On The Properties of Anisotropic Engineering Materials Based upon Orthonormal Representations

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Abstract— **A decomposition method[5] based upon orthonormal representations is reviewed and improved toexpress any anisotropic engineering tensor showing the effect of the material properties on the structures. A new decomposed form for the stress tensor (example for symmetric second rank tensor) different from the one available in the literature where the engineering understanding is improved, is presented. Numerical examples from different engineering materials serve to illustrate and verify the decomposition procedure. The norm concept of elastic constant tensor and norm ratios are used to study the anisotropy of these materials. It is shown that this method allows to investigate the elastic and mechanical properties of an anisotropic material possessing any material symmetry and determine anisotropy degree of that material. For a material given from an unknown symmetry, it is possible to determine its material symmetry type by this method.**

Index Terms— **stress tensor, elastic constant tensor, decomposition, form invariant, orthonormalized basis elements, norm.**

I. INTRODUCTION

A material is isotropic if its mechanical and elastic properties are the same in all directions. When this is not true, the material is anisotropic.

Many materials are anisotropic and inhomogeneous due to the varying composition of their constituents. Every day passed, the number of anisotropic materials is increasing by the addition of man-made anisotropic single crystals and technologically developed anisotropic materials. In order to understand the physical properties of the anisotropic materials, use of tensors by decomposing them is inevitable. Tensors are the most important mathematical entities to describe direction dependent physical properties of solids and the tensor components characterizing physical properties which must be specified without reference to any coordinate system.

The constitutive relation for linear anisotropic elasticity, defined by using stress and strain tensors, is the generalized Hooke's law [1]

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \qquad (1)$$

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This formula demonstrates the well known general linear relation between the stress tensor whose components are σ_{ij} and the strain tensor (symmetric second rank tensor) whose components are ε_{kl}. C_{ijkl} are the components of elastic constant tensor (elasticity tensor) C_{ijkl} satisfies three important symmetry restrictions. These are

$$C_{ijkl}=C_{jikl} \quad C_{ijkl}=C_{ijlk} \quad C_{ijkl}=C_{klij}, \qquad (2)$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the elastic strain energy. These restrictions reduce the number of independent elastic constants C_{ijkl} from 81 to 21. Consequently, for anisotropic materials (with triclinic symmetry) the elastic constant tensor has 21 independent components.

The indices are abbreviated according to the replacement rule given in the following TABLE [1]:

TABLE I
ABBREVIATION OF INDICES FOR FOUR AND DOUBLE INDEX NOTATIONS

four index notation	11	22	33	23, 32	13, 31	12, 12
double index notation	1	2	3	4	5	6

In literature, the works for orthonormal representation of any rank tensors can be summarized as; it was first proposed by [2], developed by [3] who gave name as integrity basis treated the strain energy function as a polynomial in the strain components and lead to determination integrity basis for invariant functions of the strain components for each one of the 32 crystallographic point groups. Using the integrity basis, orthonormal tensor basis which spans the space of elastic constants was derived. Orthonormal tensor basis is also obtained by another way which is form invariant. Reference [4] identified invariant elastic constants for each crystal class.

The purpose of the work is to review and develop the decomposition method presented in [5] for both stress tensor and elastic constant tensor. The other aim of this paper is to prove that this method is applied to even rank tensors such as symmetric second rank tensors and fourth rank tensors.

In the present paper, the decomposition method is developed for stress tensor. Next, this method is extended and applied to anisotropic elastic symmetries such as cubic, tetragonal and trigonal. As applications, numerical examples are given from the materials which exhibit cubic, tetragonal and trigonal symmetries. Norm concept and anisotropy degrees for those symmetry types are presented. Finally, in

(Advance online publication: 27 August 2012)

the last section, the results of numerical analysis are discussed and conclusions pertinent to this work are stated.

II. Decomposition Process For Stress Tensor

Stress tensor as an example to symmetric second rank tensor is decomposed. In the mechanics of continuous media i.e. in elasticity studies; the stress and strain tensors are decomposed into spherical (hydrostatic) and deviatoric parts each of which have important meanings. Besides, stress tensor is decomposed into six orthonormal parts by this method. Decomposition process is mainly based on two steps; form invariant and orthonormalized basis elements.

A. *Form Invariant*

A physical property of tensor is resolved along the triads v_1, v_2, v_3 denoting the unit vectors along the crystallographic axis [4]. The symmetry properties of the material, due to the geometric or crystallographic symmetry, can be defined by the group of orthonormal transformations which transform any of these triads v_a into its equivalent positions. When forming invariant, a physical property of tensor is also resolved along those triads. The process of resolution yields the invariants. Forming invariant is an indispensable step to construct orthonormal tensor basis needed for decomposition process, the procedure is as follows:

The form invariant expression for symmetric second rank tensor as stress tensor is,

$$\sigma_{ij} = v_{ai} v_{bj} A_{ab}, \tag{3}$$

where summation is implied by repeated indices. This expression is referred to a Cartesian system Oxyz; v_{ai} are the components of the unit vectors v_a $(a = 1,2,3)$ along the crystallographic axes. A_{ab} is invariant in the sense that when the Cartesian system is rotated to a new orientation $Ox'y'z'$, then (3) takes the following form;

$$\sigma'_{ij} = v'_{ai}\, v'_{bj}\, A_{ab}, \tag{4}$$

Where v_1, v_2, v_3 form a linearly independent basis in three dimensions but they are not necessarily always orthogonal (it is a general case). The orthogonality condition used in this work, is a particular case for both stress and elastic constant tensors so the corresponding reciprocal triads must satisfy the following relation

$$v_{ai} v_{aj} = \delta_{ij} \tag{5}$$

The expression given in (5) can be rewritten as

$$vv^T = \mathbf{I} \tag{6}$$

Where $\mathbf{I}$ is identity matrix which is

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Since $\delta_{ij} = 1$ ($i = j$) or $\delta_{ij} = 0$ ($i \neq j$). These are the orthogonality relations which are also defined in (5).

