Multistage Telescoping Decomposition Method for Solving Fractional Differential Equations

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Abstract—**The application of telescoping decomposition method, developed for ordinary differential equations, is extended to derive approximate analytical solutions of fractional differential equations. This method provides us with a simple way to adjust and control the convergence region of solution series by introducing the multistage strategy. This technique provides a sequence of functions which converges to the exact solution of the problem. The present method performs extremely well in terms of efficiency and simplicity. Some examples are solved as illustrations, using symbolic computation.**

Index Terms—**Caputo fractional derivative, Adomian polynomial, fractional differential equation.**

I. Introduction

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. In fact, a considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, are extensively used. Recently, the Adomian decomposition method and variational iteration method have been used for solving a wide range of problems (see for instance [1], [9], [7], [14], [15]).

The telescoping decomposition method was first applied in [4]. It is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. From this method, one obtains a polynomial series solution by means of an iterative procedure. A comparison between the telescoping decomposition method and Adomian decomposition method for solving ordinary differential equations is given in [4]. The fact that the telescoping decomposition method solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over the Adomian decomposition method.

In [5], the author attempts a generalization of this method for the fractional case $0 < \alpha \leq 1$ and compares it to the one given by the ADM method. Nevertheless, f is supposed to be at least of class C^1. Additionally, the solution series which converges very rapidly in most linear and nonlinear equations, in the case of a large time interval t it may produce a large error.

In what follows, we will extend this procedure for $\alpha > 0$ and prove the convergence of the algorithm. f will be supposed continuous and Lipschitzian on its second variable and while using the subdivisions of the interval, we will overcome the problem of the error and obtain a more accurate solution.

Let $\alpha > 0$, and consider the the initial value problems (IVP for short), for fractional order differential equations

$$\begin{aligned} {}^cD^\alpha u &= f(t,u), && (1)\\ u^{(k)}(0^+) &= c_k\ , \quad k = 0,1,..,m-1 && (2) \end{aligned}$$

where m is the integer defined by $m-1 < \alpha \leq m$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : \mathcal{W} = [0,T]\times[c_0-l, c_0+l] \to R$ is continuous and fulfill a Lipschitz condition with respect to the second variable; i.e., $\|f(t,u) - f(t,v)\|_\infty \leq L\,\|u-v\|_\infty$ with some constant $L > 0$ independent of t, u, and v, and $c_k \in R$, $T > 0$, $l > 0$.

The primary aim of this paper is to propose the Fractional Telescoping Decomposition Method (FTDM), a generalization of the TDM [4] and a modified form of the well-known Adomian Decomposition Method, for solving fractional linear and nonlinear initial value problems. We will use the idea of the Adomian method but avoid calculating the Adomian polynomials. In section 2 we recall some definitions of fractional integral and derivative and related basic properties which will be used in the sequel. In section 3, we present the expansion procedure of the (FTDM) and show the convergence of this method. Some numerical examples are treated in section 4 to illustrate the theoretical results and compared with the ADM method. In section 5, we introduce the Multistage Fractional Telescoping Decomposition Method a piecewise-decomposition method that provides series solutions of the FTDM in each subdivision of the interval $[0,T]$.

II. Preliminaries and notations

We now introduce notations, definitions, and preliminaries facts that we will use all along this research work. $C[a,b]$ denotes the space of continuous functions defined on $[a,b]$ and $C^n[a,b]$ denotes the class of all real valued functions defined on $[a,b]$ which have continuous nth order derivative.

Definition 2.1: [[12], [16]] The left sided Riemann-Liouville fractional integral of order α of a real function u in a finite interval $[0,T]$ in R^+ is defined as

$$I^\alpha_{0^+}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,\ \alpha > 0,\ t > 0. \quad (3)$$

where Γ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt;\ z > 0.$$

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For complementation we define $I^0_{0+} := I$ (Identity operator).

