Application of Triangular Functions to Numerical Solution of Stochastic Volterra Integral Equations

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Abstract—**In this paper, an efficient method for solving numerically stochastic Volterra integral equation is proposed. Here, we consider triangular functions and their operational matrix of integration. This method has several advantages in reducing computational burden and is more accurate than the BPFs. An error analysis is valid under fairly restrictive conditions. The method is applied to examples to illustrate the accuracy and implementation of the method.**

Keywords: Triangular functions; Operational Matrix; Stochastic operational matrix; Stochastic Volterra integral equations; Itô integral.

1 Introduction

Wide variety of problems in physics, mechanics, economics, sociology, biological lead to the Stochastic Volterra Integral Equations (SVIEs). These systems are dependent on a noise source, on a Gaussian white noise, so modeling such phenomena naturally requires the use of various stochastic Volterra integral equations.

Most SVIEs can not be solved analytically and hence it is of great importance to provide numerical solution. So, there has been a growing interest in numerical solutions of stochastic Volterra integral equations for the last years [1,7,14,15,17,18].

In the present work, we consider

$$u(t) = u_0(t) + \int_0^t k_1(s,t)u(s)ds + \int_0^t k_2(s,t)u(s)dB(s), \quad t \in [0,T), \tag{1}$$

where, $u(t)$, $u_0(t)$, $k_1(s,t)$ and $k_2(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P) with a filtration $\{F_t, t \geq 0\}$ that is increasing and right continuous and F_0 contains all P-null sets. $u(t)$ is unknown random function and $B(t)$ is a standard Brownian motion defined on the probability space and $\int_0^t k_2(s,t)u(s)dB(s)$ is the Itô integral.

Triangular Functions (TFs) have been introduced by Deb et al. (2006) in [8], and TF approximation were successfully applied for analysis of dynamic systems [8], variational problems [6], integral equations [3,5], integro-differential equations [4], Nonlinear Constrained Optimal Control Problems [9], and Volterra-Fredholm integral equations [2,12,13].

This paper is organized as follows: In the next section we review one-dimensional triangular functions. Section 3, presents stochastic concept that is used in this paper. Section 4, introduces stochastic integration operational matrix related to TFs. In Section 5, TF method is applied to solve stochastic Volterra integral equations. Section 6, investigates error analysis of method. In Section 7, numerical results are shown. Finally, Section 8, provides the conclusion.

2 Triangular functions(TFs)

In this section definitions of the TFs and their properties are reviewed. Two m-sets of triangular functions are defined over the interval $[0,T)$ as

$$T_i^1(t) = \begin{cases} 1 - \frac{t-ih}{h} & ih \leq t < (i+1))h, \\ 0 & elsewhere. \end{cases} \tag{2}$$

$$T_i^2(t) = \begin{cases} \frac{t-ih}{h} & ih \leq t < (i+1))h, \\ 0 & elsewhere. \end{cases} \tag{3}$$

where, $i = 0, ..., m-1$. m is the number of elementary functions and $h = \frac{T}{m}$. In this paper, it is assumed that $T = 1$, so, TFs are defined over $[0,1)$, and $h = \frac{1}{m}$. Moreover,

$$T_i^1(t) + T_i^2(t) = \phi_i(t), \tag{4}$$

where, $\phi_i(t)$ is the ith block pulse function:

$$\phi_i(t) = \begin{cases} 1 & (i)h \leq t < (i+1)h, \\ 0 & elsewhere. \end{cases} \tag{5}$$

From the definition of TFs, it is clear that TFs are disjoint, orthogonal, and complete [8]. Therefore it can be written

$$\int_0^1 T_i^p(t)T_j^q(t)dt = \delta_{ij}\triangle_{p,q}, \tag{6}$$

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where, δij is Kronecker delta and

$$\triangle_{p,q} = \begin{cases} \frac{h}{3} & p = q \in \{1, 2\}, \\ \frac{h}{6} & p \neq q. \end{cases} \tag{7}$$

The set of TFs may be written as a vector $T1(t)$ and $T2(t)$ of dimension m

$$T1(t) = [T_0^1(t), ..., T_{m-1}^1(t)]^T,$$

$$T2(t) = [T_0^2(t), ..., T_{m-1}^2(t)]^T, \tag{8}$$

and

$$T(t) = [T1(t), T2(t)]^T, \tag{9}$$

where, $T(t)$ is called the 1D-TF vector.