As an illustration, for the uniaxial crystal system, (3) takes the form

$$\sigma_{ij} = \sigma_1 \delta_{ij} + \sigma_2 v_{3i} v_{3j}, \tag{8}$$

where v_3 is the unique axis and σ_1 corresponds to isotropy. The first step for constructing the orthonormalized basis elements is to write the δ_{ai} in the place of v_{ai} in (3). So the following form is obtained:

$$\sigma_{ij} = \delta_{ai} \delta_{bj} A_{ab}. \tag{9}$$

Instead of the form invariant expression given in (3) for any given class and it is possible to replace the v_{ai} by the δ_{ai} to obtain the elements of the basis. According to the expression in (8), the elements of the basis are δ_{ij} and $\delta_{3i}\delta_{3j}$. Similar to (8), for monoclinic system, with v_2 normal to the $v_3 v_1 -$ plane, the form-invariant expression is

$$\sigma_{ij} = A_{11} v_{1i} v_{1j} + A_{22} v_{2i} v_{2j} + A_{33} v_{3i} v_{3j} + A_{31}(v_{3i} v_{1j} + v_{1i} v_{3j}), \tag{10}$$

By making the replacement to the above expression, the elements; $\delta_{1i}\delta_{1j}$, $\delta_{2i}\delta_{2j}$, $\delta_{3i}\delta_{3j}$, $\delta_{3i}\delta_{1j} + \delta_{1i}\delta_{3j}$ are obtained.

B. *Orthonormalized Basis Elements*

By using the (5) and orthonormalization by well known Gram-Schmidt scheme, the basis elements are

$$T_{ij}^{I} = \frac{1}{\sqrt{3}} \delta_{ij}, \quad T_{ij}^{II} = \frac{1}{\sqrt{6}} (3\delta_{3i}\delta_{3j} - \delta_{ij}),$$
$$T_{ij}^{III} = \frac{1}{\sqrt{2}} (2\delta_{1i}\delta_{1j} + \delta_{3i}\delta_{3j} - \delta_{ij}),$$
$$T_{ij}^{IV} = \frac{1}{\sqrt{2}} (\delta_{3i}\delta_{1j} - \delta_{1i}\delta_{3j}),$$
$$T_{ij}^{V} = \frac{1}{\sqrt{2}} (\delta_{1i}\delta_{2j} - \delta_{2i}\delta_{1j}),$$
$$T_{ij}^{VI} = \frac{1}{\sqrt{2}} (\delta_{3i}\delta_{2j} - \delta_{2i}\delta_{3j}). \tag{11}$$

In constructing this basis, following identity is used

$$\delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j} + \delta_{3i}\delta_{3j} = \delta_{ij}. \tag{12}$$

This is a particular case of a more general identity which is

$$v_{1i} v_{1j} + v_{2i} v_{2j} - \cos\theta (v_{1i} v_{2j} + v_{2i} v_{1j}) + \sin^2\theta (v_{3i} v_{3j}) = \sin^2\theta \delta_{ij}. \tag{13}$$

with v_{ai} is replaced by δ_{ai} and $\theta = 90^o$.

Hence the orthonormal basis elements present in (11) are obtained.

A complete orthonormal basis for the second rank symmetric tensor (i.e., stress tensor) will be the set

$\{I,II,..,VI\}$ the decomposition of σ_{ij} is given in terms of these basis elements as

$$\sigma_{ij} = \sum_k (\boldsymbol{\sigma}, T_{ij}^k) T_{ij}^k, \quad (k = I, II, ..., VI) \tag{14}$$

where $(\boldsymbol{\sigma}, T_{ij}^k)$ represents the inner product of σ_{ij} and the k^{th} elements, T_{ij}^k of the basis. It is well known that inner product is different from multiplication of two matrices. For second rank tensors, it is defined by

$$(\boldsymbol{\sigma}, T_{ij}^k) = \sum_{i=1}^{n=3} \sum_{j=1}^{n=3} \sigma_{ij} T_{ij}^k. \tag{15}$$

The orthonormal of the decomposed parts can be proved by taking inner products of orthonormalized basis elements (T_{ij}^I, T_{ij}^{II}) i.e., $(T_{ij}^I, T_{ij}^{II}) = 0$ and $(T_{ij}^I, T_{ij}^I) = 1$ and the results of inner products for other elements are the same. So this method is an orthonormal method. By using (15) and matrix forms of the orthonormalized basis elements, decomposed parts can be obtained and by adding all decomposed parts, we obtain the stress tensor which is

$$\sigma_{ij} = \frac{\sigma_{pp}}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} + \frac{1}{\sqrt{2}}(2\sigma_{11} + \sigma_{33} - \sigma_{pp}) \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{1}{\sqrt{6}}(-3\sigma_{33} + \sigma_{pp}) \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} + \frac{2\sigma_{13}}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

$$+ \frac{2\sigma_{12}}{\sqrt{2}} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2\sigma_{23}}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \tag{16}$$

In (16), $T_{ij}^k \quad (k = I, II, ..., VI)$ are the matrix parts of the orthonormalized basis elements. They are orthonormal to each other.

Furthermore stress tensor is virtually decomposed into two parts

$$\sigma_{ij} = \frac{1}{3}\sigma_{pp}\delta_{ij} + \left(\sigma_{ij} - \frac{1}{3}\sigma_{pp}\delta_{ij}\right), \tag{17}$$

where

$$\sigma_{pp} = \sigma_{11} + \sigma_{22} + \sigma_{33},$$

$\boldsymbol{\sigma}^D = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{pp}$ (deviatoric part) and $\boldsymbol{\sigma}^S = \frac{1}{3}\delta_{ij}\sigma_{pp}$ (spherical part).