Definition 2.2: [12, (section 2.2, p. 79)] The left sided Riemann-Liouville fractional derivative of order $\alpha > 0$ is

$$D^{\alpha}_{0+}u(t) := \big(\frac{d}{dt}\big)^m \left[I^{m-\alpha}_{0+}u(t)\right] \tag{4}$$

i.e

$$D^{\alpha}_{0+}u(t) := \frac{1}{\Gamma(m-\alpha)}\big(\frac{d}{dt}\big)^m \int_0^t (t-s)^{m-\alpha-1}u(s)ds. \tag{5}$$

Where $m = [\alpha] + 1$, $\alpha \notin N = \{0, 1, 2, ..\}$, $[\alpha]$ denotes the integer part of α and $(\frac{d}{dt})^m = \frac{d^m}{dt^m}$ denotes the $mth-derivative$. In particular, when $\alpha = n \in N$, then

$$D^0_{0+}u(t) = u(t)\,, D^n_{0+}u(t) = u^{(n)}(t).$$

Definition 2.3: [12, (section 2.4, p. 90)] The Caputo fractional derivative ${}^cD^{\alpha}_{0+}u(t)$ of order $\alpha > 0$ on $[0, T]$ is defined according to the above Riemann-Liouville fractional derivative by

$${}^cD^{\alpha}_{0+}u(t) := I^{m-\alpha}_{0+}\left[\big(\frac{d^m}{dt^m}\big)u(t)\right] \tag{6}$$

i.e

$${}^cD^{\alpha}_{0+}u(t) := \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\big(\frac{d}{dt}\big)^m u(s)ds. \tag{7}$$

Further, we will denote $D^{\alpha}_{0+}u(t)$ by $D^{\alpha}_0 u(t)$, $I^{\alpha}_{0+}u(t)$ by $I^{\alpha}_0 u(t)$ and ${}^cD^{\alpha}_{0+}u(t)$ by ${}^cD^{\alpha}_0 u(t)$. Also $D^{\alpha}u(t)$, $I^{\alpha}u(t)$ and ${}^cD^{\alpha}_0 u(t)$ refers to $D^{\alpha}_{0+}u(t)$, $I^{\alpha}_{0+}u(t)$ and ${}^cD^{\alpha}_0 u(t)$ respectively.

Note that

$$I^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}, \quad \alpha > 0, \quad \beta > -1 \quad , t > 0. \tag{8}$$

For non-integer α, the Caputo's fractional derivative requires the absolute integrability of the derivative of order m, and we easily recognize that in general

$$D^{\alpha}u(t) := D^m o I^{m-\alpha}u(t) \neq I^{m-\alpha} o D^m u(t) :=^c D^{\alpha}u(t) \tag{9}$$

unless the function $u(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the exchange of the m-derivative with the integral is legitimate, we have [12, (section 2.4, p. 91)]

$${}^cD^{\alpha}u(t) = D^{\alpha}[u(t) - \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k]. \tag{10}$$

In particular for $0 < \alpha < 1$ (i.e. $m = 1$) we have

$${}^cD^{\alpha}u(t) = D^{\alpha}u(t) - u(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = D^{\alpha}[u(t) - u(0^+)] \tag{11}$$

III. Adomian's decomposition method

Existence and uniqueness of solution of fractional differential equations (1) for a given initial condition (2) have been proven in [8], (see also [10, (Sect.2)]). It is shown [13, (theorem 1, section 2)], (see also [10, (lemma 2.1)]) that if the function f is continuous and fulfill a Lipschitz condition with respect to the second variable, then the initial value problem (1)–(2) is equivalent to the nonlinear Volterra integral equation of the second kind

$$u(t) = \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, u(s))ds. \tag{12}$$

In this sense, if $u(t) \in C[a, b]$ satisfies the IVP (1)–(2) , it also satisfies the integral equation (12).

Let us consider the nonlinear fractional differential equation (1) be decomposed into

$$D^{\alpha}u = f(t, u) = g(t) + h(t)u + N(t, u), \tag{13}$$

where $g, h \in C[0, T]$ and N is a nonlinear part of f. Applying the operator I^{α} to both sides of Eq.(13) yields

$$u(t) = \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + I^{\alpha}[g](t) + I^{\alpha}[h(t)u + N(t, u)](t).$$

The ADM [[1], [2]] suggests the solution be decomposed into the infinite series of components

$$u(t) = \sum_{k=0}^{\infty}u_k(t) \tag{14}$$

and the nonlinear function $N(t, u)$ of $f(t, u)$ is decomposed into a series of the so-called Adomian polynomials as follows:

$$N(t, u) = \sum_{k=0}^{\infty}A_k(u_0, .., u_k), \tag{15}$$

where the terms can be calculated recursively from

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(t, \sum_{k=0}^{n}u_k\lambda^k\right)\right]_{\lambda=0}. \tag{16}$$

