$

The proof of the theorem is completed.

Proof of Theorem 2

By checking the proof of Theorem 1, one easily sees that

$$\left\| R_j f \right\|_{hB_\alpha^{p,q}(G)} \preceq \left\| f \right\|_{B_\alpha^{p,q}(G)}.$$

To prove the theorem, now it suffices to show that

$$\left\| R_j f \right\|_{L^p(G)} \preceq \left\| f \right\|_{B_\alpha^{p,q}(G)}.$$

By the L^p boundedness of R_j for $1 < p < \infty$ proved in [CM], clearly we only need to discuss the cases $p = 1$ and $p = \infty$. If $p = 1$ and $q = \infty$, by Lemma 5, it is easy to see that

$$\left\|R_j f\right\|_{L^1(G)} \preceq \int_G \left|\int_0^\sigma t^{\frac{-1}{2}} X_j W_t * f(x)\,dt\right| dx + \|f\|_{L^1(G)}.$$

Using Integrate by part; the first term above is bounded by, (up to a constant multiple),

$$\int_G \left|\lim_{t\to 0} t^{\frac{1}{2}} X_j W_t * f(x)\right| dx$$

$$+\int_G \left|\int_0^\sigma t^{\frac{1}{2}} \frac{\partial}{\partial t} X_j W_t * f(x)\,dt\right| dx + \|f\|_{L^1(G)}.$$

By Lemma 6,

$$\int_G \left|\lim_{t\to 0} t^{\frac{1}{2}} X_j W_t * f(x)\right| dx$$

$$\preceq \liminf_{t\to 0} \int_G \left|t^{\frac{1}{2}} X_j W_t * f(x)\right| dx$$

$$\liminf_{t\to 0} \left\|t^{\frac{1}{2}} X_j W_t * f(x)\right\|_{L^1(G)} \|f\|_{L^1(G)} \preceq \|f\|_{L^1(G)}.$$

Also,

$$\left|\int_0^\sigma t^{\frac{1}{2}} \frac{\partial}{\partial t} X_j W_t * f(x)\,dt\right| dx$$

$$= \int_G \left|\int_0^\sigma t^{\frac{1}{2}} \Delta X_j W_t * f(x)\,dt\right| dx.$$

Integrating by parts k times for an even $k > \alpha$, we get

$$\int_G \left|\int_0^\sigma t^{\frac{1}{2}} \Delta X_j W_t * f(x)\,dt\right| dx$$

$$\preceq \|f\|_{L^1(G)} + \int_G \left|\int_0^\sigma t^{\frac{(k+1-\alpha)}{2}} t^{\frac{\alpha}{2}} \Delta^{\frac{k}{2}} X_j W_t * f(x)\,dt\right| dx$$

$$\preceq \|f\|_{L^1(G)} + \|f\|_{B_\alpha^{1,\infty}(G)} \int_0^\sigma t^{\frac{\alpha}{2}} dt \preceq \|f\|_{B_\alpha^{1,\infty}(G)}.$$

In the same way, we can prove the case $p = q = \infty$.

Next, we consider the case $p = 1$ and $1 \le q < \infty$. In this case, we have

$$\left\|R_j f\right\|_{L^1(G)} \preceq \|f\|_{L^1(G)}$$

$$+\left(\left|\int_0^\sigma \left\{t^{\frac{(k+1-\alpha)}{2}} \left\|\Delta^{\frac{k}{2}} X_j W_t * f\right\|_{L^1(G)}\right\}^q t^{-1} dt\right| dx\right)^{\frac{1}{q}}$$

$$\preceq \|f\|_{B_\alpha^{1,q}(G)}.$$

The proof for the case $p = \infty$ and $1 < q < \infty$ is similar and is skipped. The theorem is proved.

Proof of Theorem 3.

Without loss of generality, we may assume $z = -\beta/2$. First, we prove that

$$(4)\quad \|I_z f\|_{L^p(G)} \le C\|f\|_{B_\alpha^{p,q}(G)}$$

for any $f \in B_\alpha^{p,q}(G)$. By the definition of the Riesz potential, it is easy to see that

$$I_z f(x) \approx \int_0^\infty t^{k-\frac{\beta}{2}} \Delta^k W_t * f(x) t^{-1} dt$$

for some k satisfying $1 \ge k - \alpha/2 > 0$. By Lemma 5, it is easy to see that

$$\|I_z f\|_{L^p(G)} \preceq \left\|\int_0^\sigma t^{k-\frac{\beta}{2}-1} \Delta^k W_t * f dt\right\|_{L^p(G)} + \|f\|_{L^p(G)}.$$