Deviatoric space D consists of pure shear tensors constructing by requiring orthonormal basis which are orthogonal to each other. $\boldsymbol{\sigma}^D$ can be decomposed into five parts by orthonormal tensor basis method.

It is well known in literature that when $i = j$, $\sigma_{ij}^D = 0$, which means that the deviatoric part $\boldsymbol{\sigma}^D$ is traceless. When $\boldsymbol{\sigma}^D = 0$, (17) reduces to the form:

$$\sigma_{ij} = \frac{1}{3}\delta_{ij}\sigma_{pp}. \tag{18}$$

Here $\frac{1}{3}\sigma_{pp} = -p$, where p is the hydrostatic pressure and $\sigma_{ij} = -p\delta_{ij}$ is called pure hydrostatic state of stress.

C. *Illustrative Applications*

The pure shear tensors play a significant role in continuum mechanics. The elastic strain caused by such a pure shear stress may not be a pure shear especially in the case of elastic anisotropy. Pure shear stress fields arise in many practical cases; for example, in the torsion of linearly elastic rods or elastic-plastic bars. So it is worthwhile to pay attention to some remarkable properties of the pure shear tensors.

In the language of matrix algebra, it is equivalent to the problem of constructing sets of five mutually orthogonal singular traceless matrices. The problem is not only one of theoretical interest, it may also have some practical significance, for instance, in computational plasticity of polycrystalline metals. It might be easier to perform computer modeling of randomly oriented crystalline grains if there exists many sets of five orthogonal shears. More detailed investigation of the sets of such basis may promise mathematically motivated weight functions for the modeling of anisotropic random distributions of the oriented grains.

III. Decomposition Process For Elastic Constant Tensor

In analyzing the elastic and mechanical properties of anisotropic linear materials, elastic constant tensor is required to make up a linear constitutive relation between stress and strain tensors, each of which represents some directly detectable and measurable effect in the material (Recall Hooke's law, given in (1)). Elastic constant tensor is introduced in specification of physical properties for many anisotropic materials.

Decomposition of the elastic constant tensor into orthonormal parts, offer not only valuable insight into the tensor structure but also simplify immensely the calculations of sums, products, inverses and inner products. The decomposition method developed can be carried out for materials possessing symmetry classes such as isotropic, cubic, transversely isotropic, tetragonal (classes: $4mm$, $\bar{4}2m$, 422, $4/mmm$), trigonal (classes: 32, $3m$, $\bar{3}m$), orthorhombic and triclinic [1]. In this work, materials possessing isotropic, cubic, transversely isotropic, tetragonal, trigonal are selected for applications since important engineering materials exhibit those symmetries.

For isotropic materials, an expression for the elastic constant tensor which is different from the traditional form is also presented.

A. Form Invariant

The form invariant expression for the components of elastic constant tensor, the elastic stiffness coefficients is,

$$C_{ijkl} = v_{ai} v_{bj} v_{ck} v_{dl} A_{abcd} \tag{19}$$

Where summation is implied by repeated indices, v_{ai} are the components of the unit vectors v_a ($a = 1,2,3$) along the material direction axes. A_{abcd} is invariant in the sense that when the Cartesian system is rotated to a new orientation $Ox'y'z'$, then (19) takes the following form;

$$C'_{ijkl} = v'_{ai} v'_{bj} v'_{ck} v'_{dl} A_{abcd} \tag{20}$$

Where v_1, v_2, v_3 form a linearly independent basis in three dimensions but they are not necessarily always orthogonal (it is a general case). The orthogonality condition used in this work, is a particular case for elastic constant tensor so the corresponding reciprocal triads must satisfy the relation given in (5).

B. Orthonormalized Basis Elements

Form invariant is the necessary step in constructing orthonormal tensor basis of elasticity tensors. By appropriate use of δ_{ij}, elements of the orthonormal tensor basis can be constructed for each symmetry types [6]. Furthermore symmetry in crystal means simply invariance of the properties with respect to the transforms of some subgroup of the orthogonal group, whereas the properties of an isotropic medium are invariant with respect to all the transforms of the orthogonal group. In other words, it explains the form of C_{ijkl} tensor for any isotropic medium and it is invariant with respect to the all transforms of the orthogonal group. However there is a unique tensor that is not affected by all orthogonal transforms, it is a unique tensor, apart from a scalar factor, so C_{ijkl} can be expressed as combinations of the components δ_{ij} of that tensor with certain coefficients. There are only three different such combinations which contain four subscripts i, j, k, l namely $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$ [6]. Because of the symmetry of C_{ijkl}, i and j are permuted. So the elements takes the new form; $\delta_{ij}\delta_{kl}$ and $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. For other symmetry types, these elements are used in a suitable form, when constructing orthonomalized basis. Form-invariant expression of isotropic symmetry is formed by the following two basis elements:

$$\delta_{ij}\delta_{kl}, \qquad \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{21}$$

So, the decomposition of C_{ijkl} for triclinic system with no elastic symmetries is given in terms of its orthonormalized basis elements as

$$C_{ijkl} = \sum_K (C, A^K_{ijkl}) A^K_{ijkl}, (K = I \ldots XXI), \tag{22}$$

Where (C, A^K_{ijkl}) represents the inner product of C_{ijkl} and K^{th} elements, A^K_{ijkl}, the orthonormalized basis elements and given for each elastic symmetry types, besides, the inner products for triclinic symmetry are

C. Cubic Materials

The form invariant expression is defined for cubic material as [4]

$$C_{ijkl} = \lambda \alpha_{ijkl} + \mu \beta_{ijkl} + \rho v_{ai} v_{aj} v_{ak} v_{al} \tag{23}$$

where $\alpha_{ijkl} = \delta_{ij}\delta_{kl}$, $\beta_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ and $\tau_{ijkl} = v_{ai} v_{aj} v_{ak} v_{al}$, τ_{ijkl} is also rewritten in terms of δ_{ij} as

$$\tau_{ijkl} = (\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(\delta_{k1}\delta_{l2} + \delta_{k2}\delta_{l1}) + (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{k2}\delta_{l3} + \delta_{k3}\delta_{l2}) + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1}).$$

λ, μ and ρ are invariant elastic constants for cubic materials.

