

An Optimal Pair of Levels for Adding Relation between Two Levels of a Linking Pin Organization Structure with Two Subordinates

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Abstract—This paper proposes a model of adding relation to a linking pin organization structure where every pair of siblings in a complete binary tree of height H is adjacent such that the communication of information in the organization becomes the most efficient. For a model of adding an edge between a node with a depth M and its descendant with a depth N , we formulated the total shortening distance which is the sum of shortening lengths of shortest paths between every pair of all nodes and obtained an optimal pair of depth $(M, N)^*$ which maximizes the total shortening distance.

Index Terms—organization structure, linking pin, complete binary tree, total distance.

I. INTRODUCTION

A linking pin organization structure is a structure in which relations between members of the same section are added to a pyramid organization structure and is called System 4 in Likert's organization classification [1]. In the linking pin organization structure there exist relations between each superior and his direct subordinates and those between members which have the same immediate superior.

The linking pin organization structure can be expressed as a structure where every pair of siblings which are nodes which have the same parent in a rooted tree is adjacent, if we let nodes and edges in the structure correspond to members and relations between members in the organization respectively [2], [3]. Then the height of the linking pin organization structure expresses the number of levels in the organization, and the number of children of each node expresses the number of subordinates of each member. Moreover, the path between a pair of nodes in the structure is equivalent to the route of communication of information between a pair of members in the organization, and adding edges to the structure is equivalent to forming additional relations other than those between each superior and his direct subordinates and between members which have the same direct subordinate.

The purpose of our study is to obtain an optimal set of additional relations to the linking pin organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the structure minimizing the sum of lengths of shortest paths between every pair of all nodes.

We have obtained an optimal depth for each of the following two models of adding relations in the same level to

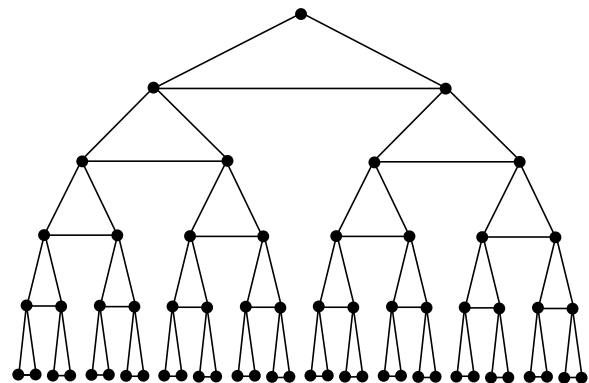


Fig. 1. An Example of a Complete Binary Linking Pin Structure of $H = 5$

a complete K -ary linking pin structure of height H where every pair of siblings in a complete K -ary tree of height H is adjacent: (i) a model of adding an edge between two nodes with the same depth and (ii) a model of adding edges between every pair of nodes with the same depth [5]. A complete K -ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have K ($K = 2, 3, \dots$) children [6]. Furthermore, we have proposed a model of adding relation between the top and a member in a complete K -ary linking pin structure of height H [7]. When an edge between the root and a node with a depth N is added, an optimal depth N^* is obtained by minimizing the sum of lengths of shortest paths between every pair of all nodes.

This paper proposes a model of adding an edge between a node with a depth M ($M = 0, 1, \dots, H - 2$) and its descendant with a depth N ($N = M + 2, M + 3, \dots, H$) in a complete binary (that is $K = 2$) linking pin structure of height H ($H = 2, 3, \dots$). This model corresponds to the formation of an additional relation between a superior and his indirect subordinate. Figure 1 shows an example of a complete binary linking pin structure of $H = 5$.

