An Application of Newton Type Iterative Method for Lavrentiev Regularization for Ill-Posed Equations: Finite Dimensional Realization

Santhosh George and Suresan Pareth

***Abstract*—In this paper, we consider, a finite dimensional realization of Newton type iterative method for Lavrentiev regularization of ill-posed equations. Precisely we consider the ill-posed equation $F(x) = f$ when the available data is f^δ with $\|f - f^\delta\| \leq \delta$ and the operator $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator defined on a real Hilbert space X. The error estimate obtained under a general source condition on $x_0 - \hat{x}$ (where x_0 is the initial guess and $\hat{x}$ is the solution of $F(x) = f$) is of optimal order. The regularization parameter α is chosen according to the adaptive method considered by Perverzev and Schock (2005). An example is provided to show the efficiency of the proposed method.**

***Index Terms*—quartic convergence, Newton Lavrentiev method, monotone operator, ill-posed problems, adaptive method.**

I. Introduction

An iteratively regularized projection method has been considered for apporoximately solving the ill-posed operator equation

$$F(x) = f \tag{1}$$

where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator (i.e., $\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in D(F)$) and X is a real Hilbert space with the inner product $\langle ., .\rangle$ and the norm $\|.\|$. It is assumed that (1) has a solution, namely $\hat{x}$ and F possesses a locally uniformly bounded Fréchet derivative $F'(x)$ for all $x \in D(F)$ (cf. [14]) i.e.,

$$\|F'(x)\| \leq C_F, \qquad x \in D(F)$$

for some constant C_F.

In application, usually only noisy data f^δ are available, such that

$$\|f - f^\delta\| \leq \delta.$$

Then the problem of recovery of $\hat{x}$ from noisy equation $F(x) = f^\delta$ is ill-posed, in the sense that a small perturbation in the data can cause large deviation in the solution. For solving (1) with monotone operators (see [7], [12], [14], [15]) one usually use the Lavrentiev regularization method. In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = f^\delta. \tag{2}$$

It is known (cf. [15], Theorem 1.1) that the equation (2) has a unique solution x_α^δ for $\alpha > 0$, provided F is Fréchet differentiable and monotone in the ball $B_r(\hat{x}) \subset D(F)$ with radius $r = \|\hat{x} - x_0\| + \delta/\alpha$. However the regularized equation (2) remains nonlinear and one may have difficulties in solving them numerically.

In [1], George and Elmahdy considered an iterative regularization method which converges linearly to x_α^δ and its finite dimensional realization in [2]. Later in [3] George and Elmahdy considered an iterative regularization method which converges quadratically to x_α^δ and its finite dimensional realization in [4].

Recall that a sequence (x_n) in X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals β, γ, such that for all $n \in N$ $\|x_n - x^*\| \leq \beta e^{-\gamma p^n}$.If the sequence (x_n) has the property that $\|x_n - x^*\| \leq \beta q^n$, $0 < q < 1$ then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate (see [8]).

Note that the method considered in [1], [2], [3] and [4] are proved using a suitably constructed majorizing sequence which heavily depends on the initial guess and hence not suitable for practical consideration.

In an attempt to avoid majorizing sequence to prove the convergence of the method considered in [1], [2], [3] and [4], the authors considered in [5], a two step iterative method for solving (1), which converges linearly to x_α^δ. Later in [11], the authors considered an application of Newton type iterative method, that converges quartically to x_α^δ. In this paper we consider, finite dimensional realization of the method considered in [11].

The organization of this paper is as follows. Section 2 describes the method and its convergence. Section 3 deals with the error analysis and parameter choice strategy. Section 4 gives the algorithm for implementing the proposed method. Numerical example and computational results are given in section 5. Finally in section 6 we summarize the key points in the paper.

II. The Method and its Convergence

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X. Our aim in this section is to obtain an approximation for x_α^δ, in the finite dimensional space $R(P_h)$, the range of P_h. For the results that follow, we impose the following conditions.

Let

$$\varepsilon_h := \|F'(x)(I - P_h)\|, \qquad \forall x \in D(F)$$

and $\{b_h : h > 0\}$ is such that $lim_{h\to 0}\frac{\|(I-P_h)x_0\|}{b_h} = 0$ and $lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if $F'(x)$ is a compact operator. Further we assume that $\varepsilon_h \leq \varepsilon_0$, $b_h \leq b_0$ and $\delta \in (0, \delta_0]$.

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A. Projection Method

We consider the following sequence defined iteratively by

$$y_{n,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)] \quad (3)$$

and

$$x_{n+1,\alpha}^{h,\delta} = y_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(y_{n,\alpha}^{h,\delta})P_h[F(y_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n,\alpha}^{h,\delta} - x_0)] \quad (4)$$

where $R_\alpha(x) := P_hF'(x)P_h + \alpha P_h$ and $x_{0,\alpha}^{h,\delta} := P_hx_0$, for obtaining an approximation for x_α^δ in the finite dimensional subspace $R(P_h)$ of X. Note that the iteration (3) and (4) are the finite dimensional realization of the iteration (3) and (4) in [11]. We will be selecting the parameter $\alpha = \alpha_i$ from some finite set

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}$$

using the adaptive method considered by Perverzev and Schock in [12].

We need the following assumptions for the convergence analysis.

Assumption 1: (cf. [14], Assumption 3) There exists a constant $k_0 \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x,u,v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x,u,v), \|\Phi(x,u,v)\| \leq k_0\|v\|\|x-u\|$.