From the above representation and disjointness property, it follows:

$$T(t).T^T(t) \simeq diag(T(t)) = \widehat{T}(t), \tag{10}$$

where, $\widehat{T}$ is a $2m \times 2m$ diagonal matrix.
The expansion of a function $f(t)$ over $[0, T)$ with respect to 1D-TFs, $i = 0, ..., m-1$ is given by

$$\begin{aligned} f(t) &\simeq \sum_{i=0}^{m-1} C1_i T_i^1(t) + \sum_{i=0}^{m-1} C2_i T_i^2(t) \\ &= C1^T.T1(t) + C2^T.T2(t) \\ &= [C1, C2]^T.[T1(t), T2(t)] = C^T.T(t), \end{aligned} \tag{11}$$

where, $C1_i$ and $C2_i$ are samples of f, for example $C1_i = f(ih)$ and $C2_i = f((i+1)h)$ for $i = 0, 1, .., m-1$, and as a result there is no need for integration. The vector C is called the 1D-TF coefficient vector.

Now, integration operational matrix of TFs is considered

$$\int_0^t T1(s)ds = P_1.T1(t) + P_2.T2(t), \tag{12}$$

$$\int_0^t T2(s)ds = P_1.T1(t) + P_2.T2(t), \tag{13}$$

where,

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m}, \tag{14}$$

$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{m\times m}. \tag{15}$$

Therefore,

$$\int_0^t T(s)ds \simeq P.T(t), \tag{16}$$

where, P is operational matrix of integration that is given by

$$P = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix}. \tag{17}$$

So, we can approximate the integral of every function as

$$\int_0^t f(s)ds \simeq \int_0^t C^T.T(s)ds \simeq C^T.P.T(t). \tag{18}$$

Assuming that $k(s, t)$ is a function of two variables. It can be expanded with respect to TFs as follows

$$k(s, t) = T^T(t)KT(s), \tag{19}$$

where, $T(s)$ and $T(t)$ are $2m_1$-dimensional and $2m_2$-dimensional triangular vectors and K is a $2m_1 \times 2m_2$ coefficient matrix of TFs. For convenience, we put $m_1 = m_2 = m$. So, matrix K can be written as

$$K = \begin{pmatrix} K11 & K12 \\ K21 & K22 \end{pmatrix}, \tag{20}$$

where, $K11$, $K12$, $K21$, and $K22$ can be computed by sampling the function $k(s, t)$ at points s_i and t_i such that $s_i = t_i = ih$, for $i = 0, 1, ..., m$. Therefore

$$(K11)_{i,j} = k(s_i, t_j); i = 0, 1, ..., m-1, j = 0, 1, ..., m-1,$$
$$(K12)_{i,j} = k(s_i, t_j); i = 0, 1, ..., m-1, j = 1, ..., m,$$
$$(K21)_{i,j} = k(s_i, t_j); i = 1, ..., m, j = 0, 1, ..., m-1,$$
$$(K22)_{i,j} = k(s_i, t_j); i = 1, ..., m, j = 1, ..., m.$$

Now, let B be a $2m \times 2m$ matrix. So, it can be similarly concluded that

$$T^T(t)BT(t) \simeq \widehat{B}T(t), \tag{21}$$

in which, $\widehat{B}$ is a 2m-vector with elements equal to the diagonal entries of matrix B. Furthermore,

$$T(t)T^T(t)X \simeq \tilde{X}T(t),$$

where, $\tilde{X} = diag(X)$ is $2m \times 2m$ diagonal matrix. The following integral can be computed

$$\int_0^1 T(t)T^T(t)dt \simeq D, \tag{22}$$

where, D is the following $2m \times 2m$ matrix

$$D = \begin{pmatrix} \frac{h}{3}I_{m\times m} & \frac{h}{6}I_{m\times m} \\ \frac{h}{6}I_{m\times m} & \frac{h}{3}I_{m\times m} \end{pmatrix}. \tag{23}$$

3 Stochastic concepts and Ito integral

Definition 3.1. (Brownian motion process). Brownian motion B(t) is a stochastic process with the following properties.