If $l_{i,j}$ ($= l_{j,i}$) denotes the distance, which is the number of edges in the shortest path from a node v_i to a node v_j ($i, j = 1, 2, \dots, 2^{H+1} - 1$) in the complete binary linking pin structure of height H , then $\sum_{i < j} l_{i,j}$ is the total distance. Furthermore, if $l'_{i,j}$ denotes the distance from v_i to v_j after adding an edge in this model, $l_{i,j} - l'_{i,j}$ is called the shortening distance between v_i and v_j , and $\sum_{i < j} (l_{i,j} - l'_{i,j})$ is called the total shortening distance. Minimizing the total distance is equivalent to maximizing the total shortening distance. When an edge between a node with a depth M and its descendant with a depth N is added to the complete binary linking pin structure of height H , an optimal depth N^* is obtained by

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maximizing the total shortening distance for each value of M [8]. Furthermore, an optimal pair of depth $(M, N)^*$ which maximizes the total shortening distance is revealed.

In Section 2 we formulate the total shortening distance of the above model. In Section 3 we show an optimal depth N^* which maximizes the total shortening distance for each value of M . In Section 4 an optimal pair of depth $(M, N)^*$ which maximizes the total shortening distance is shown.

II. FORMULATION OF TOTAL SHORTENING DISTANCE

This section formulates the total shortening distance when an edge between a node with a depth $M (M = 0, 1, \dots, H - 2)$ and its descendant with a depth $N (N = M + 2, M + 3, \dots, H)$ is added to a complete binary linking pin structure of height $H (H = 2, 3, \dots)$.

Let v_M denote the node with a depth M and let v_N denote the node with a depth N which gets adjacent to v_M . The set of descendants of v_N is denoted by V_1 . (Note that every node is a descendant of itself [6].) The set of descendants of v_M and ancestors of parent of v_N is denoted by V_2 . (Note that every node is an ancestor of itself [6].) Let V_3 denote the set obtained by removing V_1 and V_2 from the set of descendants of v_M . Let V_4 denote the set obtained by removing descendants of v_M from the set of all nodes of the complete binary linking pin structure.

The sum of shortening distances between every pair of nodes in V_1 and nodes in V_2 is given by

$$A_H(M, N) = W(H - N) \sum_{i=1}^{\lfloor \frac{N-M}{2} \rfloor} (N - M - 2i + 1), \quad (1)$$

where $W(h)$ denotes the number of nodes of a complete binary tree of height $h (h = 0, 1, 2, \dots)$, and $\lfloor x \rfloor$ denotes the maximum integer which is equal to or less than x . The sum of shortening distances between every pair of nodes in V_2 is given by

$$B(M, N) = \sum_{i=1}^{\lfloor \frac{N-M}{2} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{N-M}{2} \rfloor - i} (N - M - 2i - 2j + 1), \quad (2)$$

where we define $\sum_{i=1}^0 \cdot = 0$. The sum of shortening distances between every pair of nodes in V_1 and nodes in V_3 is given by

$$C_H(M, N) = W(H - N) \sum_{i=1}^{\lfloor \frac{N-M-1}{2} \rfloor} W(H - M - i) \times (N - M - 2i). \quad (3)$$

The sum of shortening distances between every pair of nodes in V_2 and nodes in V_3 is given by

$$D_H(M, N) = \sum_{i=1}^{\lfloor \frac{N-M-1}{2} \rfloor - 1} W(H - M - i) \times \sum_{j=1}^{\lfloor \frac{N-M-1}{2} \rfloor - i} (N - M - 2i - 2j)$$

$$+ \sum_{i=1}^{\lfloor \frac{N-M-1}{2} \rfloor} W(H - N + i - 1) \times \sum_{j=1}^{\lfloor \frac{N-M-1}{2} \rfloor - i + 1} (N - M - 2i - 2j + 2), \quad (4)$$

where we define $\sum_{i=1}^{-1} \cdot = 0$. The sum of shortening distances between every pair of nodes in V_3 is given by

$$E_H(M, N) = \sum_{i=1}^{\lfloor \frac{N-M}{2} \rfloor - 1} W(H - N + i - 1) \times \sum_{j=1}^{\lfloor \frac{N-M}{2} \rfloor - i} W(H - M - j) \times (N - M - 2i - 2j + 1). \quad (5)$$

