Systems of Singularly Perturbed Fractional Integral Equations II

Angelina M. Bijura

***Abstract*—The solution of a singularly perturbed nonlinear system fractional integral (differential) equations of order $\varsigma,\ 0 < \varsigma < 1$ is investigated. The leading order formal asymptotic solution is derived and proved to have the required properties.**

***Index Terms*—singular perturbation, fractional calculus, Volterra integral equations**

I. Introduction

CONSIDER the nonlinear singularly perturbed system of fractional order equations,

$$\begin{aligned}\varepsilon\boldsymbol{\phi}(t) \;\; &= \;\; \boldsymbol{\psi}(t;\varepsilon) + {}_0J_t^{\varsigma} f(t,\boldsymbol{\phi}(t)), \;\; 0 \le t \le T, \\ & \quad T > 0,\ 0 < \varsigma < 1,\ 0 < \varepsilon << 1. \end{aligned} \tag{1}$$

The vector function $\boldsymbol{\psi}$ is assumed to be smooth with an asymptotic expansion of the form

$$\boldsymbol{\psi}(t;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \boldsymbol{\psi}_i(t),\ \varepsilon \to 0, \tag{2}$$

where

$$\boldsymbol{\psi}_i(t) : 0 \le t \le T \to \mathbb{R}^n \in C^{\infty}(\mathbb{R}^n), \quad \boldsymbol{\psi}_i(0) = \mathbf{0}, \forall\ i.$$

The operator ${}_aJ_t^{\alpha}$, and later ${}_aD_t^{\alpha}$ are defined using the Riemann-Liouville definition. For a continuous function ϕ and for $a < \alpha < 1,\ \alpha > 0$,

$${}_aJ_t^{\alpha}\phi(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\phi(s)\, ds,\ t \ge a, \tag{3a}$$

$${}_aD_t^{\alpha}\phi(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha}\phi(s)\, ds,\ t > a. \tag{3b}$$

The analysis of the formal asymptotic solution of (1) in this paper follows that of a linear version studied in [2]. Singular perturbation is a widely studied research area for both differential and integral equations. For ordinary differential equations and integral equations, with continuous kernel, the study is extensive. For singularly perturbed Volterra equations with weakly singular kernels and fractional differential (integral) equations, the study is far from comprehensive. The disparity between the two lies in the fact that the former have formal solutions whose inner layer functions decay exponentially and the latter have inner solutions that decay algebraically. Exponential decay is easier to analyse, but there are some limitations in the analysis when algebraic decay is encountered. The problem studied in this paper exhibits an inner layer algebraic decay. For a detailed study on singular perturbation theory including practical problems and references see O'Malley [19]-[20], Smith [26], Lagerstrom [11], Kevorkian and Cole [9], Kauthen [8], Verhulst [29], Witelski and Bowen [28], and most recently Skinner [24] gave the analysis of the matched asymptotic expansion method on singularly perturbed differential equations.

The algebraic decay (behaviour) displayed by solutions of singularly perturbed Volterra equations with weakly singular kernels resembles the asymptotic behaviour of higher transcendental functions of fractional calculus. Fractional calculus is a research topic that is currently receiving huge attention from scientists and engineers. The complexity of the algebraic decay in inner layer solutions of singularly perturbed Volterra equations with weakly singular kernels, and the increased popularity of fractional order models incites the study in this paper. The novelty of the paper lies in the application of the Mittag-Leffler stability to prove algebraic decay and asymptotic correctness. The origin, theory and applications of fractional differential (and integral) equations, can be found in Oldham and Spanier [18], Eskin [5], Hilfer [7], Miller [17], Samko [23], Igor [22], Kilbas [10] and Ortigueira [21]. For specific applications for example, bioengineering see Magin [15]; transportation modeling see Schumer, Meerschaert and Baeumer [25]; control applications and system modeling see Caponetto, Dongola, Fortuna and Petráš [3]; applications in viscoelasticity see Mainardi [16]; applications in dynamics of particles, fields, media and complex systems see [27]; and references therein.

In the next section, mathematical preliminaries used throughout the paper are presented. In Section III, the formal asymptotic solution of (1) is derived. The derivation is limited to the leading order since higher order terms of the inner layer solution depend on the actual value of ς. It is also shown in Section III, using the existing literature on Volterra integral equations and fractional calculus, that the formal solution has the required properties. A basic matrix algebra is applied to accomplish this. In particular, it is shown that the inner layer solution decays to zero as the stretched variable approaches infinity, like the Mittag-Leffer function - a special function in fractional calculus. This property of the inner layer solution is then used to prove that the remainder term, obtained when the formal solution satisfies (1) approximately, is asymptotically small and therefore validates the solution. To demonstrate the methodology developed in Section III, an example of a nonlinear singularly perturbed fractional equation is presented and solved in Section IV.

Manuscript received June 27, 2012; revised September 04, 2012.
A. Bijura is with the Department of Mechanical Engineering and Mathematical Sciences, Oxford Brookes University, Oxford OX33 1HX, UK
e-mail: abijura@brookes.ac.uk.