The iterates are determined by the following recursive way:

$$\begin{cases} u_0(t) = \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + I^{\alpha}[g](t) \\ u_{n+1}(t) = I^{\alpha}[h(t)u_n + A_n](t), n \geq 0. \end{cases} \tag{17}$$

Finally, we approximate the solution by the truncated series

$$\Phi_N(t) = \sum_{n=0}^{N-1}u_n(t), \quad \lim_{N\to\infty}\Phi_N(t) = u(t). \tag{18}$$

Unfortunately one notes that this method is not very efficient for certain cases and for large enough values of n the calculation of the polynomials becomes very difficult.

IV. Main results

A. Fractional Telescoping Decomposition Method

In this section we suggest the Fractional Telescoping Decomposition Method for solving nonlinear differential equation of fractional order. We will use the idea of the Adomian method but avoid calculating the Adomian polynomials. We will consider a solution of the form $u(t) = \sum_{k=0}^{\infty} u_k(t)$, where $u_k(t)$ has to be determined sequentially upon the following algorithm:

$$u_0(t) := \sum_{k=0}^{m-1} \frac{u^{(k)}(0^+)}{k!} t^k, \tag{19}$$

$$u_1(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_0(s)) ds := I^\alpha f(t, u_0(t)), \tag{20}$$

$$u_2(t) := I^\alpha[f(t, u_0(t) + u_1(t))] - I^\alpha f[(t, u_0(t))], \tag{21}$$

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$$u_{l-1}(t) := I^\alpha[f(s, \sum_{k=0}^{l-2} u_k(s)) - f(s, \sum_{k=0}^{l-3} u_k(s))], 3 \le l \le n \tag{22}$$

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$$u_n(t) := I^\alpha \left[f(s, \sum_{k=0}^{n-1} u_k(s)) \right] - I^\alpha \left[f(s, \sum_{k=0}^{n-2} u_k(s)) \right]. \tag{23}$$

Adding (19)–(23) one obtains the nth partial sum:

$$\Phi_n(t) := \sum_{k=0}^{n} u_k(t)\, , n \ge 1, \tag{24}$$

i.e

$$\Phi_n(t) := \sum_{k=0}^{m-1} \frac{u^{(k)}(0^+)}{k!} t^k + \int_0^t \frac{(t-s)^{\alpha-1} f(s, \sum_{k=0}^{n-1} u_k(s))}{\Gamma(\alpha)} ds. \tag{25}$$

Let us note that the choice of $u_0(t)$ is not unique, it depends on the initial conditions $u^{(i)}(0), i = 0, 1, ..m-1$, and on the given problem.

B. Convergence analysis

In this section we will prove that $\sum_{k=0}^{\infty} u_k$ converges uniformly on the interval $[0, T]$ to the exact solution u of (1)–(2). We have the following lemma.

Lemma 4.1: Consider the sequence of functions u_n, $n \ge 0$, as defined in (17)–(18) and let $\mathcal{W} := [0, T] \times [u_0 - l, u_0 + l]\, , T > 0, l > 0$. If f is continuous on $\mathcal{W}$, and fulfill a Lipschitz condition with respect to the second variable, then the infinite series $\sum_{k=0}^{\infty} u_k$ converges uniformly on $[0, T]$.

Proof: First, one puts

$$\|f\| = sup_{(t,u)\in\mathcal{W}} |f(t, u)| \quad \text{and} \quad \|u\| = sup_{t\in[0,T]} |u(t)|\, . \tag{26}$$

As f is continuous on $\mathcal{W}$ and fulfill a Lipschitz condition with respect to the second variable, it follows that it exists real positives numbers M and L such that $\|f\| \le M$ and $|f(t, u) - f(t, v)| \le L \|u - v\|$ for any $t \in [0, T]$ and any $u, v \in [u_0 - l, u_0 - l]$. Now we use mathematical induction to prove the inequality

$$\|u_n\| \le \frac{L^{n-1} M t^{n\alpha}}{\Gamma(1 + n\alpha)}, \forall n \ge 1. \tag{27}$$