The sum of shortening distances between every pair of nodes in V_1 and nodes in V_4 is given by

$$F_H(M, N) = (W(H) - W(H - M))W(H - N) \times (N - M - 1). \quad (6)$$

The sum of shortening distances between every pair of nodes in V_2 and nodes in V_4 is given by

$$G_H(M, N) = (W(H) - W(H - M)) \times \sum_{i=1}^{\lfloor \frac{N-M}{2} \rfloor - 1} (N - M - 2i - 1). \quad (7)$$

The sum of shortening distances between every pair of nodes in V_3 and nodes in V_4 is given by

$$J_H(M, N) = (W(H) - W(H - M)) \times \sum_{i=1}^{\lfloor \frac{N-M-1}{2} \rfloor} W(H - N + i - 1) \times (N - M - 2i). \quad (8)$$

From the above equations, the total shortening distance $S_H(M, N)$ is given by

$$S_H(M, N) = A_H(M, N) + B(M, N) + C_H(M, N) + D_H(M, N) + E_H(M, N) + F_H(M, N) + G_H(M, N) + J_H(M, N). \quad (9)$$

III. AN OPTIMAL DEPTH N^* FOR EACH VALUE OF M

This section obtains an optimal depth N^* which maximizes the total shortening distance $S_H(M, N)$ for each value of M .

Let us classify $S_H(M, N)$ into two cases of $N = M + 2L$ where $L = 1, 2, \dots, \lfloor \frac{H-M}{2} \rfloor$ and $N = M + 2L + 1$ where $L = 1, 2, \dots, \lfloor \frac{H-M-1}{2} \rfloor$. Since the number of nodes of a complete binary tree of height h is

$$W(h) = 2^{h+1} - 1, \quad (10)$$

$S_H(M, M + 2L)$ and $S_H(M, M + 2L + 1)$ become

$$\begin{aligned}
 & S_H(M, M + 2L) \\
 = & (2^{H-M-2L+1} - 1) \sum_{i=1}^L (2L - 2i + 1) \\
 & + \sum_{i=1}^{L-1} \sum_{j=1}^{L-i} (2L - 2i - 2j + 1) \\
 & + (2^{H-M-2L+1} - 1) \sum_{i=1}^{L-1} (2^{H-M-i+1} - 1) \\
 & \times (2L - 2i) \\
 & + \sum_{i=1}^{L-2} (2^{H-M-i+1} - 1) \sum_{j=1}^{L-i-1} (2L - 2i - 2j) \\
 & + \sum_{i=1}^{L-1} (2^{H-M-2L+i} - 1) \sum_{j=1}^{L-i} (2L - 2i - 2j + 2) \\
 & + \sum_{i=1}^{L-1} (2^{H-M-2L+i} - 1) \sum_{j=1}^{L-i} (2^{H-M-j+1} - 1) \\
 & \times (2L - 2i - 2j + 1) \\
 & + (2^{H+1} - 2^{H-M+1}) (2^{H-M-2L+1} - 1) (2L - 1) \\
 & + (2^{H+1} - 2^{H-M+1}) \sum_{i=1}^{L-1} (2L - 2i - 1) \\
 & + (2^{H+1} - 2^{H-M+1}) \sum_{i=1}^{L-1} (2^{H-M-2L+i} - 1) \\
 & \times (2L - 2i) \\
 = & 2^{2H-2M-3L+3} - 2^{2H-2M-L+2} - 3 \cdot 2^{2H-M-2L+2} \\
 & + 2^{2H-M-L+3} - 2^{H-M-2L+1} - 5 \cdot 2^{H-M-L+1} \\
 & + 2^{H-M+3} - (3L - 2)2^{H+1} - L, \tag{11}
 \end{aligned}$$