(Advance online publication: 21 November 2012)

II. Mathematical Preliminaries

The following will be assumed throughout the paper:

- S_1 - $0 < \varsigma < 1$
- S_2 - The vector functions ${}_0D_t^{\varsigma}\boldsymbol{\psi} : [0,\infty) \to \mathbb{R}^n, \mathbf{f} : ([0,\infty)\times\mathbb{R}^n) \to \mathbb{R}^n$ are both C^∞ with $\boldsymbol{\psi}(0;\varepsilon) = \mathbf{0}$.
- S_3 - There exists a number $\eta > 0$ such that
$$\max_{\lambda\in\sigma\{\partial_2\mathbf{f}(0,u(0))\}}\{Re(\lambda)\} \le -\eta$$
- S_4 - For every $\lambda \in \sigma\{\partial_2 f(0,\mathbf{u}(0))\}$ the algebraic multiplicity of λ is equal to the dimension of its eigenspace

Analogous to exponential stability, the Mittag-Leffler stability is a common property used to prove stability and boundedness of solutions of fractional differential (and integral) equations. The following definition of the Mittag-Leffler stability is a special case of the one given in [14]:

Definition II.1. *The Mittag-Leffler Stability*
Consider the solution $\mathbf{y}$ *of*

$$\begin{aligned} {}_{t_0}D_t^{\alpha}\mathbf{y} = \mathbf{g}(t,\mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \\ 0<\alpha<1, \quad t_0 \le t < \infty, \end{aligned} \tag{4}$$

where $\mathbf{g} : [t_0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ *is continuous and locally Lipschitz in* $\mathbf{y}$. *The operator* D^α *is either Caputo or Riemann-Liouville fractional derivative.*
The solution of (4) *is said to be Mittag-Leffler stable if*

$$\begin{aligned} \|\mathbf{y}(t)\| \le m(\mathbf{y}_0)\,\mathrm{E}_{\alpha,1}(-\lambda(t-t_0)^\alpha), \quad \lambda > 0, \\ m(\mathbf{y}_0) \ge 0, \quad m(\mathbf{0}) = 0, \end{aligned}$$

and where m *is locally Lipschitz in* $\mathbf{y}$. *The function* $\mathrm{E}_{\alpha,\beta}$, *known as the generalized Mittag-Leffler function, is defined by*

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)}, \qquad \alpha,\beta > 0, z\in\mathbb{C}.$$

III. Formal Solution

A. Derivation

The fact that $\boldsymbol{\psi}(0;\varepsilon) = \mathbf{0}$, equation (1) can be written as

$$\begin{aligned} \varepsilon\boldsymbol{\phi}(t) = {}_0J_t^{\varsigma}\left\{{}^{\star}\boldsymbol{\psi}(t;\varepsilon) + f(t,\boldsymbol{\phi}(t))\right\}, \\ 0 \le t \le T, \quad 0<\varsigma<1, \end{aligned} \tag{5}$$

where,

$${}^{\star}\boldsymbol{\psi}(t;\varepsilon) = {}_0D_t^{\varsigma}\boldsymbol{\psi}(t;\varepsilon).$$

The solution of (1) (and hence (5)) is thought in the form of

$$\boldsymbol{\phi}(t;\varepsilon) = \boldsymbol{\phi}_{\text{out}}(t;\varepsilon) + \varphi(\varepsilon)\boldsymbol{\phi}_{\text{in}}(\tau;\varepsilon),$$

where $\boldsymbol{\phi}_{\text{out}}$ represents the outer solution, $\varphi(\varepsilon)$ scales the amplitude of the inner layer and $\boldsymbol{\phi}_{\text{in}}$ represents the inner layer solution. Both $\boldsymbol{\phi}_{\text{out}}$ and $\boldsymbol{\phi}_{\text{in}}$ are assumed to be expandable in powers of ε:

$$\boldsymbol{\phi}_{\text{out}}(t;\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i\boldsymbol{\zeta}_i(t), \ \varepsilon\to 0, \tag{6a}$$

$$\boldsymbol{\phi}_{\text{in}}(\tau;\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^{\gamma i}\boldsymbol{\xi}_i(\tau), \quad \tau = \frac{t}{\varepsilon^\gamma}, \ \varepsilon\to 0. \tag{6b}$$

To obtain the formal solution, one forms the partial sum

$$\boldsymbol{\phi}_N(t;\varepsilon) = \sum_{i=0}^{N}\varepsilon^i\boldsymbol{\zeta}_i(t) + \varphi(\varepsilon)\sum_{i=0}^{N}\varepsilon^{\gamma i}\boldsymbol{\xi}_i(\frac{t}{\varepsilon^\gamma}), \ \varepsilon\to 0,$$

and substitute it into (5). Expressing all terms in the resulting equation in terms of τ and applying the dominant balance argument, it follows that

$$\varphi(\varepsilon) = \text{ Ord}(1), \quad \varepsilon\to 0, \quad \text{and} \quad \gamma = \frac{1}{\varsigma}, \quad 0<\varsigma<1.$$

This implies the following formal asymptotic solution,

$$\boldsymbol{\phi}_N(t;\varepsilon) = \sum_{i=0}^{N}\varepsilon^i\boldsymbol{\zeta}_i(t) + \sum_{i=0}^{N}\varepsilon^{\gamma i}\boldsymbol{\xi}_i(\frac{t}{\varepsilon^\gamma}), \ \varepsilon\to 0. \tag{7}$$

Since the inner layer solution is negligible outside the layer region, one requires that for each $i\ge 0$, $\lim_{\tau\to\infty}\boldsymbol{\xi}_i(\tau) = \mathbf{0}$. In particular, as it will later be shown,

$$\boldsymbol{\xi}_i(\tau) = \text{ Ord}(\tau^{-\varsigma}), \quad \tau\to\infty, \quad \forall\ i\ge 0. \tag{8}$$