Assumption 2: There exists a continuous, strictly monotonically increasing function $\varphi : (0,a] \to (0,\infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying;

(i) $lim_{\lambda\to 0}\varphi(\lambda) = 0$,
(ii) $sup_{\lambda\geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq c_\varphi\varphi(\alpha) \qquad \forall\lambda \in (0,a]$ and
(iii) there exists $v \in X$ with $\|v\| \leq 1$ (cf. [10]) such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v.$$

$$\text{Let} \qquad e_{n,\alpha}^{h,\delta} := \|y_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{h,\delta}\|, \qquad \forall n \geq 0 \quad (5)$$

and for $0 < k_0 < \frac{2}{3(1+\frac{\varepsilon_0}{\alpha_0})}$, let $g : (0,1) \to (0,1)$ be the function defined by

$$g(t) = \frac{27k_0^3}{8}(1+\frac{\varepsilon_0}{\alpha_0})^3t^3 \qquad \forall t \in (0,1). \quad (6)$$

Hereafter we assume that $\delta \in (0,\delta_0]$ where $\delta_0 < \alpha_0$.

Let $b_0 < \dfrac{\sqrt{1+\frac{2k_0}{(1+\frac{\varepsilon_0}{\alpha_0})}(1-\frac{\delta_0}{\alpha_0})}-1}{k_0}$, $\|\hat{x}-x_0\| \leq \rho$ where $\rho < \dfrac{\sqrt{1+\frac{2k_0}{(1+\frac{\varepsilon_0}{\alpha_0})}(1-\frac{\delta_0}{\alpha_0})}-1}{k_0} - b_0$ and

$$\text{let} \quad \gamma_\rho := (1+\frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}(\rho+b_0)^2 + (\rho+b_0)\right] + \frac{\delta_0}{\alpha_0}. \quad (7)$$

Lemma 1: Let $x \in D(F)$. Then

$$\|R_\alpha^{-1}(x)P_hF'(x)\| \leq (1+\frac{\varepsilon_0}{\alpha_0}).$$

Proof. Note that,

$$\begin{aligned}
&\|R_\alpha^{-1}(x)P_hF'(x)\| \\
=\ &\sup_{\|v\|\leq 1}\|(P_hF'(x)P_h+\alpha P_h)^{-1}P_hF'(x)v\| \\
=\ &\sup_{\|v\|\leq 1}\|(P_hF'(x)P_h+\alpha P_h)^{-1}P_hF'(x) \\
&(P_h+I-P_h)v\| \\
\leq\ &\sup_{\|v\|\leq 1}\|(P_hF'(x)P_h+\alpha P_h)^{-1}P_hF'(x)(P_h)v\| + \\
&\sup_{\|v\|\leq 1}\|(P_hF'(x)P_h+\alpha P_h)^{-1}P_hF'(x)(I-P_h)v\| \\
\leq\ &(1+\frac{\varepsilon_h}{\alpha}) \\
\leq\ &(1+\frac{\varepsilon_0}{\alpha_0}).
\end{aligned}$$

Lemma 2: Let $e_0 = e_{0,\alpha}^{h,\delta}$ and γ_ρ be as in (7). Then $e_0 \leq \gamma_\rho$.

Proof. Note that,

$$\begin{aligned}
e_0 &= \|y_{0,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_h[F(P_hx_0) - f^\delta]\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_h[F(P_hx_0) - F(\hat{x}) - F'(P_hx_0) \\
&\quad (P_hx_0-\hat{x}) + F'(P_hx_0)(P_hx_0-\hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_h \\
&\quad [\int_0^1 (F'(\hat{x}+t(P_hx_0-\hat{x})) - F'(P_hx_0))(P_hx_0-\hat{x})dt \\
&\quad +F'(P_hx_0)(P_hx_0-\hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha^{-1}(P_hx_0)P_hF'(P_hx_0) \\
&\quad [\int_0^1 \Phi(\hat{x}+t(P_hx_0-\hat{x}), P_hx_0, P_hx_0-\hat{x})dt \\
&\quad +(P_hx_0-\hat{x})] + R_\alpha^{-1}(P_hx_0)P_h(F(\hat{x}) - f^\delta)\|
\end{aligned}$$

and hence by Assumption 1, Lemma 1 and the relation $\|R_\alpha^{-1}(P_hx_0)\| \leq \frac{1}{\alpha}$, we have,

$$\begin{aligned}
e_0 &\leq (1+\frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}\|P_hx_0-\hat{x}\|^2 + \|P_hx_0-\hat{x}\|\right] + \frac{\delta}{\alpha} \\
&= (1+\frac{\varepsilon_0}{\alpha_0})[\frac{k_0}{2}\|P_hx_0 - x_0 + x_0 - \hat{x}\|^2 \\
&\quad +\|P_hx_0 - x_0 + x_0 - \hat{x}\|] + \frac{\delta}{\alpha} \\
&\leq (1+\frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}(\rho+b_h)^2 + (\rho+b_h)\right] + \frac{\delta}{\alpha} \\
&\leq (1+\frac{\varepsilon_0}{\alpha_0})\left[\frac{k_0}{2}(\rho+b_0)^2 + (\rho+b_0)\right] + \frac{\delta_0}{\alpha_0} \\
&= \gamma_\rho. \qquad (8)
\end{aligned}$$

Lemma 3: Let $y_{n,\alpha}^{h,\delta}$, $x_{n,\alpha}^{h,\delta}$ and $e_{n,\alpha}^{h,\delta}$ be as in (3), (4) and (5) respectively with $\delta \in (0,\delta_0]$. Then

(a) $\|x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}\| \leq \frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})(e_{n-1,\alpha}^{h,\delta})^2$ and
(b) $\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\| \leq [1+\frac{3k_0}{2}(1+\frac{\varepsilon_0}{\alpha_0})e_{n-1,\alpha}^{h,\delta}]e_{n-1,\alpha}^{h,\delta}$.

