Various Notions about Constructive Brouwer's Fixed Point Theorem

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Abstract—**In this paper we examine the relationships among several notions about Brouwer's fixed point theorem for uniformly continuous functions from a simplex into itself in the framework of constructive mathematics à la Bishop. We compare the notions such as**

(A) a function has at most one fixed point,
(B) a function is sequentially locally non-constant,
(C) a function has locally strong fixed point,
(D) a function has locally sequentially at most one fixed point.

(A) is not sufficient to Brouwer's fixed point theorem to hold for uniformly continuous functions on a simplex. We need Brouwer's fan theorem. (B) is sufficient to Brouwer's fixed point theorem, but somewhat too strong. (D) is a weaker version of (B), but it is sufficient to Brouwer's fixed point theorem.

The main conclusion of this paper is that (C) is equivalent to (D), that is, any uniformly continuous function has a locally strong fixed point if and only if it has locally sequentially at most one fixed point.

Index Terms—**Brouwer's fixed point theorem, Locally strong fixed point, Locally sequentially at most one fixed point, Constructive mathematics.**

I. Introduction

In the previous paper [1] published in *IAENG International Journal of Applied Mathematics* we have presented a constructive proof of Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions from a compact metric space, for example, an n-dimensional simplex into itself. Uniform sequential continuity of functions assumed in [1] is equivalent to uniform continuity in classical mathematics, but the former is strictly weaker than the latter in constructive mathematics. Thus, sequential local non-constancy of functions is also sufficient to Brouwer's fixed point theorem to hold for uniformly continuous functions from a simplex into itself.

> In another paper [2] also published in *IAENG International Journal of Applied Mathematics* we have presented a constructive proof of an approximate version of the Fan-Glicksberg fixed point theorem for multi-functions (multi-valued functions) in a locally convex space, that is, the existence of an approximate fixed point of multi-functions in a locally convex space.

Sequential local non-constancy, however, may be too strong. In this paper we examine the relationships among several notions about Brouwer's fixed point theorem for uniformly continuous functions from a simplex into itself in the framework of constructive mathematics à la Bishop ([3], [4], [5]). We compare the following notions

(A) a function has at most one fixed point ([6]),
(B) a function is sequentially locally non-constant ([1]),
(C) a function has locally strong fixed point,
(D) a function has locally sequentially at most one fixed point.

(A) is not sufficient to Brouwer's fixed point theorem to hold for uniformly continuous functions on a simplex. We need Brouwer's fan theorem. (B) is sufficient to Brouwer's fixed point theorem, but somewhat too strong. To define (C) we refer to the strong maximum in [7] and the strong minimum in [8]. (D) is our original notion. It is a weaker version of (B), but it is sufficient to Brouwer's fixed point theorem. About (C) and (D) we will show the following results.

1) The existence of a locally strong fixed point of a uniformly continuous function means the existence of a fixed point which is locally unique, and also means that the function has locally sequentially at most one fixed point.
2) A uniformly continuous function which has locally sequentially at most one fixed point has a fixed point, and it means the existence of a locally strong fixed point.

Thus any uniformly continuous function has a locally strong fixed point if and only if it has locally sequentially at most one fixed point.

We require uniform continuity to functions rather than uniform sequential continuity because the former is more popular than the latter, and the main theme of this paper is comparison of the above-mentioned notions, not continuity of functions.

II. Brouwer's fixed point theorem with locally strong fixed point in constructive mathematics

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [5]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. First define finite enumerability of a set and an ε-approximation to a set. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \ldots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $x \in S$ there exists y in that ε-approximation with $|x-y| < \varepsilon$ ($|x-y|$ is the distance between x and y). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

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Let us consider an n-dimensional simplex Δ as a compact metric space. About a totally bounded set, according to Corollary 2.2.12 in [5], we have the following result.

Lemma 1: For each $\varepsilon > 0$ there exist totally bounded sets $H_1, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^{n} H_i$.

Let $x = (x_0, x_1, \dots, x_n)$ be a point in Δ with $n \geq 2$, and consider a function f from Δ into itself.