To derive the formal solution, substitute (7) into (5) giving

$$\begin{aligned} \sum_{i=0}^{N}\varepsilon^{i+1}\boldsymbol{\zeta}_i(t) + \sum_{i=0}^{N}\varepsilon^{\gamma i+1}\boldsymbol{\xi}_i(\frac{t}{\varepsilon^\gamma}) = {}_0J_t^{\varsigma}\left\{{}^{\star}\boldsymbol{\psi}(t;\varepsilon)\right\} \\ + {}_0J_t^{\varsigma}\left\{\mathbf{p}(t;\varepsilon) + \mathbf{q}(t,\tau;\varepsilon)\right\}, \end{aligned} \tag{9}$$

where,

$${}^{\star}\boldsymbol{\psi}(t;\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i{}_{\star}\boldsymbol{\psi}_i(t), \quad \varepsilon\to 0, \tag{10a}$$

$$\mathbf{p}(t;\varepsilon) = f\left(t,\sum_{i=0}^{N}\varepsilon^i\boldsymbol{\zeta}_i(t)\right), \quad \varepsilon\to 0, \tag{10b}$$

$$\begin{aligned} \mathbf{q}(t,\tau;\varepsilon) &= f\left(t,\sum_{i=0}^{N}\varepsilon^i\boldsymbol{\zeta}_i(t) + \sum_{i=0}^{N}\varepsilon^{\gamma i}\boldsymbol{\xi}_i(\tau)\right) \\ &\quad - f\left(t,\sum_{i=0}^{N}\varepsilon^i\boldsymbol{\zeta}_i(t)\right). \end{aligned} \tag{10c}$$

From above,

$$\mathbf{p}(t;\varepsilon) = \sum_{i=0}^{N}\varepsilon^i\mathbf{p}_i(t) + \text{ Ord}(\varepsilon^{N+1}), \quad \varepsilon\to 0.$$

The coefficients $\mathbf{p}_i(t)$ are given by

$$\begin{aligned} \mathbf{p}_0(t) &= f(t,\boldsymbol{\zeta}_0(t)), \\ \mathbf{p}_1(t) &= \partial_2 f(t,\boldsymbol{\zeta}_0(t))\boldsymbol{\zeta}_1(t). \text{ In general,} \\ \mathbf{p}_i(t) &= \partial_2 f(t,\boldsymbol{\zeta}_0(t))\boldsymbol{\zeta}_i(t) + \boldsymbol{\omega}_i(t), \end{aligned}$$

where $\boldsymbol{\omega}_i(t)$ is determined by $\boldsymbol{\omega}_k(t)$ for $0\le k\le i-1$. The first two terms of $\boldsymbol{\omega}_i(t)$ are:

$$\boldsymbol{\omega}_1(t) = 0, \quad \boldsymbol{\omega}_2(t) = \frac{1}{2}\partial_2^2 f(t,\zeta_0(t))\zeta_1^2(t).$$

Using the mean value theorem and (8), one can show that the function $q(t,\tau;\varepsilon)$ in (10c) satisfies

$$q(t,\tau;\varepsilon) = \text{ Ord}(\tau^{-\varsigma}), \quad \tau\to\infty. \tag{11}$$

The equation governing the outer solution is obtained from (9) by letting $\varepsilon \to 0$ and equating coefficients of equal powers of ε. Using (8) and (11) gives

$$\begin{aligned} \mathbf{0} &= {}_0J_t^{\varsigma}\left\{{}_{\star}\boldsymbol{\psi}_0(t) + f(t,\boldsymbol{\zeta}_0(t))\right\} \quad (12a)\\ \boldsymbol{\zeta}_{i-1}(t) &= {}_0J_t^{\varsigma}\left\{{}_{\star}\boldsymbol{\psi}_i(t) + \partial_2 f(t,\boldsymbol{\zeta}_0(t))\zeta_i(t) + \boldsymbol{\omega}_i(t)\right\},\\ & \qquad 0 \le t \le T, \quad i \ge 1. \quad (12b)\end{aligned}$$

To derive the equation governing the inner layer solution, express (9) in terms of τ and equate coefficient of equal powers of ε. Using (12a), the leading order inner layer equation becomes

$$\begin{aligned} \boldsymbol{\xi}_0(\tau) &= -\boldsymbol{\zeta}_0(0) + {}_0J_\tau^{\varsigma}\left\{f(0,\boldsymbol{\zeta}_0(0)+\boldsymbol{\xi}_0(\tau))\right\}\\ &\quad - {}_0J_\tau^{\varsigma}\left\{f(0,\boldsymbol{\zeta}_0(0))\right\}, \quad \tau \ge 0. \quad (13)\end{aligned}$$

Higher order equations depend on the actual values of ς.

B. The Outer Solution

The leading order outer solution $\boldsymbol{\zeta}_0(t)$, follows from (12a) and is given implicitly by

$$f(t,\boldsymbol{\zeta}_0(t)) = -{}^{\star}\boldsymbol{\psi}_0(t). \quad (14)$$

For higher order terms of the outer equation, the solution $\boldsymbol{\zeta}_i(t)$, for $i \ge 1$ follows from (12b) and is given by

$$\boldsymbol{\zeta}_{i-1}(t) = {}_0J_t^{\varsigma}\left\{{}^{\star}\boldsymbol{\psi}_i(t) + \boldsymbol{\omega}_i(t) + \partial_2 f(t,\boldsymbol{\zeta}_0(t))\boldsymbol{\zeta}_i(t)\right\}, \qquad 0 \le t \le T.$$

This is a nonlinear Volterra equation of the first kind. Since ${}^{\star}\boldsymbol{\psi}_i$, $\boldsymbol{\omega}_i$ and $\mathbf{f}$ are continuously differentiable vector functions, the existence and uniqueness of $\boldsymbol{\zeta}_i(t)$ for $t \ge 0$ and all $i \ge 1$ is guaranteed only if $\boldsymbol{\zeta}_{i-1}(0)$, $i \ge 1$. For a detailed discussion on this see for example, [6] or [13]. In the context of fractional calculus, the existence and uniqueness theorems can be found in [17], [22] and [10].