Proof. Observe that,

$$\begin{aligned}
&x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta} \\
=\ &y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h \\
&[F(y_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n-1,\alpha}^{h,\delta} - x_0)] + R_\alpha^{-1}(x_{n-1,\alpha}^{h,\delta}) \\
&P_h[F(x_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n-1,\alpha}^{h,\delta} - x_0)]
\end{aligned}$$

$$
\begin{aligned}
&= y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha} - R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h \\
&\quad [F(y^{h,\delta}_{n-1,\alpha}) - F(x^{h,\delta}_{n-1,\alpha}) + \alpha(y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha})] \\
&\quad + [R^{-1}_{\alpha}(x^{h,\delta}_{n-1,\alpha}) - R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})]P_h[F(x^{h,\delta}_{n-1,\alpha}) - f^{\delta} \\
&\quad + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)] \\
&= R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h[F'(y^{h,\delta}_{n-1,\alpha})(y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}) \\
&\quad -(F(y^{h,\delta}_{n-1,\alpha}) - F(x^{h,\delta}_{n-1,\alpha}))] + R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h \\
&\quad (F'(y^{h,\delta}_{n-1,\alpha}) - F'(x^{h,\delta}_{n-1,\alpha}))(x^{h,\delta}_{n-1,\alpha} - y^{h,\delta}_{n-1,\alpha}) \\
&:= \Gamma_1 + \Gamma_2 \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_1 &:= R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h[F'(y^{h,\delta}_{n-1,\alpha})(y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}) \\
&\quad -(F(y^{h,\delta}_{n-1,\alpha}) - F(x^{h,\delta}_{n-1,\alpha}))]
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_2 &:= R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h[F'(y^{h,\delta}_{n-1,\alpha}) - F'(x^{h,\delta}_{n-1,\alpha})] \\
&\quad (x^{h,\delta}_{n-1,\alpha} - y^{h,\delta}_{n-1,\alpha}).
\end{aligned}
$$

Note that,

$$
\begin{aligned}
&\|\Gamma_1\| \\
&= \|R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h \int_0^1 [F'(y^{h,\delta}_{n-1,\alpha}) - F'(x^{h,\delta}_{n-1,\alpha} \\
&\quad + t(y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}))](y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha})dt\| \\
&= \|R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha})P_h F'(y^{h,\delta}_{n-1,\alpha}) \int_0^1 [\phi(x^{h,\delta}_{n-1,\alpha} + \\
&\quad t(y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}), y^{h,\delta}_{n-1,\alpha}, x^{h,\delta}_{n-1,\alpha} - y^{h,\delta}_{n-1,\alpha})]dt\| \\
&\leq \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\|y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\|^2 \qquad (10)
\end{aligned}
$$

the last step follows from the Assumption 1 and Lemma 1. Similarly,

$$
\|\Gamma_2\| \leq k_0(1 + \frac{\varepsilon_0}{\alpha_0})\|y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\|^2. \qquad (11)
$$

So, (a) follows from (9), (10) and (11). And (b) follows from (a) and the triangle inequality;

$$
\|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n-1,\alpha}\| \leq \|x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha}\| + \|y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\|.
$$

THEOREM 1: Let $y^{h,\delta}_{n,\alpha}$, $x^{h,\delta}_{n,\alpha}$ be as in (3) and (4) respectively with $\delta \in (0, \delta_0]$ and $e^{h,\delta}_{n,\alpha}$, g and γ_ρ be as in equation (5), (6) and (7) respectively. Then

(a) $\|y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha}\| \leq g(e^{h,\delta}_{n-1,\alpha})e^{h,\delta}_{n-1,\alpha}$;

(b) $g(e^{h,\delta}_{n,\alpha}) \leq g(\gamma_\rho)^{4^n}$, $\quad \forall n \geq 0$;

(c) $e^{h,\delta}_{n,\alpha} \leq g(\gamma_\rho)^{\frac{4^n-1}{3}}\gamma_\rho \quad \forall n \geq 0$.

Proof. We have,

$$
\begin{aligned}
&y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha} \\
&= x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha} - R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})P_h \\
&\quad [F(x^{h,\delta}_{n,\alpha}) - f^{\delta} + \alpha(x^{h,\delta}_{n,\alpha} - x_0)] + R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha}) \\
&\quad P_h[F(y^{h,\delta}_{n-1,\alpha}) - f^{\delta} + \alpha(y^{h,\delta}_{n-1,\alpha} - x_0)] \\
&= x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha} - R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})P_h \\
&\quad [F(x^{h,\delta}_{n,\alpha}) - F(y^{h,\delta}_{n-1,\alpha}) + \alpha(x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha})] \\
&\quad + [R^{-1}_{\alpha}(y^{h,\delta}_{n-1,\alpha}) - R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})]P_h[F(y^{h,\delta}_{n-1,\alpha}) - f^{\delta} \\
&\quad + \alpha(y^{h,\delta}_{n-1,\alpha} - x_0)] \\
&= R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})P_h[F'(x^{h,\delta}_{n,\alpha})(x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha}) \\
&\quad -(F(x^{h,\delta}_{n,\alpha}) - F(y^{h,\delta}_{n-1,\alpha}))] + R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})P_h \\
&\quad [F'(x^{h,\delta}_{n,\alpha}) - F'(y^{h,\delta}_{n-1,\alpha})] \times (y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n,\alpha}) \\
&:= \Gamma_3 + \Gamma_4 \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_3 &:= R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})P_h[F'(x^{h,\delta}_{n,\alpha})(x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha}) \\
&\quad -(F(x^{h,\delta}_{n,\alpha}) - F(y^{h,\delta}_{n-1,\alpha}))]
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_4 &:= R^{-1}_{\alpha}(x^{h,\delta}_{n,\alpha})P_h[F'(x^{h,\delta}_{n,\alpha}) - F'(y^{h,\delta}_{n-1,\alpha})] \\
&\quad (y^{h,\delta}_{n-1,\alpha} - x^{h,\delta}_{n,\alpha}).
\end{aligned}
$$