Uniform continuity of functions is defined as follows;

Definition 1 (Uniform continuity): A function f is uniformly continuous in Δ if for any $x, x' \in \Delta$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{If } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \varepsilon.$$

δ depends on only ε not on x.

If f is a uniformly continuous function from Δ into itself, according to [9] and [10] it has an approximate fixed point. This means

For each $\varepsilon > 0$ there exists $x \in \Delta$ such that $|x - f(x)| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{x \in \Delta} |x - f(x)| = 0.$$

By Lemma 1 we have $\cup_{i=1}^{n} H_i = \Delta$, where n is a finite number. Since H_i is totally bounded for each i, $|x - f(x)|$ has the infimum in H_i because of its uniform continuity. Thus, we can find $H_i (1 \leq i \leq n)$ such that the infimum of $|x - f(x)|$ in H_i is 0, that is,

$$\inf_{x \in H_i} |x - f(x)| = 0,$$

for some i such that $\cup_{i=1}^{n} H_i = \Delta$.

The notion that f has at most one fixed point in [6] is defined as follows;

Definition 2 (At most one fixed point): For all $x, y \in \Delta$, if $x \neq y$, then $f(x) \neq x$ or $f(y) \neq y$.

[6] has shown that the following theorem is equivalent to Brouwer's fan theorem.

> Each uniformly continuous function from a compact metric space into itself with at most one fixed point and approximate fixed points has a fixed point.

Thus, this theorem is non-constructive.

Next, by reference to the notion of *sequentially at most one maximum* in [7], we define the property of *sequential local non-constancy* as follows;

Definition 3 (Sequential local non-constancy): There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^{m} H_i$, and if for all sequences $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1}$ in each H_i, $|x_n - f(x_n)| \longrightarrow 0$ and $|y_n - f(y_n)| \longrightarrow 0$, then $|x_n - y_n| \longrightarrow 0$.

In [1] we have presented a constructive proof of Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions. This result implies that we can constructively prove Brouwer's fixed point theorem for sequentially locally non-constant and uniformly continuous functions.

Further we define the notion that a function has *locally sequentially at most one fixed point*. It is a weaker version of sequential local non-constancy. The definition is as follows;

Definition 4 (Locally sequentially at most one fixed point): There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^{m} H_i$, and for at least one i such that $\inf_{x \in H_i} f(x) = 0$ if for all sequences $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1}$ in H_i, $|x_n - f(x_n)| \longrightarrow 0$ and $|y_n - f(y_n)| \longrightarrow 0$, then $|x_n - y_n| \longrightarrow 0$.

A fixed point of a function f is a point x^* such that $x^* = f(x^*)$ or $|x^* - f(x^*)| = 0$. In addition, by reference to the strong maximum in [7] or the strong minimum in [8], we define a locally strong fixed point of f as follows.

Definition 5 (Locally strong fixed point): Let H_i be a set such that $\cup_{i=1}^{n} H_i = \Delta$ and $\inf_{x \in H_i} f(x) = 0$. By a locally strong fixed point in H_i we mean a point $\tilde{x}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in H_i$ and $|x - f(x)| < |\tilde{x} - f(\tilde{x})| + \delta$, then $|x - \tilde{x}| < \varepsilon$.

About locally strong fixed points we get the following results.

Theorem 1: Let $\tilde{x}$ be a locally strong fixed point of f in H_i. Then,

1) $\tilde{x}$ is the unique fixed point in H_i. I call such a fixed point locally unique.
2) f has locally sequentially at most one fixed point.

Proof:

1) If $x \in H_i$ and $|x - f(x)| \leq |\tilde{x} - f(\tilde{x})|$, then $|x - \tilde{x}| < \varepsilon$ for each $\varepsilon > 0$, so $x = \tilde{x}$. Thus, $\tilde{x}$ is the unique fixed point in H_i.
2) Let $(\varepsilon_n)_{n \geq 1}$ be a decreasing sequence with $\varepsilon_n > 0$ for each n and $\varepsilon_n \longrightarrow 0$. Choose any sequence $(x_n)_{n \geq 1}$ in H_i such that $|x_n - f(x_n)| < \varepsilon_n$. Then, we have $|x_n - \tilde{x}| \longrightarrow 0$, and so $|x_n - x'_n| \longrightarrow 0$ for any such sequences $(x_n)_{n \geq 1}$ and $(x'_n)_{n \geq 1}$ in H_i. Therefore, f has locally sequentially at most one fixed point.