and

$$\begin{aligned}
 & S_H(M, M + 2L + 1) \\
 = & (2^{H-M-2L} - 1) \sum_{i=1}^L (2L - 2i + 2) \\
 & + \sum_{i=1}^{L-1} \sum_{j=1}^{L-i} (2L - 2i - 2j + 2) \\
 & + (2^{H-M-2L} - 1) \sum_{i=1}^L (2^{H-M-i+1} - 1) \\
 & \times (2L - 2i + 1) \\
 & + \sum_{i=1}^{L-1} (2^{H-M-i+1} - 1) \sum_{j=1}^{L-i} (2L - 2i - 2j + 1) \\
 & + \sum_{i=1}^L (2^{H-M-2L+i-1} - 1) \\
 & \times \sum_{j=1}^{L-i+1} (2L - 2i - 2j + 3) \\
 & + \sum_{i=1}^{L-1} (2^{H-M-2L+i-1} - 1) \sum_{j=1}^{L-i} (2^{H-M-j+1} - 1) \\
 & \times (2L - 2i - 2j + 2) \\
 & + (2^{H+1} - 2^{H-M+1}) (2^{H-M-2L} - 1) 2L
 \end{aligned}$$

$$\begin{aligned}
 & + (2^{H+1} - 2^{H-M+1}) \sum_{i=1}^{L-1} (2L - 2i) \\
 & + (2^{H+1} - 2^{H-M+1}) \sum_{i=1}^L (2^{H-M-2L+i-1} - 1) \\
 & \times (2L - 2i + 1) \\
 = & \frac{5}{3} \cdot 2^{2H-2M-3L+1} - \frac{5}{3} \cdot 2^{2H-2M-L+1} \\
 & - 3 \cdot 2^{2H-M-2L+1} + 3 \cdot 2^{2H-M-L+1} \\
 & - 2^{H-M-2L+1} - 2^{H-M-L+3} + 5 \cdot 2^{H-M+1} \\
 & - 3L \cdot 2^{H+1} - 2L. \tag{12}
 \end{aligned}$$

Lemma 1:

- (i) If $L = 1$, then $S_H(M, M + 2L) < S_H(M, M + 2L + 1)$.
- (ii) If $L \geq 2$, then $S_H(M, M + 2L) > S_H(M, M + 2L + 1)$.

Proof:

(i) If $L = 1$, then

$$\begin{aligned}
 & S_H(M, M + 2L) - S_H(M, M + 2L + 1) \\
 = & 2^{2H-M-1} \left(\frac{1}{2^{M+1}} + \frac{1}{2^{H-M-3}} - 1 \right) \\
 & - 3 \cdot 2^{H-M} + 1 \\
 < & 0. \tag{13}
 \end{aligned}$$

(ii) If $L \geq 2$, then

$$\begin{aligned}
 & S_H(M, M + 2L) - S_H(M, M + 2L + 1) \\
 = & \frac{1}{3} \cdot 2^{2H-M-L+1} \left(3 - \frac{1}{2^M} - \frac{9}{2L} + \frac{7}{2^{M+2L}} \right) \\
 & + 2^{H+2} \left(1 - \frac{1}{2^M} + \frac{1}{2^{M+1}} - \frac{1}{2^{M+L+1}} \right) + L \\
 > & 0, \tag{14}
 \end{aligned}$$

where $L = 2, 3, \dots, \lfloor \frac{H-M-1}{2} \rfloor$. The proof is complete.

Lemma 2: If $L \geq 2$, then $L^* = 2$ maximizes $S_H(M, M + 2L)$.

Proof: If $L \geq 2$, then $L^* = 2$ maximizes $S_H(M, M + 2L)$ since

$$\begin{aligned}
 & S_H(M, M + 2L) - S_H(M, M + 2L + 2) \\
 = & 2^{2H-M-L+1} \left(2 - \frac{1}{2^M} - \frac{9}{2^{L+1}} + \frac{7}{2^{M+2L+1}} \right) \\
 & + 2^H \left(6 - \frac{1}{2^{M+2L-1}} - \frac{1}{2^{M+L-2}} - \frac{1}{2^{M+L}} \right. \\
 & \left. + \frac{1}{2^{M+2L+1}} \right) + 1 \\
 > & 0, \tag{15}
 \end{aligned}$$

where $L = 2, 3, \dots, \lfloor \frac{H-M}{2} \rfloor - 1$. The proof is complete.