Analogous to the proof of (10) and (11) one can prove that

$$
\|\Gamma_3\| \leq \frac{k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\|x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha}\|^2 \qquad (13)
$$

and

$$
\|\Gamma_4\| \leq k_0(1 + \frac{\varepsilon_0}{\alpha_0})\|x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n-1,\alpha}\|^2. \qquad (14)
$$

Now (a) follows from the Lemma 3, (12), (13) and (14). Again, since for $\mu \in (0,1)$, $g(\mu t) = \mu^3 g(t)$, for all $t \in (0,1)$, by (a) we get,

$$
g(e^{h,\delta}_{n,\alpha}) \leq g(e_0)^{4^n} \qquad (15)
$$

and

$$
\begin{aligned}
e^{h,\delta}_{n,\alpha} &\leq g(e^{h,\delta}_{n-1,\alpha})e^{h,\delta}_{n-1,\alpha} \\
&\leq g^{4^{n-1}}(e_0)g(e^{h,\delta}_{n-2,\alpha})e^{h,\delta}_{n-2,\alpha} \\
&\leq g^{4^{n-1}}(e_0)g^{4^{n-2}}(e_0)g(e^{h,\delta}_{n-3,\alpha})e^{h,\delta}_{n-3,\alpha} \\
&\leq g(e_0)^{4^{n-1}+4^{n-2}+\cdots+1}e_0 \\
&\leq g(e_0)^{\frac{4^n-1}{3}}e_0 \qquad (16)
\end{aligned}
$$

provided $e^{h,\delta}_{n,\alpha} < 1$. But $e^{h,\delta}_{n,\alpha} < 1$ by Lemma 2, (6) and (16). Now (b) and (c) follow from (8), (15), (16) and the relation $g(e_0) \leq g(\gamma_\rho)$. This completes the proof of the theorem.

THEOREM 2: Suppose $0 < g(\gamma_\rho) < 1$, $r = (\frac{1}{1-g(\gamma_\rho)} + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\frac{\gamma_\rho}{1-g^2(\gamma_\rho)})\gamma_\rho$ and let the assumptions of Theorem 1 hold. Then $x^{h,\delta}_{n,\alpha}$, $y^{h,\delta}_{n,\alpha} \in B_r(P_h x_0)$ for all $n \geq 0$.

Proof. Note that by (b) of Lemma 3 we have,

$$
\begin{aligned}
\|x^{h,\delta}_{1,\alpha} - P_h x_0\| &= \|x^{h,\delta}_{1,\alpha} - x^{h,\delta}_{0,\alpha}\| \\
&\leq [1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0]e_0 \qquad (17) \\
&\leq [1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho]\gamma_\rho \\
&< r
\end{aligned}
$$

i.e., $x^{h,\delta}_{1,\alpha} \in B_r(P_h x_0)$. Again note that from (17) and (a) of Theorem 1 we get,

$$
\begin{aligned}
\|y^{h,\delta}_{1,\alpha} - P_h x_0\| &\leq \|y^{h,\delta}_{1,\alpha} - x^{h,\delta}_{1,\alpha}\| + \|x^{h,\delta}_{1,\alpha} - P_h x_0\| \\
&\leq g(e_0)e_0 + [1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0]e_0 \\
&\leq [1 + g(e_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0]e_0 \\
&\leq [1 + g(\gamma_\rho) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho]\gamma_\rho \\
&< r
\end{aligned}
$$

i.e., $y^{h,\delta}_{1,\alpha} \in B_r(P_hx_0)$. Further by (17) and (b) of Lemma 3 we have,

$$\begin{aligned}
&\|x^{h,\delta}_{2,\alpha} - P_hx_0\| \\
\leq\ & \|x^{h,\delta}_{2,\alpha} - x^{h,\delta}_{1,\alpha}\| + \|x^{h,\delta}_{1,\alpha} - P_hx_0\| \\
\leq\ & [1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e^{h,\delta}_{1,\alpha}]e^{h,\delta}_{1,\alpha} + [1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0]e_0 \\
\leq\ & [1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})g(e_0)e_0]g(e_0)e_0 \\
& +[1 + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0]e_0 \\
\leq\ & [1 + g(e_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0(1 + g^2(e_0))]e_0 \qquad (18) \\
\leq\ & [1 + g(\gamma_\rho) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho(1 + g^2(\gamma_\rho))]\gamma_\rho \\
<\ & r
\end{aligned}$$

and by (18) and (a) of Theorem 1
we have,

$$\begin{aligned}
&\|y^{h,\delta}_{2,\alpha} - P_hx_0\| \\
\leq\ & \|y^{h,\delta}_{2,\alpha} - x^{h,\delta}_{2,\alpha}\| + \|x^{h,\delta}_{2,\alpha} - P_hx_0\| \\
\leq\ & g(e^{h,\delta}_{1,\alpha})e^{h,\delta}_{1,\alpha} + [1 + g(e_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0 \\
& (1 + g^2(e_0))]e_0 \\
\leq\ & g^5(e_0)e_0 + [1 + g(e_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0 \\
& (1 + g^2(e_0))]e_0 \\
\leq\ & [1 + g(e_0) + g^5(e_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0 \\
& (1 + g^2(e_0))]e_0 \\
\leq\ & [1 + g(e_0) + g^2(e_0) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})e_0 \\
& (1 + g^2(e_0))]e_0 \\
\leq\ & [1 + g(\gamma_\rho) + g^2(\gamma_\rho) + \frac{3k_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\gamma_\rho \\
& (1 + g^2(\gamma_\rho))]\gamma_\rho \\
<\ & r
\end{aligned}$$

i.e., $x^{h,\delta}_{2,\alpha},\ y^{h,\delta}_{2,\alpha} \in B_r(P_hx_0)$. Continuing this way one can prove that $x^{h,\delta}_{n,\alpha},\ y^{h,\delta}_{n,\alpha} \in B_r(P_hx_0), \forall n \geq 0$. This completes the proof.

The main result of this section is the following Theorem.