(i) (Independence of increments) $B(t) - B(s)$, for $t > s$, is independent of the past.

(ii) (Normal increments) $B(t)-B(s)$ has Normal distribution with mean 0 and variance $t-s$. This implies (taking s = 0) that $B(t) - B(0)$ has $N(0,t)$ distribution.

(iii) (Continuity of paths) $B(t), t \geq 0$ are continuous functions of t.

Definition 3.2. Let $\{N(t)\}_{t\geq 0}$ be an increasing family of σ-algebras of sub-sets of Ω. A process $g(t,\omega)$ from $[0,\infty)\times\Omega$ to R^n is called $N(t)$-adapted if for each $t \geq 0$ the function $\omega \longrightarrow g(t,\omega)$ is $N(t)$-measurable [16].

Definition 3.3. Let $\nu = \nu(S,T)$ be the class of functions $f(t,\omega) : [0,\infty)\times\Omega \longrightarrow R$ such that,

(i) $(t,\omega) \longrightarrow f(t,\omega)$, is $B\times\mathcal{F}$-measurable, where B denotes the Borel σ-algebra on $[0,\infty)$ and $\mathcal{F}$ is the σ-algebra on Ω.

(ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the σ-algebra generated by the random variables $B(s)$; $s \leq t$.

(iii) $E\big[\int_S^T f^2(t,\omega)dt\big] < \infty$.

Proof. see [16]

Definition 3.4. (The Itô integral), [16]. Let $f \in \nu(S,T)$, then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t,\omega)dB(t)(\omega) = \lim_{n\to\infty}\int_S^T \phi_n(t,\omega)dB(t)(\omega),$$

$$(limit \quad in \quad L^2(P))$$

where, ϕ_n is a sequence of elementary functions such that

$$E\big[\int_S^T (f(t,\omega)-\phi_n(t,\omega))^2dt\big] \to 0, \quad as \quad n\to\infty.$$

Theorem 3.5. *(The Itô isometry). Let $f \in \nu(S,T)$, then*

$$E\big[(\int_S^T f(t,\omega)dB(t)(\omega))^2\big] = E\big[\int_S^T f^2(t,\omega)dt\big].$$

Proof. [16]

Definition 3.6. (1-dimensional Itô processes), [16]. Let $B(t)$ be 1-dimensional Brownian motion on $(\Omega,\mathcal{F},P)$. A 1-dimensional Itô process (stochastic integral) is a stochastic process $X(t)$ on $(\Omega,\mathcal{F},P)$ of the form

$$X(t) = X(0) + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB(s),$$

or

$$dX(t) = udt + vdB(t), \tag{24}$$

where

$$P\big[\int_0^t v^2(s,\omega)ds < \infty, \quad for \quad all \quad t\geq 0\big] = 1,$$

$$P\big[\int_0^t \mid u(s,\omega)\mid ds < \infty, \quad for \quad all \quad t\geq 0\big] = 1.$$

Theorem 3.7. *(The 1-dimensional Itô formula). Let $X(t)$ be an Itô process given by (1) and $g(t,x) \in C^2([0,\infty)\times R)$, then*

$$Y(t) = g\big(t,X(t)\big),$$

is again an Itô process, and

$$dY(t)$$

$$= \frac{\partial g}{\partial t}\big(t,X(t)\big)dt + \frac{\partial g}{\partial x}\big(t,X(t)\big)dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\big(t,X(t)\big)\big(dX(t)\big)^2, \tag{25}$$

where $(dX(t))^2 = (dX(t))(dX(t))$ is computed according to the rules,

$$dt.dt = dt.dB(t) = dB(t).dt = 0, \quad dB(t).dB(t) = dt. \tag{26}$$

Proof. see [16].
Furthermore, we will apply several times the usual elementary equalities

$$(\sum_{i=1}^m a_i)^r \leq m^{r-1}\sum_{i=1}^m a_i^r, \qquad a_i > 0, r\in N, \tag{27}$$

and

$$|a+b|^r \leq (2^{r-1}\vee 1)(|a|^r+|b|^r), \quad r\geq 0. \tag{28}$$

Also $\|.\|$ is notation of

$$\|f(t)\|^2 = \int_0^1 |f(t)|^2dt.$$

Lemma 3.8. *(The Gronwall inequality) Let $\alpha,\beta \in [t_0,T]\to R$ be integral with*

$$0 \leq \alpha(t) \leq \beta(t) + L\int_{t_0}^t \alpha(s)ds.$$

for $t\in[t_0,T]$ where, $L > 0$. Then

$$\alpha(t) \leq \beta(t) + L\int_{t_0}^t e^{L(t-s)}\beta(s)ds, \quad t\in[t_0,T].$$

For more details see [10,11,16,19].