For cubic materials, the decomposition of C_{ijkl} for cubic system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = (C, A^I_{ijkl}) A^I_{ijkl} + (C, A^{II}_{ijkl}) A^{II}_{ijkl} + (C, A^{III}_{ijkl}) A^{III}_{ijkl}, \tag{24}$$

Where (C, A^K_{ijkl}) denotes the inner product of C_{ijkl} and

$$A^I_{ijkl} = \frac{1}{3}\alpha_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}, \quad A^{II}_{ijkl} = \frac{1}{6\sqrt{5}}(3\beta_{ijkl} - 2\alpha_{ijkl}),$$

$A^{III}_{ijkl} = \frac{1}{2\sqrt{30}}(5\tau_{ijkl} - 3\beta_{ijkl} + 2\alpha_{ijkl})$, which are orthonormalized basis elements for cubic materials. The inner products are

$$(C, A^I_{ijkl}) = \frac{1}{3}[(C_{11} + C_{22} + C_{33}) + 2(C_{12} + C_{13} + C_{23})], \tag{25}$$

$$(C, A^{II}_{ijkl}) = \frac{1}{6\sqrt{5}}[4(C_{11} + C_{22} + C_{33}) + 12(C_{44} + C_{55} + C_{66}) - 4(C_{12} + C_{13} + C_{23})]. \tag{26}$$

$$(C, A^{III}) = \frac{1}{2\sqrt{30}}[-4(C_{11} + C_{22} + C_{33}) + 8(C_{44} + C_{55} + C_{66}) + 4(C_{12} + C_{13} + C_{23})]. \tag{27}$$

D. Tetragonal Materials

The form invariant expression for tetragonal materials [4]

$$C_{ijkl} = \lambda_1\alpha_{ijkl} + \lambda_2\beta_{ijkl} + \lambda_3\gamma_{ijkl} + \lambda_4\delta_{ijkl} + \lambda_5\varepsilon_{ijkl} + \lambda_6\tau_{ijkl} \tag{28}$$

Where $\gamma_{ijkl} = \delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}$, $\delta_{ijkl} = \delta_{ij}\delta_{k3}\delta_{l3} + \delta_{i3}\delta_{j3}\delta_{kl}$

and $\varepsilon_{ijkl} = (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{k2}\delta_{l3} + \delta_{k3}\delta_{l2}) + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1})$.

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 are invariant elastic constants for tetragonal system. The representation of C_{ijkl} for tetragonal materials is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = (C, A^{I}_{ijkl})A^{I}_{ijkl} + (C, A^{II}_{ijkl})A^{II}_{ijkl} + (C, A^{III}_{ijkl})A^{III}_{ijkl} + (C, A^{IV}_{ijkl})A^{IV}_{ijkl} + (C, A^{V}_{ijkl})A^{V}_{ijkl} + (C, A^{VI}_{ijkl})A^{VI}_{ijkl}, \tag{29}$$

Where $A^{III}_{ijkl} = \frac{1}{6\sqrt{5}}(15\gamma_{ijkl} - \beta_{ijkl} - \alpha_{ijkl})$,

$$A^{IV}_{ijkl} = \frac{1}{12}(9\delta_{ijkl} - 15\gamma_{ijkl} + \beta_{ijkl} - 5\alpha_{ijkl}),$$

$$A^{V}_{ijkl} = \frac{1}{4}(2\varepsilon_{ijkl} - \delta_{ijkl} + 3\gamma_{ijkl} - \beta_{ijkl} + \alpha_{ijkl}),$$

$$A^{VI}_{ijkl} = \frac{1}{2\sqrt{2}}(2\tau_{ijkl} - \varepsilon_{ijkl} - \delta_{ijkl} + 3\gamma_{ijkl} - \beta_{ijkl} + \alpha_{ijkl}).$$

Which are orthonormalized basis elements for tetragonal materials.

Since first two orthonormalized basis elements of tetragonal materials are the same as isotropic symmetry[5], inner products are also identical, the other inner products for tetragonal symmetry are

$$(C, A^{III}) = \frac{1}{6\sqrt{5}}[-3C_{11} - 3C_{22} + 12C_{33} - 2C_{12} - 2C_{13} - 2C_{23} - 4C_{44} - 4C_{55} - 4C_{66}],$$

$$(C, A^{IV}) = \frac{1}{12}[-3C_{11} - 3C_{22} - 10C_{12} + 8C_{13} + 8C_{23} + 4C_{44} + 4C_{55} + 4C_{66}],$$

$$(C, A^{V}) = \frac{1}{4}[-C_{11} - C_{22} + 2C_{12} + 4C_{44} + 4C_{55} - 4C_{66}]. \tag{30}$$

E. Trigonal Materials

For trigonal materials, the form invariant expression is [4]

$$C_{ijkl} = \lambda_1\alpha_{ijkl} + \lambda_2\beta_{ijkl} + \lambda_3\gamma_{ijkl} + \lambda_4\delta_{ijkl} + \lambda_5\varepsilon_{ijkl} + \lambda_6\psi_{ijkl} \tag{31}$$

where

$$\psi_{ijkl} = (\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(\delta_{k1}\delta_{l3} + \delta_{k3}\delta_{l1}) + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{k1}\delta_{l2} + \delta_{k2}\delta_{l1}) + (\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{k1}\delta_{l1} - \delta_{k2}\delta_{l2}) + (\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})(\delta_{k2}\delta_{l3} - \delta_{k3}\delta_{l2}),$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 are invariant elastic constants for trigonal system. The decomposition of C_{ijkl} for trigonal materials is given in terms of its orthonormalized basis elements as

$$C_{ijkl} = \sum_K (C, A^K_{ijkl})A^K_{ijkl}, (K = I...VI) \tag{32}$$

where $A^{VI}_{ijkl} = \frac{1}{4}\psi_{ijkl}$ which is the last orthonormalized basis elements for trigonal system.