THEOREM 3: Let $0 < g(\gamma_\rho) < 1$, $y^{h,\delta}_{n,\alpha}$ and $x^{h,\delta}_{n,\alpha}$ be as in (3) and (4) respectively with $\delta \in (0, \delta_0]$ and assumptions of the Theorem 2 hold. Then $(x^{h,\delta}_{n,\alpha})$ is Cauchy sequence in $B_r(P_hx_0)$ and converges to $x^{h,\delta}_\alpha \in \overline{B_r(P_hx_0)}$. Further $P_h[F(x^{h,\delta}_\alpha) + \alpha(x^{h,\delta}_\alpha - x_0)] = P_hf^\delta$ and

$$\|x^{h,\delta}_{n,\alpha} - x^{h,\delta}_\alpha\| \leq Ce^{-\gamma 4^n}$$

where $C = (\frac{1}{1-g(\gamma_\rho)^4} + \frac{3k_0\gamma_\rho}{2}(1 + \frac{\varepsilon_0}{\alpha_0})\frac{1}{1-g^2(\gamma_\rho)^4}g(\gamma_\rho)^{4^n})\gamma_\rho$ and $\gamma = -\log g(\gamma_\rho)$.

Proof. Using the relation (b) of Lemma 3 and (c) of Theorem 1, we obtain,

$$\begin{aligned}
&\|x^{h,\delta}_{n+m,\alpha} - x^{h,\delta}_{n,\alpha}\| \\
\leq\ & \sum_{i=0}^{m-1} \|x^{h,\delta}_{n+i+1,\alpha} - x^{h,\delta}_{n+i,\alpha}\| \\
\leq\ & \sum_{i=0}^{m-1} \left[1 + \frac{3k_0e_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})g(e_0)^{4^{n+i}}\right] g(e_0)^{4^{n+i}}e_0 \\
\leq\ & [(1 + g(e_0)^4 + g(e_0)^{4^2} + \cdots + g(e_0)^{4^m}) \\
& +\frac{3k_0e_0}{2}(1 + \frac{\varepsilon_0}{\alpha_0})(1 + g^2(e_0)^4 + g^2(e_0)^{4^2} + \cdots \\
& +g^2(e_0)^{4^m})g(e_0)^{4^n}]g(e_0)^{4^n}e_0 \\
\leq\ & [(1 + g(\gamma_\rho)^4 + g(\gamma_\rho)^{4^2} + \cdots + g(\gamma_\rho)^{4^m}) \\
& +\frac{3k_0\gamma_\rho}{2}(1 + \frac{\varepsilon_0}{\alpha_0})(1 + g^2(\gamma_\rho)^4 + g^2(\gamma_\rho)^{4^2} + \cdots \\
& +g^2(\gamma_\rho)^{4^m})g(\gamma_\rho)^{4^n}]g(\gamma_\rho)^{4^n}\gamma_\rho \\
\leq\ & Cg(\gamma_\rho)^{4^n} \\
\leq\ & Ce^{-\gamma 4^n}.
\end{aligned}$$

Thus $x^{h,\delta}_{n,\alpha}$ is a Cauchy sequence in $B_r(P_hx_0)$ and hence it converges, say to $x^{h,\delta}_\alpha \in \overline{B_r(P_hx_0)}$.
Observe that,

$$\begin{aligned}
&\|P_h[F(x^{h,\delta}_{n,\alpha}) - f^\delta + \alpha(x^{h,\delta}_{n,\alpha} - x_0)]\| \\
=\ & \|R_\alpha(x^{h,\delta}_{n,\alpha})(x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n,\alpha})\| \\
\leq\ & \|R_\alpha(x^{h,\delta}_{n,\alpha})\|\|x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n,\alpha}\| \\
=\ & \|(P_hF'(x^{h,\delta}_{n,\alpha})P_h + \alpha P_h)\|e^{h,\delta}_{n,\alpha} \\
\leq\ & (C_F + \alpha)g(\gamma_\rho)^{4^n}\gamma_\rho. \qquad (19)
\end{aligned}$$

Now by letting $n \to \infty$ in (19) we obtain

$$P_h[F(x^{h,\delta}_\alpha) + \alpha(x^{h,\delta}_\alpha - x_0)] = P_hf^\delta. \qquad (20)$$

This completes the proof.

III. Error Bounds Under Source Conditions

The objective of this section is to obtain an error estimate for $\|x^{h,\delta}_{n,\alpha} - \hat{x}\|$ under a source condition on $x_0 - \hat{x}$.

Proposition 1: Let $F : D(F) \subseteq X \to X$ be a monotone operator in X. Let $x^{h,\delta}_\alpha$ be the solution of (20) and $x^h_\alpha := x^{h,0}_\alpha$. Then

$$\|x^{h,\delta}_\alpha - x^h_\alpha\| \leq \frac{\delta}{\alpha}.$$

Proof. The result follows from the monotonicity of F and the relation;

$$P_h[F(x^{h,\delta}_\alpha) - F(x^h_\alpha) + \alpha(x^{h,\delta}_\alpha - x^h_\alpha)] \quad = \quad P_h(f^\delta - f).$$

THEOREM 4: Let $\rho < \frac{2}{k_0(1+\frac{\varepsilon_0}{\alpha_0})}$ and $\hat{x} \in D(F)$ be a solution of (1). And let Assumption 1, Assumption 2 and the assumptions in Proposition 1 be satisfied. Then

$$\|x^h_\alpha - \hat{x}\| \leq \tilde{C}(\varphi(\alpha) + \frac{\varepsilon_h}{\alpha})$$

where $\tilde{C} := \frac{\max\{1, \rho + \|\hat{x}\|\}}{1-(1+\frac{\varepsilon_0}{\alpha_0})\frac{k_0}{2}\rho}$.