4 Stochastic integral operational matrix for triangular functions

The Ito integral of each T_i^1 and T_i^2 is defined by

$$\int_0^t T_i^1(s)dB(s)$$

$$= \begin{cases} 0 & 0 \le t < ih, \\ (i+1)[B(t)-B(ih)] - \int_{ih}^{t} \frac{s}{h} dB(s) & ih \le t < (i+1)h, \\ (i+1)[B((i+1)h)-B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h} dB(s) & (i+1)h \le t < T, \end{cases} \quad (29)$$

for $i = 0, ..., m-1$.

$$\int_0^t T_i^2(s)dB(s)$$

$$= \begin{cases} 0 & 0 \le t < ih, \\ -i[B((t)-B(ih)] + \int_{ih}^{t} \frac{s}{h} dB(s) & ih \le t < (i+1)h, \\ -i[B((i+1)h)-B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h} dB(s) & (i+1)h \le t < T, \end{cases} \quad (30)$$

for $i = 0, ..., m-1$.

We approximate $B(t) - B(ih)$, by $B((i+0.5)h) - B(ih)$, at mid-point of $[ih, (i+1)h)]$. Also, $\int_{ih}^{t} \frac{s}{h} dB(s)$ with $\int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s)$. As a result, $\int_0^t T_i^1(s)dB(s)$, and $\int_0^t T_i^2(s)dB(s)$ with 1D-TF are taken in the vector form as

$$\int_0^t T_i^1(s)dB(s)$$

$$\simeq [0, 0, \ldots, 0,$$

$$(i+1)[B((i+0.5)h)-B(ih)] - \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s),$$

$$(i+1)[B((i+1)h)-B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h} dB(s), \ldots,$$

$$(i+1)[B((i+1)h)-B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h} dB(s)](T1+T2),$$

in which the ith component is $(i+1)[B((i+0.5)h) - B(ih)] - \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s)$ and

$$\int_0^t T_i^2(s)dB(s)$$

$$\simeq [0, 0, \ldots, 0, -i[B((i+0.5)h)-B(ih)] + \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s),$$

$$-i[B((i+1)h)-B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h} dB(s), \ldots,$$

$$-i[B((i+1)h)-B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h} dB(s)](T1+T2).$$

Considering following definitions,

$$\alpha(i) := (i+1)[B((i+0.5)h)-B(ih)] - \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s),$$

$$\beta(i) := (i+1)[B((i+1)h)-B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h} dB(s),$$

$$\gamma(i) := -i[B((i+0.5)h)-B(ih)] + \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s),$$

$$\rho(i) := -i[B((i+1)h)-B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h} dB(s),$$

stochastic operational matrix of integration is given by

$$P1_S = \begin{pmatrix} \alpha(0) & \beta(0) & \beta(0) & \ldots & \beta(0) \\ 0 & \alpha(1) & \beta(1) & \ldots & \beta(1) \\ 0 & 0 & \alpha(2) & \ldots & \beta(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \beta(m-2) \\ 0 & 0 & 0 & \ldots & \alpha(m-1) \end{pmatrix}_{m\times m}, \quad (31)$$

$$P2_S = \begin{pmatrix} \gamma(0) & \rho(0) & \rho(0) & \ldots & \rho(0) \\ 0 & \gamma(1) & \rho(1) & \ldots & \rho(1) \\ 0 & 0 & \gamma(2) & \ldots & \rho(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \rho(m-2) \\ 0 & 0 & 0 & \ldots & \gamma(m-1) \end{pmatrix}_{m\times m}, \quad (32)$$