Since first five orthonormalized basis elements of trigonal system are the same as transversely isotropic symmetry [5], inner products are also common, the last inner product for trigonal system are

$$(C, A^{VI}) = 2C_{56} + C_{14} - C_{24}. \tag{33}$$

IV. Numerical Analysis

Let us consider the decomposition of the elastic constant tensor in the following materials.

A. For Aluminium Antimonide(AlSb)

AlSb possesses cubic symmetry. In this symmetry, four three-fold axes arranged like the body diagonals of a cube. There are three independent elastic constants for cubic symmetry which are C_{11}, C_{12}, C_{44}. The elastic coefficients in GPa for AlSb are presented as[7]

$$C_{ij} = \begin{bmatrix} 87.7 & 43.4 & 43.4 & 0 & 0 & 0 \\ 43.4 & 87.7 & 43.4 & 0 & 0 & 0 \\ 43.4 & 43.4 & 87.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 40.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 40.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 40.8 \end{bmatrix} \tag{34}$$

By using this method, the formula given in (24) should be applied. For this reason inner products must be calculated as

$$(C, A^{I}) = 174.5, (C, A^{II}) = 149.1, (C, A^{III}) = 40.86. \tag{35}$$

The symmetric fourth rank tensor for AlSb can be represented in the form

$$C_{ijkl} = 174.5A^{I}_{ijkl} + 149.1A^{II}_{ijkl} + 40.86A^{III}_{ijkl} \tag{36}$$

If the orthonormalized basis elements for cubic symmetry are inserted into the right-hand side of (36), the identical matrix in (34) can be obtained, which shows the validity of the decomposed terms.

Isotropic part of AlSb is

$$I = 174.5A^{I}_{ijkl} + 149.1A^{II}_{ijkl} \tag{37}$$

If the related orthonormalized basis elements are put into (37), isotropic part is found in matrix form as

$$I = \begin{bmatrix} 102.6 & 35.9 & 35.9 & 0 & 0 & 0 \\ 35.9 & 102.6 & 35.9 & 0 & 0 & 0 \\ 35.9 & 35.9 & 102.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 33.34 & 0 & 0 \\ 0 & 0 & 0 & 0 & 33.34 & 0 \\ 0 & 0 & 0 & 0 & 0 & 33.34 \end{bmatrix} \tag{38}$$

The cubic part of the material is

$$C = 40.86A^{III}_{ijkl} \tag{39}$$

When the corresponding basis element is inserted into (39), cubic part is found as

$$C=\begin{bmatrix} -14.9 & 7.5 & 7.5 & 0 & 0 & 0 \\ 7.5 & -14.9 & 7.5 & 0 & 0 & 0 \\ 7.5 & 7.5 & -14.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.5 \end{bmatrix}. \quad (40)$$

If the matrices given in (38) and (40) are added, the original matrix for Aluminium Antimonide given in (34) can be constructed.

B. For Zircon

Zircon is an example for tetragonal symmetry. There are six independent elastic constants for tetragonal symmetry which are $C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, C_{66}$. The elastic coefficients in GPa for it are given as[8]

$$C_{ij}=\begin{bmatrix} 284 & 73 & 119 & 0 & 0 & 0 \\ 73 & 284 & 119 & 0 & 0 & 0 \\ 119 & 119 & 309 & 0 & 0 & 0 \\ 0 & 0 & 0 & 77.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 77.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 47.7 \end{bmatrix} \quad (41)$$

The formula given in (29) is used to apply the method. For this reason inner products must be calculated as presented below

$$(C,A^{I})=499.67,\ (C,A^{II})=350,\ (C,A^{III})=-58.64,$$
$$(C,A^{IV})=20.4,\ (C,A^{V})=53.12,\ (C,A^{VI})=48.66 \quad (42)$$

The symmetric fourth rank tensor for Zircon can be represented in the form

$$C_{ijkl}=499.67A^{I}_{ijkl}+350A^{II}_{ijkl}-58.64A^{III}_{ijkl}+20.4A^{IV}_{ijkl}+$$
$$53.12A^{V}_{ijkl}+48.66A^{VI}_{ijkl} \quad (43)$$

If the orthonormalized basis elements of tetragonal symmetry are substituted into (43), identical matrix in (41) is obtained, which exhibits the validity of the decomposed terms. Isotropic part of Zircon is

$$I=499.67A^{I}_{ijkl}+350A^{II}_{ijkl} \quad (44)$$

If the corresponding orthonormalized basis elements are put into (44), isotropic part is found as

$$I=\begin{bmatrix} 270.92 & 114.37 & 114.37 & 0 & 0 & 0 \\ 114.37 & 270.92 & 114.37 & 0 & 0 & 0 \\ 114.37 & 114.37 & 270.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 78.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 78.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.27 \end{bmatrix} \quad (45)$$

The cubic part of the material is

$$C=-58.64A^{III}_{ijkl} \quad (46)$$

If the orthonormalized basis element stands for cubic symmetry, substituted into (46), cubic part is found as

$$C=\begin{bmatrix} 21.4 & -10.7 & -10.7 & 0 & 0 & 0 \\ -10.7 & 21.4 & -10.7 & 0 & 0 & 0 \\ -10.7 & -10.7 & 21.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10.7 \end{bmatrix}. \quad (47)$$

Lastly, the tetragonal part of the material is

$$Tet=20.4A^{IV}_{ijkl}+53.12A^{V}_{ijkl}+48.66A^{VI}_{ijkl} \quad (48)$$

If the corresponding orthonormalized basis elements for tetragonal symmetry are put into (48), tetragonal part is found as

$$Tet=\begin{bmatrix} -8.33 & -30.67 & 15.33 & 0 & 0 & 0 \\ -30.67 & -8.33 & 15.33 & 0 & 0 & 0 \\ 15.33 & 15.33 & 16.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9.93 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9.93 & 0 \\ 0 & 0 & 0 & 0 & 0 & -19.87 \end{bmatrix}. \quad (49)$$

If the matrices given in (45), (47) and (49) are summed up, the original matrix for Zircon in (41) is represented in terms of its orthonormal decomposed parts.