C. The Inner Layer Solution

The application of the Taylor theorem is used to write (13) as

$$\begin{aligned} \boldsymbol{\xi}_0(\tau) &= -\boldsymbol{\zeta}_0(0) + {}_0J_\tau^{\varsigma}\left\{\partial_2 f(0,\boldsymbol{\zeta}_0(0))\boldsymbol{\xi}_0(\tau)\right\}\\ &\quad + {}_0J_\tau^{\varsigma}\left\{\mathbf{h}(\boldsymbol{\xi}_0(\tau))\right\}, \quad \tau \ge 0, \quad (15)\end{aligned}$$

where $\partial_2 f \in \mathbb{R}^{n\times n}$, and $\mathbf{h}(\boldsymbol{\xi}_0(\tau)) = \mathrm{Ord}(\boldsymbol{\xi}_0(\tau))$, $\tau \to 0, \quad \tau \to \infty$.

Following the assumptions in Section 2, the existing literature of fractional calculus can be applied to show that equation (13) has a unique solution $\boldsymbol{\xi}_0(\tau)$ which is continuous and bounded for all $\tau \ge 0$. For details see [10], [17] and [22].

The remainder of this subsection demonstrates that indeed the solution $\boldsymbol{\xi}_0$ satisfies (8), the main result that was assumed in the derivation of the formal solution.

Equation (15) is a perturbed linear equation. The unperturbed equation,

$$\boldsymbol{\varrho}(\tau) = \boldsymbol{\varrho}(0) + {}_0J_\tau^{\varsigma} A\boldsymbol{\varrho}(\tau), \quad \tau \ge 0, \quad (16)$$

where $A \in \mathbb{R}^{n\times n}$ has been shown in [2] to satisfy

$$\boldsymbol{\varrho}(\tau) = \mathrm{O}(\tau^{-\varsigma}), \quad \tau \to \infty. \quad (17)$$

Levin [12] proved that solutions of the unperturbed and perturbed systems are asymptotically equivalent. Therefore, the solution $\boldsymbol{\xi}_0(\tau)$ of (15) approaches zero as τ approaches infinity. To show that indeed this solution satisfies condition (8) which is used in the derivation of the formal solution, it will be shown that the solution $\boldsymbol{\xi}_0(\tau)$ is Mittag-Leffer stable.

Equation (15) is sub-linear. To determine the asymptotic behaviour, one may suppress the dependence of $\mathbf{h}$ on $\boldsymbol{\xi}_0$ and writes the equation as

$$\boldsymbol{\xi}_0(\tau) = -\boldsymbol{\zeta}_0(0) + {}_0J_\tau^{\varsigma}\left\{\partial_2 f(0,\boldsymbol{\zeta}_0(0))\boldsymbol{\xi}_0(\tau) + \boldsymbol{\chi}(\tau)\right\}, \qquad \tau \ge 0, \quad (18)$$

where

$$\boldsymbol{\chi}(\tau) = \mathrm{O}(\boldsymbol{\xi}_0(\tau)), \quad \tau \to 0, \ \tau \to \infty. \quad (19)$$

Taking the Laplace transform on both sides of (18) one has

$$\Xi(s) = -s^{\varsigma-1}\left(s^{\varsigma}I_n - M\right)^{-1}\boldsymbol{\zeta}_0(0) + \left(s^{\varsigma}I_n - M\right)^{-1}\mathbf{X}(s),$$

where,

$$\begin{aligned} \Xi(s) &= \mathcal{L}\left(\boldsymbol{\xi}_0(\tau)\right),\\ \mathbf{X}(s) &= \mathcal{L}\left(\boldsymbol{\chi}(\tau)\right),\\ I_n & \quad \text{is the identity matrix of size n,}\\ M &= \partial_2 f(0,\boldsymbol{\zeta}_0(0)), \ \text{an } n\times n \text{ matrix,}\\ \mathcal{L} & \quad \text{is the laplace operator.}\end{aligned}$$

By condition S_4, M is diagonalisable. If λ is an eigenvalue of M with corresponding eigenvector $\mathbf{v}$ then $\frac{1}{s^{\varsigma}-\lambda}$ is an eigenvalue of $\left(s^{\varsigma}I_n - M\right)^{-1}$ corresponding to the same eigenvector, $\mathbf{v}$. It follows that the matrix $\left(s^{\varsigma}I_n - M\right)^{-1}$ is also diagonalisable and therefore an invertible matrix P and a diagonal matrix $D(s)$ exist such that

$$\Xi(s) = -s^{\varsigma-1}PD(s)P^{-1}\boldsymbol{\zeta}_0(0) + PD(s)P^{-1}\mathbf{X}(s),$$

where

$$D(s) = \begin{pmatrix} \frac{1}{s^{\varsigma}-\lambda_1} & 0 & \dots & \dots & 0\\ 0 & \frac{1}{s^{\varsigma}-\lambda_2} & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \dots & \vdots\\ 0 & 0 & \dots & \dots & \frac{1}{s^{\varsigma}-\lambda_n}\end{pmatrix}.$$

Here, λ_i is an i-th eigenvalue of M and $\frac{1}{s^{\varsigma}-\lambda_i}$ is an i-th eigenvalue of $\left(s^{\varsigma}I_n - M\right)^{-1}$. The matrices P and P^{-1} are independent of s.