$$\int_0^t T1(s)dB(s) = P1_s.T1(t) + P1_s.T2(t), \quad (33)$$

$$\int_0^t T2(s)dB(s) = P2_S.T1(t) + P2_S.T2(t). \qquad (34)$$

Therefore

$$\int_0^t T(s)dB(s) \simeq P_S(T1+T2) = P_S.T(t), \qquad (35)$$

where, P_S the stochastic operational matrix of integration in the 1D-TF domain, can be represent as

$$P_S = \begin{pmatrix} P1_S & P1_S \\ P2_S & P2_S \end{pmatrix}. \qquad (36)$$

So, the Itô integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)dB(s) \simeq \int_0^t C^T T(s)dB(s) \simeq C^T P_S T(t). \qquad (37)$$

.

5 Application of TFs to solve SVIEs

Here this method is applied for Eq.(1)

$$u(t) = u_0(t) + \int_0^t k_1(s,t)u(s)ds + \int_0^t k_2(s,t)u(s)dB(s)$$

$$t \in [0,T],$$

we approximate function $u(t), u_0(t), k_1(s,t), k_2(s,t)$ by TFs,

$$u(t) \simeq \bar{u}(t) = U^T T(t) = T^T(t)U, \qquad (38)$$

$$u_0(t) \simeq U_0^T T(t) = T^T(t)U_0, \qquad (39)$$

$$k_1(s,t) \simeq T^T(t)K_1T(s), \qquad (40)$$

$$k_2(s,t) \simeq T^T(t)K_2T(s), \qquad (41)$$

where, the vectors U, U_0, and matrices K_1, K_2 are TFs coefficient of u, u_0, k_1 and k_2 respectively.
Substituting (38-41) into (1), we get

$$T^T(t)U \simeq T^T(t)U_0 + \int_0^t T^T(t)K_1T(s)T^T(s)Uds +$$

$$\int_0^t T(t)^T K_2 T(s)T^T(s)UdB(s).$$

Using previous relations

$$T^T(t)U \simeq T^T(t)U_0 + T^T(t)K_1\int_0^t T(s)T^T(s)Uds +$$

$$T^T(t)K_2\int_0^t T(s)T^T(s)dB(s)$$

$$= T^T(t)U_0 + T^T(t)K_1\widetilde{U}PT(t) + T^T(t)K_2\widetilde{U}P_ST(t), \qquad (42)$$

where, $\widetilde{U} = diag(U)$, $K_1\widetilde{U}P$ and $K_2\widetilde{U}P_S$ are $2m \times 2m$ matrices. Eq.(21) gives

$$T^T(t)K_1\widetilde{U}PT(t) \simeq \widehat{B_1}^T T(t) = T^T(t)\widehat{B_1},$$

$$T^T(t)K_2\widetilde{U}P_ST(t) \simeq \widehat{B_2}^T T(t) = T^T(t)\widehat{B_2}^T,$$

in which, $\widehat{B_1}$ and $\widehat{B_2}$ are 2m-vectors with components equal to the diagonal entries of the matrices $K_1\widetilde{U}P$, $K_2\widetilde{U}P_S$ respectively. $\widehat{B_1}$, $\widehat{B_2}$ can be written as

$$\widehat{B_1} = \Pi U,$$

$$\widehat{B_2} = \Pi_S U,$$

where, Π and Π_S are $2m \times 2m$ matrices with components

$$\Pi_{i,j} = (K_1)_{i,j}P_{j,i}, \quad i,j = 1...m,$$

$$(\Pi_S)_{i,j} = (K_2)_{i,j}(P_S)_{j,i}, \quad i,j = 1...m.$$

Then,

$$T^T(t)U \simeq T^T(t)U_0 + T^T(t)\Pi.U + T^T(t)\Pi_S.U,$$

by replacing $\simeq$ with $=$, it gives

$$(I - \Pi - \Pi_s)U = U_0. \qquad (43)$$

After solving the linear system (43) U is calculated and as a result $u(t)$ of (38) is approximated.