Proof. Let $M := \int_0^1 F'(\hat{x} + t(x^h_\alpha - \hat{x}))dt$. Then from the relation

$$P_h[F(x^h_\alpha) - F(\hat{x}) + \alpha(x^h_\alpha - x_0)] = 0$$

we have,

$$(P_hMP_h + \alpha P_h)(x^h_\alpha - \hat{x}) = P_h\alpha(x_0 - \hat{x}) + P_hM(I - P_h)\hat{x}.$$

Hence,

$$\begin{aligned} & x_\alpha^h - \hat{x} \\ = \; & [(P_hMP_h + \alpha P_h)^{-1}P_h - (F'(\hat{x}) + \alpha I)^{-1}]\alpha(x_0 - \hat{x}) \\ & +(F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\ & +(P_hMP_h + \alpha P_h)^{-1}P_hM(I - P_h)\hat{x} \\ = \; & (P_hMP_h + \alpha P_h)^{-1}P_h[F'(\hat{x}) - M + M(I - P_h)] \\ & (F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\ & +(F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\ & +(P_hMP_h + \alpha P_h)^{-1}P_hM(I - P_h)\hat{x} \\ := \; & \zeta_1 + \zeta_2 \end{aligned} \tag{21}$$

where

$$\begin{aligned} \zeta_1 \; := \; & (P_hMP_h + \alpha P_h)^{-1}P_h[F'(\hat{x}) - M + M(I - P_h)] \\ & (F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \end{aligned}$$

and

$$\begin{aligned} \zeta_2 \; := \; & (F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) + (P_hMP_h + \alpha P_h)^{-1} \\ & P_hM(I - P_h)\hat{x}. \end{aligned}$$

Observe that,

$$\begin{aligned} & \|\zeta_1\| \\ \leq \; & \|(P_hMP_h + \alpha P_h)^{-1}P_h \int_0^1 [F'(\hat{x}) - F'(\hat{x} \\ & +t(x_\alpha^h - \hat{x}))]dt(F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x})\| \\ & +\|(P_hMP_h + \alpha P_h)^{-1}P_hM(I - P_h) \\ & (F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x})\| \\ \leq \; & \|(P_hMP_h + \alpha P_h)^{-1}P_h \\ & \int_0^1 [F'(\hat{x} + t(x_\alpha^h - \hat{x}))(P_h + I - P_h) \\ & \phi(\hat{x}, \hat{x} + t(x_\alpha^h - \hat{x}), (F'(\hat{x}) + \alpha I)^{-1}\alpha(x_0 - \hat{x}))]dt\| \\ & +\frac{\varepsilon_h}{\alpha}\rho \\ \leq \; & (1 + \frac{\varepsilon_h}{\alpha})\frac{k_0}{2}\rho\|x_\alpha^h - \hat{x}\| + \frac{\varepsilon_h}{\alpha}\rho \\ \leq \; & (1 + \frac{\varepsilon_0}{\alpha_0})\frac{k_0}{2}\rho\|x_\alpha^h - \hat{x}\| + \frac{\varepsilon_h}{\alpha}\rho \end{aligned} \tag{22}$$

and

$$\|\zeta_2\| \leq \phi(\alpha) + \frac{\varepsilon_h}{\alpha}\|\hat{x}\|. \tag{23}$$

The result now follows from (21), (22) and (23).

THEOREM 5: Let $x_{n,\alpha}^{h,\delta}$ be as in (4). And the assumptions in Theorem 3 and Theorem 4 hold. Then

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \leq Ce^{-\gamma 4^n} + max\{1, \tilde{C}\}(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}).$$

Proof., Observe that,

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \leq \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| + \|x_\alpha^{h,\delta} - x_\alpha^h\| + \|x_\alpha^h - \hat{x}\|$$

so, by Proposition 1, Theorem 3 and Theorem 4 we obtain,

$$\begin{aligned} \|x_{n,\alpha}^{h,\delta} - \hat{x}\| \; & \leq \; Ce^{-\gamma 4^n} + \frac{\delta}{\alpha} + \tilde{C}(\varphi(\alpha) + \frac{\varepsilon_h}{\alpha}) \\ & \leq \; Ce^{-\gamma 4^n} + max\{1, \tilde{C}\}(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}). \end{aligned}$$

Let

$$n_\delta := min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta + \varepsilon_h}{\alpha}\right\} \tag{24}$$

and

$$C_0 = C + max\{1, \tilde{C}\}. \tag{25}$$

THEOREM 6: Let n_δ and C_0 be as in (24) and (25) respectively. And let $x_{n_\delta,\alpha}^{h,\delta}$ be as in (4) and the assumptions in Theorem 5 be satisfied. Then

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C_0(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}). \tag{26}$$

A. A priori choice of the parameter

Note that the error estimate $\varphi(\alpha) + \frac{\delta+\varepsilon_h}{\alpha}$ in (26) is of optimal order if $\alpha_\delta := \alpha(\delta, h)$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta + \varepsilon_h$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta + \varepsilon_h = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$. In view of the above observations and (26) we have the following.

THEOREM 7: Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 6 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$ and let n_δ be as in (24). Then

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

B. An adaptive choice of the parameter

In this subsection, we present a parameter choice rule based on the balancing principle studied in [9], [12]. In this method, the regularization parameter α is selected from some finite set

$$D_N(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, N\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \cdots, N$, we have

$$\|x_{n_i,\alpha_i}^{h,\delta} - x_{\alpha_i}^{h,\delta}\| \leq C\frac{\delta + \varepsilon_h}{\alpha_i}, \quad \forall i = 0, 1, \cdots N.$$

Let $x_i := x_{n_i,\alpha_i}^{h,\delta}$. In this paper we select $\alpha = \alpha_i$ from $D_N(\alpha)$ for computing x_i, for each $i = 0, 1, \cdots, N$.