Using S_3, the infinity norm of a matrix and the supremum norm of a vector, one obtains

$$\|\Xi(s)\|_\infty \le \alpha_0\|\boldsymbol{\zeta}_0(0)\|_\infty\frac{s^{\varsigma-1}}{s^{\varsigma}+\eta} + \alpha_0\frac{1}{s^{\varsigma}+\eta}\|\mathbf{X}(s)\|_\infty,$$

where

$$\alpha_0 = \|P\|_\infty\|P^{-1}\|_\infty.$$

Let $\mathbf{K}(s)$ be a positive vector function such that there is a positive vector $\mathbf{k}(\tau)$ which satisfies

$$\mathcal{L}\left(\mathbf{k}(\tau)\right) = \mathbf{K}(s).$$

Then

$$\|\Xi(s)\|_\infty = \alpha_0\|\boldsymbol{\zeta}_0(0)\|_\infty\frac{s^{\varsigma-1}}{s^{\varsigma}+\eta} + \alpha_0\frac{1}{s^{\varsigma}+\eta}\|\mathbf{X}(s)\|_\infty - \mathbf{K}(s).$$

Applying the inverse Laplace transform on both sides, one gets,

$$\|\boldsymbol{\xi}_0(\tau)\|_\infty = \alpha_0 \|\boldsymbol{\zeta}_0(0)\|_\infty \,\mathrm{E}_\varsigma(-\eta\tau^\varsigma) + \alpha_0 \int_0^\tau (\tau-\sigma)^{\varsigma-1}\,\mathrm{E}_{\varsigma,\varsigma}(-\eta(\tau-\sigma)^\varsigma)\|\chi(\sigma)\|_\infty\, d\sigma - \mathbf{k}(\tau),$$

where $\mathrm{E}_{\alpha,\beta},\ \alpha,\beta > 0$ is the Mittag-Lefler function defined in Section 2. It then follows that,

$$\|\boldsymbol{\xi}_0(\tau)\|_\infty \le \alpha_0 \|\boldsymbol{\zeta}_0(0)\|_\infty \,\mathrm{E}_\varsigma(-\eta\tau^\varsigma) + \alpha_0 \int_0^\tau (\tau-\sigma)^{\varsigma-1}\,\mathrm{E}_{\varsigma,\varsigma}(-\eta(\tau-\sigma)^\varsigma)\|\chi(\sigma)\|_\infty\, d\sigma. \qquad (20)$$

Using (17), (19) and Levin's [12] result, one can show that,

$$\|\chi(\tau)\|_\infty \le \|\boldsymbol{\zeta}_0(0)\|_\infty \tau^{-\varsigma}, \quad \tau \to \infty.$$

Applying this inequality in (20) yields,

$$\|\boldsymbol{\xi}_0(\tau)\|_\infty \le \alpha_0 \|\boldsymbol{\zeta}_0(0)\|_\infty \,\mathrm{E}_\varsigma(-\eta\tau^\varsigma) + \alpha_0 \|\boldsymbol{\zeta}_0(0)\|_\infty \int_0^\tau (\tau-\sigma)^{-\varsigma}\sigma^{\varsigma-1}\,\mathrm{E}_{\varsigma,\varsigma}(-\eta(\tau-\sigma)^\varsigma)\, d\sigma.$$

The integration of the Mittag-Leffler function, see for example [22], gives the required result,

$$\|\boldsymbol{\xi}_0(\tau)\|_\infty \le \alpha_0 \|\boldsymbol{\zeta}_0(0)\|_\infty (1+\Gamma(\varsigma))\,\mathrm{E}_{\varsigma,1}(-\eta\tau^\varsigma). \qquad (21)$$

This proves that the solution of the leading inner layer equation (13) is Mittag-Leffler stable. The asymptotic expansion of the Mittag-Leffler function, originally from [4] and later in several research articles including [22] and [1] validates (8).

D. The Remainder Term

Let $\mathbf{u}_0(t;\varepsilon) = \boldsymbol{\zeta}_0(t) + \boldsymbol{\xi}_0(\frac{t}{\varepsilon^\gamma})$. Suppose that vector $\mathbf{u}_0(t;\varepsilon)$ satisfies (1) approximately with a remainder, $\boldsymbol{\rho}_0(t;\varepsilon)$. Then

$$\begin{aligned}\varepsilon \mathbf{u}_0(t;\varepsilon) &= \boldsymbol{\psi}(t;\varepsilon) + {}_0J_t^\varsigma f(t,\mathbf{u}_0(t;\varepsilon)) - \boldsymbol{\rho}_0(t;\varepsilon), \\ & \quad 0 \le t \le T,\ T>0,\ 0<\varepsilon<<1.\end{aligned}$$

Equivalently,

$$\begin{aligned}\boldsymbol{\rho}_0(t;\varepsilon) &= \boldsymbol{\psi}(t;\varepsilon) + {}_0J_t^\gamma f(t,\mathbf{u}_0(t;\varepsilon)) - \varepsilon\mathbf{u}_0(t;\varepsilon), \\ & \quad 0 \le t \le T,\ T>0,\ 0<\varepsilon<<1. \qquad (22)\end{aligned}$$

The asymptotic correctness of the derived formal solution, in terms of the leading order solution, is presented in the theorem below.

Theorem III.1. *Suppose* S_1, S_2, S_3 *and* S_4 *hold. Then the residual* $\boldsymbol{\rho}_0(t;\varepsilon)$ *given in* (22) *satisfies,*

$$|\boldsymbol{\rho}_0(t;\varepsilon)| \le \kappa_0\ \varepsilon, \quad \varepsilon \to 0,$$

uniformly for all $0 \le t \le T,\ \ T>0$, *for some fixed positive constant* κ_0 *which does not depend on* ε.