6 Error analysis

In following theorems for simplicity we assume $T = 1$ and $h = \frac{1}{m}$. Moreover

$$f(x) \simeq \widehat{f}_m(x) = \sum_{i=0}^{m-1} C1_iT1_i(x) + C2_iT2_i(x)$$

$$= [f(0), f(h), ..., f((m-1)h)]T1 + [f(h), f(2h), ..., f(mh)]T2$$

$$= C1^T T1 + C2^T T2. \qquad (44)$$

Theorem 6.1. *Assume that:*

(1) $f(x)$ *is continuous on* $[0,1]$ *and twice differentiable in* $(0,1)$,

(2) $\overline{f_m(x)}$ *are correspondingly to TFs,*

(3) $|f''(x)| < M$ *for every* $x \in [0,1]$.

Then

$$\| f(x) - \overline{f_m(x)} \| = O(h^2).$$

Proof: Suppose $t_i = \frac{i}{m} = ih$ and $I_i = [t_i, t_{i+1}]$. The representation error when $f(x)$ is represented in a series of TFs over every subinterval $[t_i, t_{i+1}]$, $i = 0, ..., m-1$ is

$$e_i(x) = f(x) - C1_i^T T1_i(x) - C2_i^T T2_i(x),$$

where

$$C1_i = f(ih), \qquad C2_i = f((i+1)h).$$

Base on Taylor's series, the error for $f(x) = C(constant)$ and $f(x) = x$, $f(x) = x^2$ are computed.

It is obvious that $f(x) = C$; $e_i(x) = 0$.
So, this error for $f(x) = x$ in interval $[t_i, t_{i+1}]$, is computed by

$$e_i(x)_{[t_i,t_{i+1}]} = |x - C1_i T1 - C2_i T2|$$

$$= |x - [ih(1 - \frac{(x-ih)}{h}) - ((i+1)h)\frac{(x-ih)}{h}]| = 0.$$

Then the error for $f(x) = x^2$ is

$$e_i(x)_{[t_i,t_{i+1}]}$$

$$= |x^2 - [(ih)^2(1 - \frac{(x-ih)}{h}) - ((i+1)h)^2\frac{(x-ih)}{h}]|$$

$$= |x^2 + i^2h^2 - 2hxi - hx + ih^2|$$

$$\leq \frac{h^2}{4}.$$

So, the error with TFs is h^2M, $x \in [ih, (i+1)h]$.

$$\| e_i(x) \|^2 = \int_{t_i}^{t_{i+1}} |e_i(x)|^2 dx = \int_{t_i}^{t_{i+1}} h^4M^2 dx = M^2h^5,$$

$$\| e \|^2$$

$$= \int_0^1 e^2(x)dx = \int_0^1 (\sum_{i=0}^{m-1} e_i(x))^2 dx = \sum_{i=0}^{m-1} \int_0^1 e_i^2(x)dx$$

$$= \sum_{i=0}^{m-1} \| e_i \|^2 = m.M^2h^5 = M^2h^4.$$

Hence,

$$\| f(x) - \overline{f_m(x)} \| = O(h^2).$$

Now we assume that $f(x,y)$ is a twice differentiable function on $D = [0,1) \times [0,1)$ such that second partial differential is bounded. We define the error between f(x,y) and its 2D-TFs expansion, $f_{i,j}$, over every subregion D_{ij}, as follows:

$$e_{ij}(x,y) = f(x,y) - f_{ij}.$$

where,

$$D_{ij} := \{(x,y)|t_i \leq x \leq t_{i+1}, t_j \leq x \leq t_{j+1}\}.$$

For $f(x,y) = C(constant)$, this error is zero, and also, for $f(x,y) = ax + by, a, b \in R$.
Now, we compute the error for

$$f(x,y) = ax^2 + by^2 + Cxy.$$

Using previous relation, error for $f(x,y) = x^2$ and $f(x,y) = y^2$ are Mh^2. Then error for $f(x,y) = xy$ is

$$|xy - (ih(1 - \frac{(x-ih)}{h}) + ((i+1)h)\frac{(x-ih)}{h})(jh(1 - \frac{(y-jh)}{h}) + ((j+1)h)\frac{(y-jh)}{h})| = 0.$$