Lemma 3:

- (i) If $M = 0$ and $H = 4$, then $S_H(M, M + 3) > S_H(M, M + 4)$.
- (ii) If $M = 0$ and $H \geq 5$, then $S_H(M, M + 3) < S_H(M, M + 4)$.
- (iii) If $M \geq 1$, then $S_H(M, M + 3) > S_H(M, M + 4)$.

Proof:

(i) If $M = 0$ and $H = 4$, then

$$S_H(M, M + 3) - S_H(M, M + 4) = 2 > 0. \quad (16)$$

(ii) If $M = 0$ and $H \geq 5$, then

$$S_H(M, M + 3) - S_H(M, M + 4) = 2^{2H-3} \left(\frac{17}{2^H} - 1 \right) < 0. \quad (17)$$

(iii) If $M \geq 1$, then

$$\begin{aligned} & S_H(M, M + 3) - S_H(M, M + 4) \\ &= 2^{2H-M-2} \left(1 - \frac{1}{2^{M-2}} + \frac{1}{2^{M-1}} + \frac{1}{2^{M+1}} \right) \\ &\quad + 2^{H-M-3} + 2^{H+1} \\ &> 0, \end{aligned} \quad (18)$$

The proof is complete.

Theorem 4: Let N^* maximize $S_H(M, N)$ for each value of M , then we have the following:

- (i) If $M = H - 2$, then $N^* = M + 2$.
- (ii) If $M = H - 3$, then $N^* = M + 3$.
- (iii) If $M \leq H - 4$, then we have the following:
 - (a) If $M = 0$ and $H = 4$, then $N^* = M + 3$.
 - (b) If $M = 0$ and $H \geq 5$, then $N^* = M + 4$.
 - (c) If $M \geq 1$, then $N^* = M + 3$.

Proof:

- (i) If $M = H - 2$, then $N^* = M + 2$ trivially.
- (ii) If $M = H - 3$, then $N^* = M + 3$ since $S_H(M, M + 2) < S_H(M, M + 3)$ from (i) of Lemma 1.
- (iii) If $M \leq H - 4$, then $N^* = M + 3$ for $N \leq M + 3$ from (i) of Lemma 1 and $N^* = M + 4$ for $N \geq M + 4$ from (ii) of Lemma 1 and Lemma 2.
 - (a) If $M = 0$ and $H = 4$, then $N^* = M + 3$ since $S_H(M, M + 3) > S_H(M, M + 4)$ from (i) of Lemma 3.
 - (b) If $M = 0$ and $H \geq 5$, then $N^* = M + 4$ since $S_H(M, M + 3) < S_H(M, M + 4)$ from (ii) of Lemma 3.
 - (c) If $M \geq 1$, then $N^* = M + 3$ since $S_H(M, M + 3) > S_H(M, M + 4)$ from (iii) of Lemma 3.

The proof is complete.

IV. AN OPTIMAL PAIR OF DEPTH $(M, N)^*$

This section seeks an optimal pair of depth $(M, N)^*$ which maximizes the total shortening distance $S_H(M, N)$.

Lemma 5:

- (i) In the case of $M = 0$ we have the following:
 - (a) If $H = 2$, then $(M, N)^* = (0, 2)$.
 - (b) If $H = 3, 4$, then $(M, N)^* = (0, 3)$.
 - (c) If $H \geq 5$, then $(M, N)^* = (0, 4)$.
- (ii) In the case of $M \geq 1$ we have the following:
 - (a) If $H = 3$, then $(M, N)^* = (1, 3)$.
 - (b) If $H \geq 4$, then $(M, N)^* = (1, 4)$.

Proof:

- (i) In the case of $M = 0$.