The result is true for $n = 1$, since for all $t \in [0, T]$ $|u_1(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_0(s)) ds \right|$ $=|I^\alpha f(t, u_0(t))| \le \|f\| I^\alpha(1) \le \|f\| \frac{t^\alpha}{\Gamma(1+\alpha)} \le M \frac{t^\alpha}{\Gamma(1+\alpha)}$.
Suppose that for all $k \ge 2$,

$$\|u_k\| \le \frac{L^{k-1} M t^{k\alpha}}{\Gamma(1 + k\alpha)}, \tag{28}$$

and show that the inequality holds for $k + 1$. Recall that: $u_{k+1}(t) := \int_0^t \frac{(t-s)^{\alpha-1}\left[f(s, \sum_{j=0}^{k} u_j(s)) - f(s, \sum_{j=0}^{k-1} u_j(s))\right]}{\Gamma(\alpha)} ds.$

Since f is continuous and Lipschitzian with respect to the second variable, it follows:

$$\begin{aligned} &\left| f(s, \sum_{j=0}^{k} u_j(s)) - f(s, \sum_{j=0}^{k-1} u_j(s)) \right| \\ \le\; & L \left| \sum_{j=0}^{k} u_j(s) - \sum_{j=0}^{k-1} u_j(s) \right| \\ \le\; & L\, |u_k(s)| \le L \|u_k\|\, . \end{aligned}$$

Thus, in view of (25), (28) and (8) we have for all t in $[0, T]$:

$$\begin{aligned} |u_{k+1}(t)| &\le \int_0^t \left| \frac{(t-s)^{\alpha-1}\left[f(s, \sum_{j=0}^{k} u_j(s)) - f(s, \sum_{j=0}^{k-1} u_j(s))\right]}{\Gamma(\alpha)} \right| ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L \|u_k\| ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L \frac{L^{k-1} M s^{k\alpha}}{\Gamma(1+k\alpha)} ds \\ &\le \frac{L L^{k-1} M}{\Gamma(1+k\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{s^{k\alpha}}{\Gamma(\alpha)} ds \\ &\le \frac{L^k M}{\Gamma(1+k\alpha)} I^\alpha(t^{k\alpha}) \\ &\le \frac{L^k M}{\Gamma(1+k\alpha)} \frac{\Gamma(1+k\alpha) t^{(k+1)\alpha}}{\Gamma(1+\alpha(k+1))} \\ &\le \frac{L^k M t^{(k+1)\alpha}}{\Gamma(1+\alpha(k+1))}, \end{aligned}$$

which is the desired result.

In order to finish the proof, we need to verify that the series converges. This, however, is a well known result since

$$\sum_{n=0}^{\infty}\frac{L^{n-1}Mt^{n\alpha}}{\Gamma(1+n\alpha)} = L^{-1}M\sum_{n=0}^{\infty}\frac{(Lt^{\alpha})^n}{\Gamma(1+n\alpha)} = L^{-1}ME_{\alpha}(Lt^{\alpha}),$$

where $E_\alpha(Lt^\alpha)$ is the the Mittag–Leffler function of order α , evaluated at Lt$^{\alpha}$. ■

Theorem 4.2: Consider the initial value problem (1)–(2), and the sequence of functions $u_n(t)$, $n \geq 0$, as defined in (19)–(23). If f is continuous on $\mathcal{W}$, and fulfill a Lipschitz condition with respect to the second variable, then the infinite series $\sum_{k=0}^{\infty} u_k$ converges to the exact solution of the initial value problem (1)–(2).

Proof: Recall that,

$$u(t) = \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds,$$

and

$$\Phi_n(t) = \sum_{k=0}^{n} u_k(t)\,, n \geq 1.$$

Define the sequence of the functions h_n by

$$h_n(s) := f(s,\sum_{k=0}^{n}u_k(s)) = f(s,\Phi_n(s))\,, n \geq 1.$$

By lemma 4.1, we have that Φ_n converge uniformly in the interval $[0,T]$ to some function Φ, and as f is uniformly continuous on $\mathcal{W}$, then the sequence h_n converges uniformly to $f(s,\Phi)$. Therefore,

$$\sum_{k=0}^{\infty}u_k(t) := \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + \lim_{n\to\infty}\int_0^t\frac{(t-s)^{\alpha-1}h_n(s)}{\Gamma(\alpha)}ds$$

$$:= \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + \int_0^t\frac{(t-s)^{\alpha-1}\lim_{n\to\infty}h_n(s)}{\Gamma(\alpha)}ds$$

$$:= \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + \int_0^t\frac{(t-s)^{\alpha-1}f(s,\lim_{n\to\infty}\Phi_n(s))}{\Gamma(\alpha)}ds$$

$$:= \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k + \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\Phi(s))ds,$$

and as $\Phi(t) := \lim_{n\to\infty}\Phi_n(t) := \sum_{k=0}^{\infty}u_k(t)$, for all t in $[0,T]$, it follows that $\Phi := \sum_{k=0}^{\infty}u_k$ is solution of the problem (1)–(2). ■

V. Numerical examples

The numerical results are discussed in this section based on two nonlinear fractional differential equations.