This leads

$$\| e_{ij}(x,y) \|^2 = \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |e_{i,j}(x,y)|^2 dydx$$

$$= \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} h^4M^2 dydx = M^2h^6.$$

$$\| e(x,y) \|^2 = \int_0^1 \int_0^1 e^2(x,y)dydx$$

$$= \int_0^1 \int_0^1 (\sum_{i=0}^{m-1} \sum_{j=0}^{m-1} e_{ij}(x,y))^2 dydx$$

$$= \int_0^1 \int_0^1 (\sum_{i=0}^{m-1} \sum_{j=0}^{m-1} e_{ij}^2(x,y))dydx +$$

$$2\sum_{i<i_1} \sum_{j<j_1} \int_0^1 \int_0^1 e_{i,j}(x,y)e_{i_1,j_1}(x,y)dydx.$$

Since for $i < i_1$, $j < j_1$, we have

$$D_{i,j} \cap D_{i_1,j_1} = \{\},$$

then

$$\| e(x,y) \|^2 = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \int_0^1 \int_0^1 e_{ij}^2(x,y))dydx$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \| e_{ij} \|^2 \leq m^2M^2h^6 = M^2h^4.$$

Hence,

$$\| e(x,y) \|^2 = O(h^2),$$

where, $e(x,y) = \overline{f_m(x,y)} - f(x,y)$.

Theorem 6.2. *Let $u(t)$ and $\bar{u}(t)$ be solution of equation (1) and (38), respectively, and assume*
1. $\| u(t) \| < C$.
2. $\| k_i \| < C \quad i = 1, 2.$
Then

$$sup_{0\leq t\leq T}(E(\| (u(t) - \bar{u}(t)) \|)^2)^{1/2} = O(h^2), \quad t \in [0,1].$$

Table 1: Mean, standard deviation and confidence interval for error mean in Example 1 with m=8.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	*Upper*
30	0.0037718893	0.0027234708	0.0027973078	0.0047464708
50	0.0036791366	0.0019663909	0.0031340808	0.0042241924
100	0.0037348421	0.0024004921	0.0032643456	0.0042053386
200	0.0038060617	0.0020049289	0.0035281927	0.0040839307
500	0.0036943720	0.0021958184	0.0035019000	0.0038868440
1000	0.0037556983	0.0022554441	0.0036159044	0.0038954922

Table 2: Mean, standard deviation and confidence interval for error mean in Example 1 with m=16.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	*Upper*
30	0.0030521307	0.0018854114	0.0023774449	0.0037268165
50	0.0032942542	0.0019295053	0.0027594226	0.0038290858
100	0.0030189793	0.0023418020	0.0025599861	0.0034779725
200	0.0030228432	0.0020486679	0.0027389123	0.0033067741
500	0.0032408144	0.0023818067	0.0030320398	0.0034495890

Proof: We have

$$u(t) - \bar{u}(t) = u_0(t) - \bar{u_0}(t) +$$

$$\int_0^t k_1(s,t)u(s) - \bar{k_1}(s,t)\bar{u}(s)ds +$$

$$\int_0^t k_2(s,t)u(s) - \bar{k_2}(s,t)\bar{u}(s)dB(s).$$

So,

$$E(\| u - \bar{u} \|^2) \leq$$

$$3[E(\| (u_0 - \bar{u_0}) \|^2) + E(\| \int_0^t (k_1 u - \bar{k_1}\bar{u})ds \|^2)$$

$$+E(\| \int_0^t (k_2 u - \bar{k_2}\bar{u})dB(s) \|^2)] \tag{45}$$

$$\leq 3[E(\| (u_0 - \overline{u_0}) \|^2) + (\int_0^t E(\| k_1 u - \overline{k_1}\overline{u} \|^2)ds) +$$

$$(\int_0^t E(\| k_2 u - \overline{k_2}\overline{u}) \|^2)ds],$$

by the Cauchy-Schwartz inequality and the linearity of Ito integrals in their integrands.
The first term satisfies by last theorem,

$$E(\| u_0 - \overline{u_0}) \|^2) \leq E(C^2 h^4) = O(h^4),$$

now,

$$\| (k_i(s,t)u(t) - \overline{k_i}(s,t)\overline{u(t)} \|^2 \leq$$

$$2 \| (k_i - \overline{k_i})u \|^2 + 2 \| \overline{k_i}(\overline{u} - u) \|^2 \leq$$