Proof

Consider (22),

$$\begin{aligned}\boldsymbol{\rho}_0(t;\varepsilon) &= \boldsymbol{\psi}(t;\varepsilon) + {}_0J_t^\varsigma f(t,\boldsymbol{\zeta}_0(t)+\boldsymbol{\xi}_0(t/\varepsilon^\gamma)) \\ & \quad -\varepsilon(\boldsymbol{\zeta}_0(t)+\boldsymbol{\xi}_0(t/\varepsilon^\gamma)).\end{aligned}$$

Using (14) and (13) expressed in t, one obtains

$$\begin{aligned}\boldsymbol{\rho}_0(t;\varepsilon) &= {}_0J_t^\varsigma \left\{f(t,\boldsymbol{\zeta}_0(t)+\boldsymbol{\xi}(t/\varepsilon^\gamma)) - f(t,\boldsymbol{\zeta}_0(t))\right\} \\ & \quad - {}_0J_t^\varsigma \left\{f(0,\boldsymbol{\zeta}_0(0)+\boldsymbol{\xi}(t/\varepsilon^\gamma)) - f(0,\boldsymbol{\zeta}_0(0))\right\}.\end{aligned}$$

The Taylor theorem then yields,

$$\begin{aligned}\boldsymbol{\rho}_0(t;\varepsilon) &= {}_0J_t^\varsigma \left\{\partial_2 f(t,\boldsymbol{\zeta}_0(t)) - \partial_2 f(0,\boldsymbol{\zeta}_0(0))\right\} \boldsymbol{\xi}(t/\varepsilon^\gamma) \\ & \quad + o(\boldsymbol{\xi}(t/\varepsilon^\gamma)), \quad \varepsilon \to 0.\end{aligned}$$

From this equation it is clear that for fixed $t>0$, $\boldsymbol{\rho}(t;\varepsilon) \to 0$, as $\varepsilon \to 0$. To prove the theorem, apply (21) into the above equation to get

$$\|\boldsymbol{\rho}_0(t;\varepsilon)\|_\infty \le \alpha_0 \|\boldsymbol{\zeta}_0(0)\|_\infty (1+\Gamma(\varsigma))\ {}_0J_t^\varsigma \|\partial_2 f(t,\boldsymbol{\zeta}_0(t)) - \partial_2 f(0,\boldsymbol{\zeta}_0(0))\|_\infty \,\mathrm{E}_{\varsigma,1}(-\eta t^\varsigma/\varepsilon),\ \varepsilon \to 0.$$

Using the formula

$$\frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\,\mathrm{E}_{\alpha,1}(\lambda t^\alpha) = \mathrm{E}_{\alpha,1}(\lambda x^\alpha) - 1, \qquad \alpha>0,\ \lambda \in \mathbb{C},$$

one obtains

$$\begin{aligned}\|\boldsymbol{\rho}_0(t;\varepsilon)\|_\infty &\le \frac{\varepsilon}{\eta}\alpha_0\alpha_1 \|\boldsymbol{\zeta}_0(0)\|_\infty (1+\Gamma(\varsigma)) \\ & \quad \left\{1-\mathrm{E}_{\varsigma,1}(-\eta t^\varsigma/\varepsilon)\right\},\ \varepsilon \to 0.\end{aligned}$$

Here,

$$\alpha_1 = \|\partial_2 f(t,\boldsymbol{\zeta}_0(t)) - \partial_2 f(0,\boldsymbol{\zeta}_0(0))\|_\infty,\ \ 0 \le t \le T,\ T>0.$$

An equivalent inequality is

$$\|\boldsymbol{\rho}_0(t;\varepsilon)\|_\infty \le \frac{\varepsilon}{\eta}\alpha_0\alpha_1 \|\boldsymbol{\zeta}_0(0)\|_\infty (1+\Gamma(\varsigma)),$$

which is the required result. Therefore the formal asymptotic solution is asymptotically correct.

IV. EXAMPLE

Consider the following example of a singularly perturbed nonlinear fractional integral equation of order $\frac{1}{2}$.

$$\varepsilon \boldsymbol{\phi}(t) = {}_0J_t^{\frac{1}{2}} \left\{ \begin{pmatrix} \frac{\sqrt{\pi}}{2} \\ 0 \\ -\frac{\pi}{2} \end{pmatrix} \right\} + {}_0J_t^{\frac{1}{2}} \left\{ \begin{pmatrix} -\phi_1(t)+\phi_2(t) \\ 0.5\phi_1(t)-\phi_2(t)-\phi_1(t)\phi_3(t) \\ -\phi_3(t)+\phi_1(t)\phi_2(t) \end{pmatrix} \right\}, \qquad (23)$$

where

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad \text{and} \quad t \ge 0.$$

It is not difficult to check that all assumptions in Section 2 are satisfied by equation (23).

A. Formal Solution

The leading order outer solution is given, from (14), by

$$\begin{pmatrix} -\frac{\sqrt{\pi}}{2} \\ 0 \\ \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{\zeta_0}(t) + \begin{pmatrix} 0 \\ -\zeta_{01}(t)\zeta_{03}(t) \\ \zeta_{01}(t)\zeta_{02}(t) \end{pmatrix},$$

where,

$$\boldsymbol{\zeta_0} = \begin{pmatrix} \zeta_{01} \\ \zeta_{02} \\ \zeta_{03} \end{pmatrix}.$$

The solution of this system is unique and is given by

$$\boldsymbol{\zeta_0}(t) = \begin{pmatrix} \sqrt{\pi} \\ \frac{\sqrt{\pi}}{2} \\ 0 \end{pmatrix}, \quad t \geq 0. \tag{24}$$

Since $\boldsymbol{\zeta_0}(t) \neq \mathbf{0}$, $\boldsymbol{\zeta_i}(t),\ i \geq 1$ cannot be determined. Therefore, to all orders, the outer solution is given by (24).