(a) If $H = 2$, then $(M, N)^* = (0, 2)$ from (i) of Theorem 4.

(b) If $H = 3$, then $(M, N)^* = (0, 3)$ from (ii) of Theorem 4. If $H = 4$, then $(M, N)^* = (0, 3)$ from (iii)-(a) of Theorem 4.

(c) If $H \geq 5$, then $(M, N)^* = (0, 4)$ from (iii)-(b) of Theorem 4.

(ii) In the case of $M \geq 1$.

(a) If $H = 3$, then $(M, N)^* = (1, 3)$ trivially.

(b) If $H \geq 4$, then as follows. Let $Q_H(M)$ denote the total shortening distance when $N = M + 3$, so that we have

$$\begin{aligned} Q_H(M) &\equiv S_H(M, M + 3) \\ &= -5 \cdot 2^{2H-2M-2} + 3 \cdot 2^{2H-M-1} + 11 \cdot 2^{H-M-1} \\ &\quad - 3 \cdot 2^{H+1} - 2 \end{aligned} \quad (19)$$

for $M = 1, 2, \dots, H - 3$. From

$$\begin{aligned} & Q_H(M + 1) - Q_H(M) \\ &= 2^{2H-M-2} \left(\frac{1}{2^{M-1}} + \frac{1}{2^M} + \frac{1}{2^{M+1}} + \frac{1}{2^{M+2}} - 3 \right) \\ &\quad - 11 \cdot 2^{H-M-2} \\ &< 0 \end{aligned} \quad (20)$$

for $M = 1, 2, \dots, H - 4$, and

$$S_H(1, 4) - S_H(H - 2, H) = 7 \cdot 2^{2H-4} \left(1 - \frac{3}{2^{H-2}} \right) + 5 > 0, \quad (21)$$

we have $(M, N)^* = (1, 4)$.

The proof is complete.

Theorem 6:

- (i) If $H = 2$, then $(M, N)^* = (0, 2)$.
- (ii) If $H = 3$, then $(M, N)^* = (0, 3)$.
- (iii) If $H \geq 4$, then $(M, N)^* = (1, 4)$.

Proof:

- (i) If $H = 2$, then $(M, N)^* = (0, 2)$ trivially.
- (ii) If $H = 3$, then $(M, N)^* = (0, 3)$ from (i)-(b) and (ii)-(a) of Lemma 5, and

$$S_H(0, 3) - S_H(1, 3) = 1 > 0. \quad (22)$$

(iii) If $H = 4$, then $(M, N)^* = (1, 4)$ from (i)-(b) and (ii)-(b) of Lemma 5, and

$$S_H(0, 3) - S_H(1, 4) = -4 < 0. \quad (23)$$

If $H \geq 5$, then $(M, N)^* = (1, 4)$ from (i)-(c) and (ii)-(b) of Lemma 5, and

$$\begin{aligned} & S_H(0, 4) - S_H(1, 4) \\ &= 2^{2H-4} \left(\frac{1}{2^{H-3}} + \frac{1}{2^{H-1}} + \frac{1}{2^{2H-5}} - 1 \right) - 2 \\ &< 0. \end{aligned} \quad (24)$$

The proof is complete.

V. CONCLUSIONS

This study considered the addition of relation to a linking pin organization structure such that the communication of information between every member in the organization becomes the most efficient. For a model of adding an edge between a node with a depth M and its descendant with a depth N to a complete binary linking pin structure of height H where every pair of siblings in a complete binary tree of height H is adjacent, we obtained an optimal depth N^* which maximizes the total shortening distance for each value of M . Furthermore, an optimal pair of depth $(M, N)^*$ which maximizes the total shortening distance is revealed.

Theorem 6 shows that the most efficient manner of adding relation between a superior and his indirect subordinate is to add the relation between the top and a member of the second level below the top, that between the top and a member of the third level below the top, or that between a member of the first level below the top and a member of the fourth level below the top depending on the number of levels in the organization structure.

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