C. For Haematite

Haematite is a trigonal material which exhibits trigonal symmetry. There are six independent elastic constants for trigonal symmetry which are $C_{11}, C_{12}, C_{13}, C_{14}, C_{33}, C_{44}$.

The elastic constant data for Haematite are presented as[7]

$$C_{ij}=\begin{bmatrix} 242 & 54.9 & 15.7 & -12.7 & 0 & 0 \\ 54.9 & 242 & 15.7 & 12.7 & 0 & 0 \\ 15.7 & 15.7 & 228 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & 85.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 85.3 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 93.55 \end{bmatrix} \quad (50)$$

By using the formula given in (32), inner products are calculated as

$$(C,A^{I})=294.87,\ (C,A^{II})=422.8,\ (C,A^{III})=4.08,$$
$$(C,A^{IV})=-57.77,\ (C,A^{V})=-16.5,\ (C,A^{VI})=-50.8. \quad (51)$$

The symmetric fourth rank tensor for Haematetite can be represented in the form

$$C_{ijkl}=294.87A^{I}_{ijkl}+422.8A^{II}_{ijkl}+4.08A^{III}_{ijkl}-57.77A^{IV}_{ijkl}-$$
$$16.5A^{V}_{ijkl}-50.8A^{VI}_{ijkl} \quad (52)$$

When orhonormalized basis elements for trigonal materials are substituted into the right-hand side of (52), identical matrix in (50), which shows the validity of the decomposed terms.

Isotropic part of Haematite is

$$I = 294.87A^{I}_{ijkl} + 422.8A^{II}_{ijkl} \tag{53}$$

If the related orthonormalized basis elements are put into (53), isotropic part is found as

$$I = \begin{bmatrix} 224.35 & 35.26 & 35.26 & 0 & 0 & 0 \\ 35.26 & 224.35 & 35.26 & 0 & 0 & 0 \\ 35.26 & 35.26 & 224.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 94.54 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 94.54 \end{bmatrix} \tag{54}$$

The transversely isotropic part of the material is

$$TI = 4.08A^{III}_{ijkl} - 57.77A^{IV}_{ijkl} - 16.5A^{V}_{ijkl} \tag{55}$$

When orthonormalized basis elements stands for transversely isotropic part are substituted into (55) transversely isotropic part is found as

$$TI = \begin{bmatrix} 17.65 & 19.64 & -19.56 & 0 & 0 & 0 \\ 19.64 & 17.65 & -19.56 & 0 & 0 & 0 \\ -19.56 & -19.56 & 3.65 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9.24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9.24 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.99 \end{bmatrix}. \tag{56}$$

The trigonal part of the material is

$$TR = -50.8A^{VI}_{ijkl} \tag{57}$$

If we put the appropriate values into (57), trigonal part is found as

$$TR = \begin{bmatrix} 0 & 0 & 0 & -12.7 & 0 & 0 \\ 0 & 0 & 0 & 12.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -12.7 & 12.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12.7 \\ 0 & 0 & 0 & 0 & -12.7 & 0 \end{bmatrix}. \tag{58}$$

The matrices given in (54), (56),and (58) are the decomposed parts of the original matrix for Haematite given in (50). So (50) is represented by summation of (54), (56) and (58).

V. The Norm Concept And Anisotropy Degree

The norm concept for elastic constant tensor is described, norm and norm ratios as well as the measure of `nearness' of the nearest isotropic tensor are computed for several examples from various anisotropic materials possessing elastic symmetries such as cubic, tetragonal and trigonal. These computations are used to compare and assess the anisotropy in various anisotropic materials by means of strength or magnitude and also determine the `nearness' of the nearest isotropic tensor for the materials with lower symmetry types.

Norm is an invariant of the material. Because of this property, it can be used as a parameter representing and comparing the overall effect of a certain property of anisotropic materials of the same or different symmetry. So comparison of magnitudes of the norms give a valuable information about the origin of the physical property under examination. If the norm value of a material is large, it has more effective property than the other materials of the same symmetry type. Euclidean norm is used for computations as a measure in this work. Euclidean norm also represents the stiffness effect in the material like fiber-reinforced composites.

Euclidean norm of a Cartesian tensor is defined as the square root of the contracted product over all the indices with itself, which is given as follows

$$N = \|C\| = \{C_{ijkl\ldots}C_{ijkl\ldots}\}^{\frac{1}{2}} \tag{59}$$

Since the basis constructed in this thesis is orthonormal and $C_{ijkl\ldots}$ is in the space spanned by that orthonormal basis $\{A^K\}$, it is straightforward to see that, now the norm

$$N = \|C\| = \{\sum_K (C, A^K_{ijkl})^2\}^{\frac{1}{2}} \tag{60}$$

The norm of nearest isotropic tensor, denoted by C^o_{iikl}, of C_{ijkl} is therefore

$$N_i = \|C^o\| = \{\sum_{K=I}(C^o, A^K_{ijkl})^2\}^{\frac{1}{2}}, (K = I, II) \tag{61}$$

In similar way, with respect to the tensor C_{ijkl}, the nearest tensors of other symmetry classes within the class spanned by the basis $\{A^K\}$ can be read off from the representation and their norms may be computed according to (60). By using the norms, the nearest isotropic tensors of lower symmetries such as cubic, tetragonal and trigonal can be found via the following formula [3]

$$\varepsilon^o = \frac{\|C\| - \|C^o\|}{\|C\|} \tag{62}$$

Where ε^o is a scalar constant independent of the rotation of the axes. It is a measure of `nearness' of the nearest isotropic tensor.