THEOREM 8: (cf. [14], Theorem 3.1) Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\alpha_i}$. Let the assumptions of Theorem 6 and Theorem 7 hold and let

$$l := max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\alpha_i}\right\} < N,$$

$$k := max\{i : \|x_i - x_j\| \leq 4C_0\frac{\delta + \varepsilon_h}{\alpha_j}, \quad j = 0, 1, 2, \cdots, i\}.$$

Then $l \leq k$ and $\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta + \varepsilon_h)$ where $c = 6C_0\mu$.

IV. Implementation of adaptive choice rule

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 1 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i\alpha_0, i = 0, 1, 2, \cdots, N$.

A. Algorithm

1. Set $i = 0$.
2. Choose $n_i := min\left\{n : e^{-\gamma 4^n} \leq \frac{\delta+\varepsilon_h}{\alpha_i}\right\}$.
3. Solve $x_i := x^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (3) and (4).
4. If $\|x_i - x_j\| > 4C_0\frac{\delta+\varepsilon_h}{\alpha_j}, j < i$, then take $k = i-1$ and return x_k.
5. Else set $i = i+1$ and go to 2.

V. Numerical Example

In this section we consider the example considered in [14] for illustrating the algorithm considered in section IV. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $dimV_n = n+1$. Precisely we choose V_n as the linear span of $\{v_1, v_2, \cdots, v_{n+1}\}$ where $v_i, i = 1, 2, \cdots, n+1$ are the linear splines in a uniform grid of $n+1$ points in $[0,1]$. Note that $x^{h,\delta}_{n,\alpha},\ y^{h,\delta}_{n,\alpha} \in V_n$. So $y^{h,\delta}_{n,\alpha} = \sum_{i=1}^{n+1}\xi^n_i v_i$ and $x^{h,\delta}_{n,\alpha} = \sum_{i=1}^{n+1}\eta^n_i v_i$, where ξ^n_i and $\eta^n_i, i = 1, 2, \cdots, n+1$ are some scalars. Then from (3) we have

$$(P_hF'(x^{h,\delta}_{n,\alpha}) + \alpha)(y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha}) = P_h[f^\delta - F(x^{h,\delta}_{n,\alpha}) + \alpha(x^{h,\delta}_{0,\alpha} - x^{h,\delta}_{n,\alpha})]. \quad (27)$$

Observe that $(y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha})$ is a solution of (27) if and only if $(\overline{\xi^n - \eta^n}) = (\xi^n_1 - \eta^n_1, \xi^n_2 - \eta^n_2, \cdots, \xi^n_{n+1} - \eta^n_{n+1})^T$ is the unique solution of

$$(Q_n + \alpha B_n)(\overline{\xi^n - \eta^n}) = B_n[\bar{\mu^n} - F_{h1} + \alpha(X_0 - \bar{\eta^n})] \quad (28)$$

where

$$\begin{aligned} Q_n &= [\langle F'(x^{h,\delta}_{n,\alpha})v_i, v_j\rangle], i, j = 1, 2\cdots, n+1 \\ B_n &= [\langle v_i, v_j\rangle], i, j = 1, 2\cdots, n+1 \\ \bar{\mu^n} &= [f^\delta(t_1), f^\delta(t_2), \cdots, f^\delta(t_{n+1})]^T \\ F_{h1} &= [F(x^{h,\delta}_{n,\alpha})(t_1), F(x^{h,\delta}_{n,\alpha})(t_2), \cdots, F(x^{h,\delta}_{n,\alpha})(t_{n+1})]^T \\ X_0 &= [x_0(t_1), x_0(t_2), \cdots, x_0(t_{n+1})]^T \end{aligned}$$

and $t_1, t_2, \cdots, t_{n+1}$ are the grid points. Further from (4) it follows that

$$(P_hF'(y^{h,\delta}_{n,\alpha}) + \alpha)(x^{h,\delta}_{n+1,\alpha} - y^{h,\delta}_{n,\alpha}) = P_h[f^\delta - F(y^{h,\delta}_{n,\alpha}) + \alpha(x^{h,\delta}_{0,\alpha} - y^{h,\delta}_{n,\alpha})] \quad (29)$$

and hence $(x^{h,\delta}_{n+1,\alpha} - y^{h,\delta}_{n,\alpha})$ is a solution of (29) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta^{n+1}_1 - \xi^n_1, \eta^{n+1}_2 - \xi^n_2, \cdots, \eta^{n+1}_{n+1} - \xi^n_{n+1})^T$ is the unique solution of

$$(T_n + \alpha B_n)(\overline{\eta^{n+1} - \xi^n}) = B_n[\bar{\mu^n} - F_{h2} + \alpha(X_0 - \bar{\xi^n})] \quad (30)$$

where

$$\begin{aligned} T_n &= [\langle F'(y^{h,\delta}_{n,\alpha})v_i, v_j\rangle], i, j = 1, 2\cdots, n+1 \\ F_{h2} &= [F(y^{h,\delta}_{n,\alpha})(t_1), F(y^{h,\delta}_{n,\alpha})(t_2), \cdots, F(y^{h,\delta}_{n,\alpha})(t_{n+1})]^T. \end{aligned}$$

Note that (28) and (30) are uniquely solvable as Q_n and T_n are positive definite matrix (i.e., $xQ_nx^T > 0$ and $xT_nx^T > 0$ for all non-zero vector x) and B_n is an invertible matrix.

EXAMPLE 4: (see [14], section 4.3) Let $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1 \\ (1-s)t, & 0 \leq t \leq s \leq 1 \end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x - y\rangle = \int_0^1\left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds\right] \times (x-y)(t)dt \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)u^2(s)w(s)ds. \quad (31)$$

Note that for $u, v > 0$,

$$\begin{aligned} [F'(v) - F'(u)]w &= 3\int_0^1 k(t,s)u^2(s)ds \times \frac{\int_0^1 k(t,s)[v^2(s) - u^2(s)]w(s)ds}{\int_0^1 k(t,s)u^2(s)ds} \\ &:= F'(u)\Phi(v,u,w) \end{aligned}$$

where $\Phi(v,u,w) = \frac{\int_0^1 k(t,s)[v^2(s)-u^2(s)]w(s)ds}{\int_0^1 k(t,s)(u(s))^2ds}$.