Using (24), the leading order inner layer equation follows from (13) as,

$$\begin{aligned} \boldsymbol{\xi_0}(\tau) &= \begin{pmatrix} -\sqrt{\pi} \\ -\frac{\sqrt{\pi}}{2} \\ 0 \end{pmatrix} + {}_0J_\tau^{\frac{1}{2}} \left\{ \begin{pmatrix} 0 \\ -\xi_{01}(\tau)\xi_{03}(\tau) \\ \xi_{01}(\tau)\xi_{02}(\tau) \end{pmatrix} \right\} \\ &+ {}_0J_\tau^{\frac{1}{2}} \left\{ \begin{pmatrix} -1 & 1 & 0 \\ 0.5 & -1 & -\sqrt{\pi} \\ \frac{\sqrt{\pi}}{2} & \sqrt{\pi} & -1 \end{pmatrix} \boldsymbol{\xi_0}(\tau) \right\} \end{aligned} \tag{25}$$

for $\tau \geq 0$ where,

$$\boldsymbol{\xi_0} = \begin{pmatrix} \xi_{01} \\ \xi_{02} \\ \xi_{03} \end{pmatrix}.$$

Similar to (23), this equation is nonlinear, but it is simpler than (23) in the sense that the parameter, ε, does not appear explicitly in the equation.

The eigenvalues of the matrix in (25) are $-1.53626, -0.73187 + 1.69035i$, and $-0.73187 - 1.69035i$. This implies that the matrix in equation (25) is diagonalisable, the property used in Section III-C to prove the asymptotic behaviour of the inner layer solution.

The solution $\boldsymbol{\xi}_0(\tau)$ of (25) is unique, continuous and bounded. Asymptotically, by Levin [12], this solution is equivalent to the solution of the linear version which decays to zero as τ approaches infinity. The linear version satisfies,

$$\|\boldsymbol{\xi}_0(\tau)\| \leq \|\boldsymbol{\xi}_0(0)\|\tau^{-\frac{1}{2}},\ \tau \to \infty. \tag{26}$$

To show that the solution $\boldsymbol{\xi_0}(\tau)$ satisfies (21) and hence (8), one can show that each component of $\boldsymbol{\xi_0}(\tau)$ satisfies (21). Do to this, express equation (25) in an equivalent form:

$$\begin{aligned} \xi_{01}(\tau) &= -\sqrt{\pi} + {}_0J_\tau^{\frac{1}{2}} \left\{-\xi_{01}(\tau) + \xi_{02}(\tau)\right\}, && \text{(27a)} \\ \xi_{02}(\tau) &= -\frac{\sqrt{\pi}}{2} + {}_0J_\tau^{\frac{1}{2}} \left\{-\xi_{02}(\tau) + \varpi(\tau)\right\}, && \text{(27b)} \\ \xi_{03}(\tau) &= {}_0J_\tau^{\frac{1}{2}} \left\{-\xi_{03}(\tau) + \vartheta(\tau)\right\}, && \text{(27c)} \end{aligned}$$

where

$$\begin{aligned} \varpi(\tau) &= 0.5\ \xi_{01}(\tau) - \sqrt{\pi}\ \xi_{03}(\tau) - \xi_{01}(\tau)\xi_{03}(\tau), \\ \vartheta(\tau) &= \frac{\sqrt{\pi}}{2}\ \xi_{01}(\tau) + \sqrt{\pi}\ \xi_{02}(\tau) + \xi_{01}(\tau)\xi_{02}(\tau). \end{aligned}$$

It follows that each ξ_{0i} is continuous and bounded as $\tau \to \infty$ and so are ϖ and ϑ. Thus each of these functions; $\xi_{01},\ \xi_{02},\ \xi_{03},\ \varpi$ and ϑ has a laplace transform. Now consider a more general form of (27),

$$\xi_{0i}(\tau) = \nu_{0i} + {}_0J_\tau^{\frac{1}{2}} \left\{-\xi_{0i}(\tau) + \psi_{0i}(\tau)\right\}, \quad i = 1, 2, 3, \tag{28}$$

where

$$\begin{array}{lcl} \nu_{01}(\tau) = -\sqrt{\pi}, & & \psi_{01}(\tau) = \xi_{02}(\tau), \\ \nu_{02}(\tau) = -\frac{\sqrt{\pi}}{2}, & \text{and} & \psi_{02}(\tau) = \varpi(\tau), \\ \nu_{03}(\tau) = 0, & & \psi_{03}(\tau) = \vartheta(\tau). \end{array}$$

Taking the laplace transform on both sides of (28) gives,

$$\Xi_{0i}(s) = \frac{\nu_{0i}}{s^{\frac{1}{2}}(s^{\frac{1}{2}} + 1)} + \frac{1}{(s^{\frac{1}{2}} + 1)}\Psi_{0i}(s),$$

where $\Xi(s)_{0i} = \mathcal{L}(\xi_{0i}(\tau))$ and $\Psi(s)_{0i} = \mathcal{L}(\psi_{0i}(\tau))$.