It is obvious that the anisotropy of the material, for instance, the symmetry group of the material and the anisotropy of the measured property depicted in the same materials may be quite different. Clearly, the property tensor must show, at least, the symmetry of the material. For instance, a property which is measured in a material can almost be isotropic but the material symmetry group itself may have very few symmetry elements. We know that, for isotropic materials, the elastic constant tensor has two scalar (isotropic) parts, so the norm of the elastic constant tensor for isotropic materials depends only on the norm of the scalar parts, i.e., $N = N_i$. so the ratio $N_i / N = 1$ for isotropic materials. For cubic materials, the elastic constant tensor has the same two parts that consisting the isotropic symmetry and a third which is designated as the anisotropic part, hence we define two ratios: N_i / N for the isotropic parts and N_a / N for the anisotropic part. For lower symmetry type materials such as tetragonal and trigonal, the elastic constant tensor additionally contains more anisotropic parts, so we can define N_a / N for all the anisotropic parts.

Although the norm ratios of different parts represent the anisotropy of that particular part, they can also be used to asses and compare the anisotropy degree of a material property as a whole.

The following significant notes are taken into account when we have evaluated the computed results in following tables. These notes are:

1. It can be used as a parameter representing and comparing the overall effect of a certain property of anisotropic materials of the same or different symmetry. If the norm value of a material is large, it has more effective property than the other materials of the same symmetry type.

2. When N_i is the largest among norms of the decomposed parts, if the norm ratio N_i / N is closer to one, the material property is closer to isotropic.

3. When N_i is not the largest or not present, norm of the other parts can be used as a criterion. But in this case the situation is reverse; if the norm ratio value is larger than the others, the material property is more anisotropic.

In following sections, several examples from cubic, tetragonal and trigonal symmetries are presented.

A. Materials from Cubic Symmetry

Elastic constants of cubic materials are given in TABLE II. The units are in GPa.

TABLE II
ELASTIC CONSTANT DATA OF CUBIC MATERIALS

Cubic Media	C_{11}	C_{12}	C_{44}
AlSb[7]	87.7	43.4	40.8
Indium Phosphide(InP)[7]	102	58	46
Gallium Arsenide(GaAs)[7]	118	53.5	59.4
Gallium Antimonide(GaSb)[7]	88.4	40.3	43.4
Indium Arsenide(InAs)[7]	84.4	46.4	39.6
Gallium Phosphide(GaP)[7]	142	63	71.6

For cubic materials, the norm and norm ratios, ε^o (the anisotropy degrees) are computed in order to determine which one is close to isotropy or anisotropy. The results for norm, norm ratios and the measure of `nearness' of the nearest isotropic tensor are presented in the following TABLE.

TABLE III
THE NORM AND NORM RATIOS (THE ANISOTROPY DEGREES) FOR CUBIC MATERIALS

Cubic Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
AlSb	229.5	40.9	233.1	0.985	0.175	0.016
InP	272.1	52.6	277.1	0.982	0.190	0.018
GaAs	312.6	59.5	318.3	0.982	0.187	0.018
GaSb	232.4	42.4	236.2	0.984	0.180	0.016
InAs	226	45.1	230.5	0.981	0.196	0.019
GaP	375.3	70.3	381.9	0.983	0.184	0.017

According to the calculated results in TABLE III, the most isotropic material among the other six materials is Aluminium Antimonide (AlSb). Since mathematically, $\frac{N_i}{N}$ for AlSb is so close to 1 that implies the closeness to the isotropic behaviour of the cubic materials which agrees with the physical understanding of the materials with cubic symmetry. This case is also verified by taking into account the results of ε^o which is closer to 0 than those of other five materials which indicates that AlSb is nearest to isotropy among the other materials. The most anisotropic material is selected as Indium Arsenide (InAs). Since the value of $\frac{N_i}{N}$ for InAs is the smallest and in reverse manner, the value of $\frac{N_a}{N}$ for InAs is the largest among the cubic materials. This case shows that the property of Indium Arsenide is the most anisotropic.

B. Materials from Tetragonal Symmetry

Elastic constants of tetragonal materials are given in TABLE IV. The units are in GPa.

TABLE IV
ELASTIC CONSTANT DATA OF TETRAGONAL MATERIALS

Tetragonal Media	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	C_{66}
Zircon, $ZrSiO_4$ (metamict)[8]	284	73	119	309	77.5	47.7
Indium-cadmium alloy, In-3.42 at %Cd[9]	44.8	41	40.5	44.1	6.86	11.3
Ammonium dihydrogen arsenate (piezoel.), NH_4 H_2 ASO_4 [10]	62.2	8.6	18.4	29.6	6.69	6.22
Rolled steel[7]	284	96	112	269	82.1	68.9
Indium bismuth(InBi)[7]	51.1	37	32	34.6	19.8	15.9

The norm and norm ratios, ε^o (the anisotropy degrees) for tetragonal materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm,

norm ratios and the measure of `nearness' of the nearest isotropic tensor are summarized in TABLE V.

TABLE V
THE NORM AND NORM RATIOS (THE ANISOTROPY DEGREES) FOR TETRAGONAL MATERIALS

Tetragonal Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Zircon, $ZrSiO_4$ (metamict)	610.08	95.1	617.45	0.988	0.154	0.012
Indium-cadmium alloy, In-3.42 at %Cd	128.5	15.7	129.48	0.993	0.121	0.007
Ammonium dihydrogen arsenate (piezoel.), NH_4 H_2ASO_4	95.7	38.5	103.1	0.928	0.373	0.070
Rolled steel	611.5	36.1	612.5	0.998	0.059	0.002
Indium bismuth(InBi)	128	31.77	131.9	0.971	0.241	0.029

Due to the numerical results in TABLE V, rolled steel exhibits the most isotropic property among the others. On the other hand, by taking into account the ratio $\frac{N_a}{N}$, Ammonium dihydrogen arsenate (piezoel.) shows the most anisotropic property among the other tetragonal materials.