Example 5.1: Consider the following nonlinear fractional differential equation with variable coefficients ([11]):

$$^{c}D^{\alpha}u + tu^2 = \frac{32}{21\Gamma(3/4)}t^{7/4} + t^5 \quad , \quad u(0)=0. \quad (29)$$

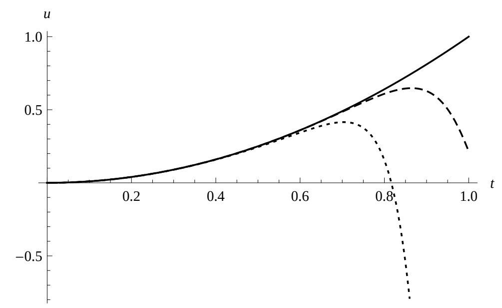

Fig. 1. Approximation solution for equation (29) with α=1/4, using ADM: Dotted line, FTDM: Dashed line, and exact solution: Solid line.

The exact solution for equation (29) with $\alpha = 1/4$ is $u(t) = t^2$.

Solution by using the ADM method: In view of (13), we have $g(t) = \frac{32}{21\Gamma(3/4)}t^{7/4}+t^5$, $h(t)=0$ and $N(t,u) = -tu^2$. Thus, according to (17), we have the following recursive relation:

$$\begin{cases} u_0(t) = I^{\alpha}(\frac{32}{21\Gamma(3/4)}t^{7/4}+t^5) \\ u_{n+1}(t) = I^{\alpha}(A_n)(t), n \geq 0. \end{cases} \quad (30)$$

According to this algorithm, one has

$$\begin{aligned}
u_0(t) &= I^{\frac{1}{4}}(\frac{32t^{7/4}}{21\Gamma(3/4)}+t^5) = t^2 + 0.649137t^{\frac{21}{4}},\\
u_1(t) &= I^{\frac{1}{4}}(A_0(t)) = -I^{\frac{1}{4}}(t(u_0(t))^2)\\
&= -0.649137t^{\frac{21}{4}} - 0.752091t^{\frac{17}{2}} - 0.225799t^{\frac{47}{4}},\\
u_2(t) &= I^{\frac{1}{4}}(A_1(t)) = -2I^{\frac{1}{4}}(tu_0(t)u_1(t))\\
&= 1.110223.10^{-16}t^{\frac{21}{4}} + 0.752091t^{\frac{17}{2}}\\
&\quad +1.03183t^{\frac{47}{4}} + 0.228051t^{15} - 0.272273t^{\frac{73}{4}}\\
&\quad -0.157046t^{\frac{43}{2}} - 0.0227726t^{\frac{99}{4}},\\
u_3(t) &= I^{\frac{1}{4}}(A_2(t)) = -2I^{\frac{1}{4}}(tu_0(t)u_2(t)) - I^{\frac{1}{4}}(t(u_1(t))^2)\\
&= 2.2204461.10^{-16}t^{\frac{21}{4}} - 1.6931572.10^{-16}t^{\frac{17}{2}}\\
&\quad -0.80603t^{\frac{47}{4}} - 1.04212t^{15} + 0.0527271t^{\frac{73}{4}}\\
&\quad +0.408836t^{\frac{43}{2}} - 0.127119t^{\frac{99}{4}} - 0.139551t^{28}\\
&\quad +0.163155t^{\frac{125}{4}} + 0.155279t^{\frac{69}{2}} + 0.0137661t^{\frac{151}{4}}\\
&\quad -0.0296237t^{41} - 0.0143403t^{\frac{177}{4}} - 0.0027192t^{\frac{95}{2}}\\
&\quad -0.000193939t^{\frac{203}{4}}.
\end{aligned}$$