Observe that

$$\begin{aligned} \Phi(v,u,w) &= \frac{\int_0^1 k(t,s)[v^2(s) - u^2(s)]w(s)ds}{\int_0^1 k(t,s)u^2(s)ds} \\ &= \frac{\int_0^1 k(t,s)[u(s) + v(s)][v(s) - u(s)]w(s)ds}{\int_0^1 k(t,s)u^2(s)ds}. \end{aligned}$$

So Assumption 2 satisfies with $k_0 \geq \left\|\frac{\int_0^1 k(t,s)[u(s)+v(s)]ds}{\int_0^1 k(t,s)u^2(s)ds}\right\|$.

In our computation, we take $f(t) = \frac{6sin(\pi t)+sin^3(\pi t)}{9\pi^2}$ and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = sin(\pi t).$$

We use

$$x_0(t) = sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + sin^2(\pi t)]}{4\pi^2}$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

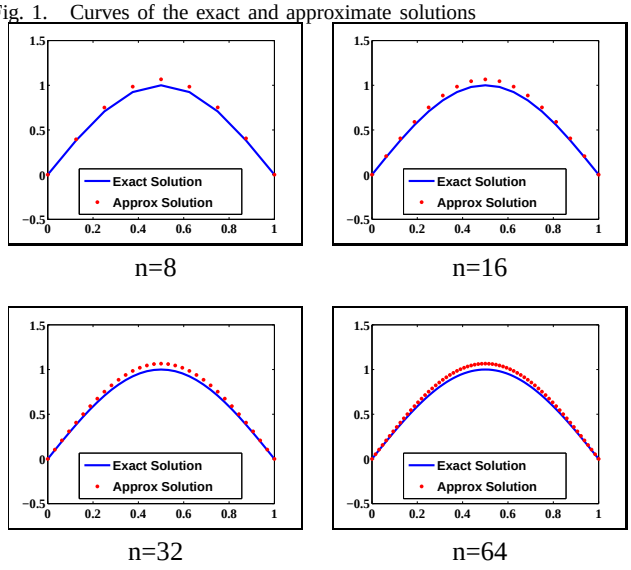
$$x_0 - \hat{x} = \varphi(F'(\hat{x}))\frac{1}{4}$$

where $\varphi(\lambda) = \lambda$.

For the operator $F'(.)$ defined in (31), $\varepsilon_h = O(n^{-2})$ (cf. [6]). Thus we expect to obtain the rate of convergence $O((\delta + \varepsilon_h)^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.1)(\delta + \varepsilon_h)$, $\mu = 1.1$, $\rho = 0.11$, $\gamma_\rho = 0.7818$ and $g(\gamma_\rho) = 0.99$. The results of the computation are presented in Table 1. The plots of the exact solution and the approximate solution obtained are given in Figures 1 and 2.

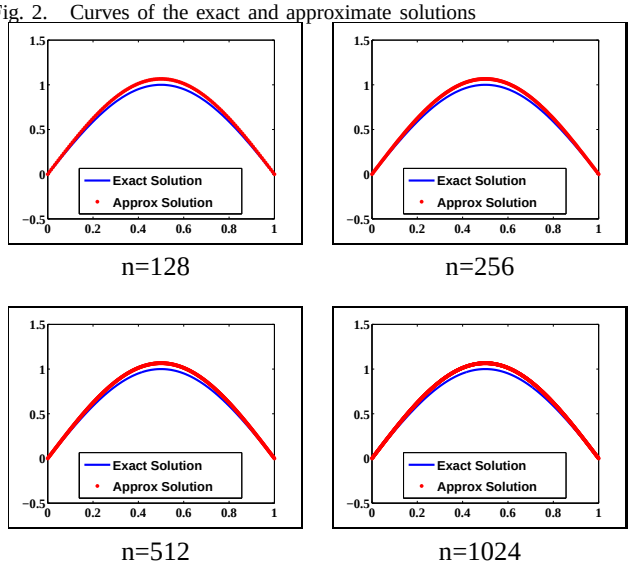
Fig. 1. Curves of the exact and approximate solutions



n=8 n=16

n=32 n=64

Fig. 2. Curves of the exact and approximate solutions



n=128 n=256

n=512 n=1024

TABLE I
ITERATIONS AND CORRESPONDING ERROR ESTIMATES

n	k	n_k	$\delta+\varepsilon_h$	α	$\|x_k-\hat{x}\|$	$\frac{\|x_k-\hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	2	4	0.0135	0.0180	0.0356	0.3065
16	2	4	0.0134	0.0178	0.0432	0.3736
32	2	4	0.0133	0.0178	0.0450	0.3897
64	2	4	0.0133	0.0177	0.0455	0.3938
128	2	4	0.0133	0.0177	0.0456	0.3948
256	2	4	0.0133	0.0177	0.0456	0.3950
512	19	5	0.0133	0.0897	0.0456	0.3951
1024	27	6	0.0133	0.1923	0.0456	0.3951

VI. CONCLUSION

We have suggested and analyzed the finite dimensional realization of the iterative method considered in [11] for obtaining an approximate solution for nonlinear ill-posed operator equation $F(x) = f$ when the operator $F : D(F) \subseteq X \to X$ defined on a real Hilbert space X is monotone, and the available data is f^δ with $\|f - f^\delta\| \leq \delta$. Using a general source condition on $x_0 - \hat{x}$ we obtained an optimal order error estimate. The regularization parameter α is chosen according to the balancing principle considered by Perverzev and Schock (2005). The numerical results provided confirm the efficiency of the method.

ACKNOWLEDGMENT

S.Pareth thanks National Institute of Technology, Karnataka, India for the financial support.

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