C. Materials from Trigonal Symmetry

Elastic constants of trigonal materials are presented in TABLE VI. The units are in GPa.

TABLE VI
ELASTIC CONSTANT DATA OF TRIGONAL MATERIALS

Trigonal Media	C_{11}	C_{12}	C_{13}	C_{14}	C_{33}	C_{44}
Haematite,Fe_2O_3 [7]	242	54.9	15.7	-12.7	228	85.3
Antimony[11]	99.4	30.9	26.4	21.6	44.5	39.5
Magnesite,$MgCO_3$ [12]	259	75.6	58.8	-19	156	54.8
As-Sb at % As 25.5[13]	106.7	48.4	28.5	18.8	48	40.8
Arsenic[14]	130.2	30.3	64.3	-3.71	58.7	22.5

The norm and norm ratios, ε^o (the anisotropy degrees) for trigonal materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm, norm ratios and the measure of `nearness' of the nearest isotropic tensor are summarized in TABLE VII.

TABLE VII
THE NORM AND NORM RATIOS (THE ANISOTROPY DEGREES) FOR TRIGONAL MATERIALS

Trigonal Media	N_i	N_a	N	$\frac{N_i}{N}$	$\frac{N_a}{N}$	ε^o
Haematite,Fe_2O_3	515.5	78.8	521.5	0.989	0.151	0.012
Antimony	202.3	100.9	226	0.895	0.446	0.105
Magnesite,$MgCO_3$	457.5	116.5	472.1	0.969	0.247	0.031
As-Sb at % As 25.5	214.4	97	235.3	0.911	0.412	0.089
Arsenic	250.4	85.4	264.5	0.946	0.323	0.054

From TABLE VII, it is understood that Haematite is the most isotropic material among the others by comparing the ratio $\frac{N_i}{N}$ and ε^o. Besides among trigonal materials, Antimony is the most anisotropic material by investigating the effect of the ratio $\frac{N_a}{N}$.

VI. RESULTS AND CONCLUSION

The decomposition methods of tensors have many applications in different subjects of engineering. In the mechanics of continuous media, for instance, in elasticity studies; the stress and strain tensors are decomposed into spherical (hydrostatic) and deviatoric parts each of which have important meanings. From (17), it is obvious that stress tensor is decomposed into spherical (hydrostatic pressure) part which is the first term of (16) and the deviatoric part which is the sum of the other five terms of (16). It is decomposed into traceless tensors, each of them is related to shearing which represents a general symmetric second rank tensor (stress and strain tensors).

Each of the six tensor parts has physical meanings and all decomposed parts form an orthonormal set. The first part of equation (16) represents the spherical (hydrostatic pressure) effect which is connected to the change of volume without change of shape through the bulk modulus. The second and third part represent combined simple extension or contraction along the various symmetry axes. The second part is a special case of biaxial stress which is plane stressed state. This part could be, for example, the stresses which are produced by torsional loading in a shaft. For Mohr's circle construction, the center coincides with the origin of axes and a rotation of 90^o (on the circle) leads to a state of stress in which the normal stresses are zero. This rotation is equivalent to a 45^o rotation in the body (real space). The magnitude of the shear stress at this orientation is equal to the radius of the circle. It shows at once that if the axes are turned through 45^o about Ox_3 (the axis of shear) then, the normal stresses vanish.

This is why the stress is named as pure shear stress and the tensor takes the form of fourth, fifth and sixth parts respectively[15]. Pure shear stress fields arise in many practical cases; for example, these type of stresses occur in the torsion of linearly elastic rods[16]. Furthermore the components of the third part are proportional to 1, 1, -2. This case may be an example for cylindrical shear. It is axisymmetric with respect to the Ox_3 axis which means invariant under a rotation about it. Thick walled cylindrical pressure vessels are one of the most typical applications of these type of stresses. The internal pressure will cause stresses in the material such that the hoop stress component is twice as much as the axial stress components (radial stresses and longitudinal stresses) for the cylindrical pressure vessel[17]. But the hoop stress and axial stresses are in different directions. This cause an advantage for engineering materials that can be made stronger in one direction than another (the property of anisotropy). Last three parts represent simple shearing in the symmetry planes. The sum of these three parts correspond to state of pure shear stresses which is Cauchy stress tensor. It is traceless and symmetric. Pure shear stress state has been widely described in recent studies such as Blinowski and Rychlewski[16], Hayes and Laffey[18].It should be noted that symmetric second rank tensors such as stress tensor and strain tensors are important subjects to understand the idea behind mechanics and elasticity. This is why decomposing them into orthonormal parts plays a significant role.

This method helps us to figure out the physical meanings of these tensors by decomposing them into six parts which introduces a new form of decomposition.

This decomposition method for elastic constant tensor have many applications in various subjects of science (atomic and molecular physics and the physics of condensed matter) and engineering.

Moreover, for very valuable materials (diamonds, quartz) used in mining, it is difficult to measure its elastic constants because of its small samples. Applying this decomposition procedure, it is possible to specify the elastic constants of these types of materials.

Representation of elastic constant tensor in terms of its orthonormal parts by this method provides a deeper understanding about elastic and mechanical behavior of anisotropic engineering materials. It also has more significant effects on many applications in different fields such as

1) investigation of the pure shear and pure longitudinal wave propagation in different anisotropic engineering materials.

2) study the effect of angle orientation of fibers and the material properties of fibers and the material properties of fiber and matrix on the stiffness of the composite.

3) determination of material symmetry type.

4) computation of norm and norm ratios for assessing and comparing the anisotropic properties of materials.

Finally, I hope this paper prepares interested readers to appreciate a deep understanding of application of this method to stress tensor as an example of symmetric second rank tensor and general review of the method [5] based upon orthonormal representations.

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