When $q = \infty$ and $p \ne \infty$, let ε be a small positive number such that $p\varepsilon$ is sufficiently small in a way to be determined. Then

$$\left|\int_0^\sigma t^{k-\frac{\beta}{2}} \Delta^k W_t * f(x) t^{-1} dt\right|$$

$$\preceq \left\{\int_0^\sigma t^{\left(k-\frac{\beta}{2}\right)p-1-p\varepsilon} \left|\Delta^k W_t * f(x)\right|^p dt\right\}^{\frac{1}{p}}.$$

Thus, up to a positive constant, $\|I_z f\|_{L^p(G)}^p$ is dominated by

$$\sup_{0<s\le\sigma}\left(s^{k-\frac{\alpha}{2}} \left\|\Delta^k W_s * f\right\|_{L^p(G)}\right)^p \int_0^\sigma t^{p\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)-1-p\varepsilon} dt + \|f\|_{L^p(G)}$$

We now have

$$\|I_z f\|_{L^p(G)} \preceq \|f\|_{B_\alpha^{p,\infty}(G)}$$

by choosing a small $\varepsilon > 0$.

When $q = \infty$ and $p = \infty$, $\|I_z f\|_{L^\infty(G)}$ is bounded by

$$\|f\|_{L^\infty(G)} + C\int_0^\sigma t^{k-\frac{\beta}{2}-1} \left\|\Delta^k W_t * f\right\|_{L^\infty(G)} dt$$

$$\preceq \|f\|_{B_\alpha^{\infty,\infty}(G)} \left\{1 + \int_0^\sigma t^{\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)-1} dt\right\} \preceq \|f\|_{B_\alpha^{\infty,\infty}(G)}.$$

When $1 \le q < \infty$ and $1 \le p \le \infty$. Using the Minkowski inequality, we have

$$\|I_z f\|_{L^p(G)} \preceq \|f\|_{L^p(G)} + \int_0^\sigma t^{k-\frac{\beta}{2}-1} \left\|\Delta^k W_t * f\right\|_{L^p(G)} dt.$$

For $q=1$,

$$\int_0^\sigma t^{k-\frac{\beta}{2}-1} \left\| \Delta^k W_t * f \right\|_{L^p(G)} dt$$
$$\preceq \int_0^\sigma t^{k-\frac{\alpha}{2}-1} \left\| \Delta^k W_t * f \right\|_{L^p(G)} dt \approx \| f \|_{hB_\alpha^{p,1}(G)} .$$

For $1 < q < \infty$, we use Hölder's inequality to obtain

$$\| I_z f \|_{L^p(G)} \le C_1 \| f \|_{L^p(G)}$$
$$+C_2 \left(\int_0^\sigma t^{q\left(k-\frac{\alpha}{2}\right)-1} \left\| \Delta^k W_t * f \right\|_{L^p(G)}^q dt \right)^{\frac{1}{q}} \approx \| f \|_{hB_\alpha^{p,q}(G)} ,$$

where

$$C_2 = \left(\int_0^\sigma t^{q'\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)-1} dt \right)^{\frac{1}{q'}} .$$

Thus (4) is proved. To complete the proof of Theorem 3, it remains to show that

$$\| I_z f \|_{hB_{\alpha+\beta}^{p,q}(G)} \le C \| f \|_{B_\alpha^{p,q}(G)} .$$

We skip the proof, since it is similar to those of Theorem 1.

References

[1] B. Blank and D. Fan, Hardy spaces on compact Lie groups, Ann. Fac. Sci. Toulouse Math. 6 (1997), No. 3, 429-479.
[2] J. Chen and D. Fan, Central Multipliers on Compact Lie Groups, Math Z. DOI:10,1007/s00209-009-0618-4.
[3] J. Chen and D. Fan, A Bilinear Fractional Integral on Compact Lie Groups, to appear in Canad. Math. Bull.
[4] J. Chen, D. Fan and L. Sun, Hardy Spaces Estimates for Wave Equation on Compact Lie Groups, Jour. Func. Analysis, 259 (2010), 3230-3264.
[5] J.L. Clerc, Bochner-Riesz means of H^{p} functions (0<p<1), Lecture Notes in Math. 1234, (1987), 86-107.
[6] M. Cowling, A.M. Mantero and F. Ricci, Pointwise estimates for some kernels on compact Lie groups, Rend. Circ. Mat. Palerma, XXXI (1982), 145-158.
[7] S. Meda and R. Pini, Lipschitz spaces on compact Lie groups, Monatsh Math. 105 (1988), 177-191.
[8] E.M. Stein, Topics in Harmonic Analysis, Ann. of Math. Studies #63, (1970), Princeton University Press, Princeton, NJ.
[9] E.M. Stein, Harmonic Analysis: Real Variable Method, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, New Jersey, 1993.
[10] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, New Jersey, 1970.