$$C.(\| k_i - \overline{k_i} \|^2) + C.(\| (\overline{u} - u) \|^2), \quad i = 1, 2,$$

furthermore,

$$\| k_i - \overline{k_i} \|^2 = O(h^4) \quad i = 1, 2.$$

Hence

$$E(\| u - \overline{u} \|^2) \leq$$

$$3[E(\| (u_0 - \overline{u_0}) \|^2) + \int_0^t E(\| (k_1 u - \overline{k_1}\overline{u}) \|^2 \, ds) +$$

$$\int_0^t E(\| (k_2 u - \overline{k_2}\overline{u}) \|^2)ds] \leq$$

$$CE(\| u_0 - \overline{u_0} \|^2) + C\int_0^t E(\| k_1 - \overline{k_1} \|^2)ds +$$

$$C\int_0^t E(\| k_2 - \overline{k_2} \|^2)ds + C\int_0^t E(\| (u - \overline{u}) \|^2)ds,$$

then by Gronwall's inequality, we get

$$E(\| (u - \overline{u})^2 \|)) \leq Ch^4.$$

7 Numerical examples

In this section, the theoretical results of the previous sections are used for numerical examples. Let X_i denote the triangular coefficient of exact solution of the given example, and Y_i be the triangular coefficient of computed solution by the presented method. In these examples error is taken as

$$\|E\|_\infty = \max_{1 \leq i \leq m} |X_i - Y_i|.$$

Example 1. *Consider the following linear stochastic Volterra integral equation,*

$$u(t) = \frac{1}{3} + \int_0^t ln(s+1)u(s)ds + \int_0^t \sqrt{ln(s+1)}u(s)dB(s),$$

Table 3: Mean, standard deviation and confidence interval for error mean in Example 2 with m=8.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	*Upper*
30	0.0071270253	0.0009796141	0.0067764748	0.0074775757
50	0.0070807614	0.0009460076	0.0068185414	0.0073429813
100	0.0071393177	0.0007080540	0.0070005391	0.0072780962
200	0.0071454450	0.0008471962	0.0070280295	0.0072628604
500	0.0070885786	0.0007660474	0.0070214315	0.0071557256
1000	0.0070377540	0.0007304459	0.0069924804	0.0070830275

Table 4: Mean, standard deviation and confidence interval for error mean in Example 2 with m=16.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	*Upper*
30	0.0059570103	0.0007245428	0.0056977360	0.0062162846
50	0.0060677570	0.0006453775	0.0058888675	0.0062466465
100	0.0060400389	0.0005996983	0.0058254355	0.0060605173
200	0.0059407980	0.0007160603	0.0058415571	0.0060400389
500	0.0059240266	0.0006894356	0.0058635949	0.0059844583

$$t \in [0, 0.5), \qquad (46)$$

with the exact solution

$$u(t) = \frac{1}{3} e^{-\frac{1}{2}t + \frac{1}{2}tln(t+1) + \frac{1}{2}ln(t+1) + \int_0^t \sqrt{ln(s+1)}dB(s)},$$

for $0 \le t < 0.5$.

The numerical results are shown in Table 1 and Table 2. In tables, n is the number of iterations, $\overline{x}_E$ is error mean, and s_E is standard deviation of error.

Example 2. [15] *Consider the following linear stochastic Volterra integral equation,*

$$u(t) = \frac{1}{12} + \int_0^t cos(s)u(s)ds + \int_0^t sin(s)u(s)dB(s)$$

$$t \in [0, 0.5), \qquad (47)$$

with the exact solution

$$u(t) = \frac{1}{12} e^{-\frac{t}{4} + sin(t) + \frac{sin(2t}{8} + \int_0^t sin(s)dB(s)},$$

for $0 \le t < 0.5$.

The numerical results are shown in Table 3 and Table 4.

8 Conclusion

As some SVIEs can not be solved analytically, in this article we present a new technique for solving SVIEs numerically. Here, triangular functions and their operational matrix of integration are considered. The benefits of this method are lower cost of setting up the system of equations without any integration, moreover, the computational cost of operations is low. Also, convergence of this method is faster than BPFs [15] and order of convergence is $O(h^2)$. These advantages make the method easier to apply. Efficiency of this method and good reasonable degree of accuracy is confirmed by two numerical examples.

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