The inverse laplace transform then yields,

$$\xi_{0i}(\tau) = \nu_{0i}\,\mathrm{E}_{\frac{1}{2},1}(-\tau^{\frac{1}{2}}) \ + \int_0^\tau (\tau - \sigma)^{-\frac{1}{2}}\,\mathrm{E}_{\frac{1}{2},\frac{1}{2}}(-(\tau - \sigma)^{\frac{1}{2}})\psi_{0i}(\sigma)\,d\sigma.$$

It follows that,

$$\|\xi_{0i}(\tau)\| \leq |\nu_{0i}|\,\mathrm{E}_{\frac{1}{2},1}(-\tau^{\frac{1}{2}}) \ + \int_0^\tau (\tau - \sigma)^{-\frac{1}{2}}\,\mathrm{E}_{\frac{1}{2},\frac{1}{2}}(-(\tau - \sigma)^{\frac{1}{2}})\|\psi_{0i}(\sigma)\|\,d\sigma,$$

Using (26), one can show that,

$$\|\psi_{0i}(\tau)\| \leq |\nu_{0i}|\sigma_{0j}\tau^{\frac{1}{2}}, \quad \tau \to \infty,\ i, j = 1, 2, 3.$$

Here, σ_{0j} is a constant which depends on $\xi_{0i}(\tau)$ for $j \neq i$.

Substituting this inequality into the integral and perform integration yields the required result, that

$$\|\xi_{0i}(\tau)\| \leq |\nu_{0i}|(1 + \sigma_{0j}\sqrt{\pi})\,\mathrm{E}_{\frac{1}{2},1}(-\tau^{\frac{1}{2}}), \quad i = 1, 2, 3.$$

Although the solution of (25) could not be determined explicitly, the solution of (23) for values of $t > 0$, away from the inner layer is given by (24).

References

[1] Bijura, A. M., *Singularly perturbed Volterra integral equations with weakly singular kernels*, Int. J. Math. Math. Sci., 30(3), (2002), pp. 129–143.

[2] Bijura, A. M., *Systems of singular perturbed fractional integral equations*, Journal of Integral Equations and Applications., 24(2), (2012), pp. 195–211.

[3] Caponetto, R., Dongola, G. & Petráš, I., *Fractional Order Systems - Modeling and Control Applications,* World Scientific Series on Nonlinear Science Series A, Vol. 72, 2010.

[4] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G., *Higher Transcendental Functions, Vol. 3,* McGraw-Hill Book Co., New York, 1955.

[5] Eskin, G. I., *Boundary Value Problems for Elliptic Pseudodifferential Equations*, Transl. Math. Monographs, 52, American Mathematical Society, 1981.

[6] Gripenberg, G. & Londen, S.-O. & Staffans, O., *Volterra Integral and Functional Equations*, Encyclopedia of Mathematics and Its Applications, vol. 34, Cambridge University Press, Cambridge, 1990.

[7] Hilfer, R., Ed. *Applications of Factional Calculus in Physics*, World Scientific, River Edge, NJ, USA, 2000.

[8] Kauthen, J. -P., *A survey on singularly perturbed Volterra equations,* Appl. Numer. Math., 24 (1997), pp. 95–114.

[9] Kevorkian, J. K. & Cole, J. D., *Multiple Scale and Singular Perturbation Methods,* Springer-Verlag New York Inc, USA, 1996.

[10] Kilbas, A. A., Srivastava, H. M. & Trujillo, J. J., *Theory and Applications of Fractional Differential Equations, Volume 204 (North-Holland Mathematics Studies),* Elsevier, 2006.

[11] Lagerstrom, P. A., *Matched Asymptotyic Expansions: Ideas and Techniques*, Springer-Verlag New York Inc, USA, 1988.

[12] Levin, J. J., *Nonlinearly perturbed Volterra equations*, Tôhoku Math. Journ., 32 (1980), pp. 317–335.

[13] Linz, P., *Analytical and Numerical Methods for Volterra Equations,* SIAM, Philadelphia, 1985.

[14] Li, Y., Chen, Y. & Podlubny, I., *Mittag-Leffler stability of fractional order nonlinear dynamic systems*, Automatica, 45(2009), pp. 1965–1969.

[15] Magin, R., *Fractional Calculus in Bioengineering,* Begell House Inc. Redding, CT, 2006.

[16] Mainardi, F., *Fractional Calculus and Waves in Linear Viscoelasticity - An Introduction to Mathematical Models.* Imperial College Press, London, 2010.

[17] MILLER, K. S. & ROSS, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc. 1993.

[18] OLDHAM, K. B. & SPANIER, J., *The Fractional Calculus*, Academic Press Inc., 1974.

[19] O'MALLEY, R. E., *Introduction to Singular Perturbation*, Academic Press, New York and London, 1974.

[20] O'MALLEY, R. E., *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag, New York, 1991.

[21] ORTIGUEIRA, M. D., *Fractional Calculus for Scientists and Engineers*, Lecture Notes in Electrical Engineering, 84, Springer Sciences + Business Media, Inc., USA, 2011.

[22] PODLUBNY, I., *Fractional Differential Equations*, Academic Press, 1999.

[23] SAMKO, S. G., KILBAS, A. A. & MARICHEV, O. I., *Fractional Integral and Derivatives: Theory and Application*, Gordon and Breach Science Publisher, Switzerland, UK, USA, 1993.

[24] SKINNER, L. A., *Singular Perturbation Theory*, Springer Sciences + Business Media, LLC, 2011.

[25] SCHUMER, R., MEERSCHAERT, M. M. & BAEUMER, B., *Fractional advection-dispersion equations for modeling transport at the Earth surface*, Journal of Geographical Research, 114 (2009), p.F00A07, doi:10.1029/2008JF001246

[26] SMITH, D. R., *Singular Perturbation Theory; An Introduction with Applications*, Cambridge University Press, Cambridge, New York, 1985.

[27] TARASOV, V. E., *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media (Nonlinear Physical Science)*, Springer-Verlag Berlin Heidelberg, 2010.

[28] WITELSKI, T & BOWEN, M., *Problems in Singular Perturbation Theory*, Scholarpedia, 2009, 4(4):3951.

[29] VERHULST, F., *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics (Texts in Applied Mathematics)*, Springer Sciences + Business Media, Inc., United States of America, 2005.