■

Now we show the following lemma.

Lemma 2: Let H_i be a set such that $\cup_{i=1}^{n} H_i = \Delta$ and $\inf_{x \in H_i} f(x) = 0$, and f be a uniformly continuous function from an n-dimensional simplex into itself. If the following property holds:

> For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in H_i$, $|x - f(x)| < \varepsilon$ and $|y - f(y)| < \varepsilon$, then $|x - y| \leq \delta$.

Then, f has a fixed point in H_i and has a locally strong fixed point in H_i.

Proof: Choose a sequence $(x_n)_{n \geq 1}$ in H_i such that $|x_n - f(x_n)| \longrightarrow 0$. Compute N such that $|x_n - f(x_n)| < \delta$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|x_m - x_n| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(x_n)_{n \geq 1}$ is a Cauchy sequence in H_i, and converges to a limit $\tilde{x} \in H_i$. The continuity of f yields $|\tilde{x} - f(\tilde{x})| = 0$, that is, $f(\tilde{x}) = \tilde{x}$, and so $\tilde{x}$ is a fixed point of f. By the above property if $|x - f(x)| < \varepsilon$ and $|\tilde{x} - f(\tilde{x})| = 0$, then $|x - \tilde{x}| \leq \delta$. $\tilde{x}$ is a locally strong fixed point. ■

Finally we show the following theorem.

Theorem 2: Let f be a uniformly continuous function from an n-dimensional simplex Δ into itself. Assume $\inf_{x\in H_i}|x-f(x)|=0$ for some $H_i\subset\Delta$ defined above. If f has locally sequentially at most one fixed point, then f has a locally strong fixed point.

Proof: Choose a sequence $(z_m)_{m\geq 1}$ in H_i such that $|f(z_m)-z_m|\longrightarrow 0$. In view of Lemma 2, it is enough to prove that the following property holds.

For each $\delta>0$ there exists $\varepsilon>0$ such that if $x,y\in H_i$, $|x-f(x)|<\varepsilon$ and $|y-f(y)|<\varepsilon$, then $|x-y|\leq\delta$.

Assume that the set

$$K=\{(x,y)\in H_i\times H_i:\ |x-y|\geq\delta\}$$

is nonempty and compact (see Theorem 2.2.13 of [5]). Since the mapping $(x,y)\longrightarrow\max(|x-f(x)|,|y-f(y)|)$ is uniformly continuous, by Corollary 2.2.7 in [5] we can construct an increasing binary sequence $(\lambda_m)_{m\geq 1}$ such that

$$\lambda_m=0\Rightarrow\inf_{(x,y)\in K}\max(|x-f(x)|,|y-f(y)|)<2^{-m},$$

$$\lambda_m=1\Rightarrow\inf_{(x,y)\in K}\max(|x-f(x)|,|y-f(y)|)>2^{-m-1}.$$

It suffices to find m such that $\lambda_m=1$. In that case, if $|x-f(x)|<2^{-m-1}$ and $|y-f(y)|<2^{-m-1}$, we have $(x,y)\notin K$ and $|x-y|\leq\delta$. Assume $\lambda_1=0$. If $\lambda_m=0$, choose $(x_m,y_m)\in K$ such that $\max(|x_m-f(x_m)|,|y_m-f(y_m)|)<2^{-m}$, and if $\lambda_m=1$, set $x_m=y_m=z_m$. Then, $|x_m-f(x_m)|\longrightarrow 0$ and $|y_m-f(y_m)|\longrightarrow 0$, so $|x_m-y_m|\longrightarrow 0$. Computing M such that $|x_m-y_m|<\delta$, we must have $\lambda_m=1$. ■

We have completed the proof that any uniformly continuous function has a locally strong fixed point if and only if it has locally sequentially at most one fixed point.

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