Finally one has an approximation of the first four terms as follows.

$$\begin{aligned}
\Phi_{3_{ADM}}(t) \quad : \quad &= \sum_{n=0}^{3}u_n(t) = t^2 + 4.4408921x10^{-16}t^{\frac{21}{4}}\\
&\quad -7.7715612x10^{-16}t^{\frac{17}{2}} + 9.15934x10^{-16}t^{\frac{47}{4}}\\
&\quad -1.53519t^{15} - 1.75243t^{\frac{73}{4}}\\
&\quad -0.720433t^{\frac{43}{2}} - 0.104597t^{\frac{99}{4}} + ...
\end{aligned}$$

Solution by using the FTDM method: According to (19)–(23), we have the initial guess
$u_0(t) = 0$, that lead to the following terms

$$
\begin{aligned}
u_1(t) &= I^{\frac{1}{4}} f(t, u_0(t)) = t^2 + 0.649137 t^{\frac{21}{4}}, \\
u_2(t) &= I^{\frac{1}{4}} f(t, u_0(t) + u_1(t)) - I^{\frac{1}{4}} f(t, u_0(t)) \\
&= -0.649137 t^{\frac{21}{4}} - 0.752091 t^{\frac{17}{2}} - 0.2258 t^{\frac{47}{4}}, \\
u_3(t) &= I^{\frac{1}{4}} f(t, u_0(t) + u_1(t) + u_2(t)) \\
&\quad - I^{\frac{1}{4}} f(t, u_0(t) + u_1(t)) \\
&= 1.110223024626.10^{-16} t^{\frac{21}{4}} + 0.752091 t^{\frac{17}{2}} \\
&\quad + 1.03183 t^{\frac{47}{4}} + 0.228051 t^{15} - 0.272273 t^{\frac{73}{4}} \\
&\quad - 0.157046 t^{\frac{43}{2}} - 0.0227726 t^{\frac{99}{4}},
\end{aligned}
$$

and the solution, approximation of the first four terms, is given by

$$
\begin{aligned}
\Phi_{3_{FTDM}}(t) \quad : \quad &= \sum_{n=0}^{3} u_n(t) = t^2 + 5.551115123.10^{-16} t^{\frac{21}{4}} \\
&\quad - 5.676534909795.10^{-16} t^{\frac{17}{2}} + 0.80603 t^{\frac{47}{4}} \\
&\quad + 0.228051 t^{15} - 0.272273 t^{\frac{73}{4}} \\
&\quad - 0.157046 t^{43/2} - 0.0227726 t^{\frac{99}{4}} + \ldots
\end{aligned}
$$

The approximation solution $\Phi_3(t)$ of Eq. (29) have been plotted in Fig. 1 and it is easy to deduce that the one obtained from FTDM is more efficient than the ADM method compared with the exact value.

Let us note that, as mentioned previously (subsection IV-A), the choice of the initial value is not unique. For example, to simplify the calculus and accelerate the convergence of the series solution, one can choose the initial guess $u_0(t) = I^{\frac{1}{4}}(\frac{32t^{7/4}}{21\Gamma(3/4)} + t^5) = t^2 + 0.649137 t^{\frac{21}{4}}$, and working all along the procedure with $f(t, u) \leftrightarrow N(t, u) = -tu^2$.

Example 5.2: Consider the fractional Riccati equation

$$
{}^c D^{\alpha} u = 1 - u^2 \quad , \quad u(0) = 0. \tag{31}
$$

Solution by using the FTDM method: According to (19)–(23), we have the initial guess
$u_0(t) = 0$ that lead to the following terms
$u_1(t) = I^{\alpha} f(t, u_0(t)) = I^{\alpha} f(t, 0) = I^{\alpha}(1) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$,
$\mathrm{u}_2(t) = I^{\alpha} f(t, u_0(t) + u_1(t)) - I^{\alpha} f(t, u_0(t))$
$u_2(t) = I^{\alpha} f(t, \frac{t^{\alpha}}{\Gamma(\alpha+1)}) - I^{\alpha}(1)$
$u_2(t) = I^{\alpha} f(t, \frac{t^{\alpha}}{\Gamma(\alpha+1)}) - \frac{t^{\alpha}}{\Gamma(\alpha+1)}$,
$u_3(t) = I^{\alpha} f(t, u_0(t) + u_1(t) + u_2(t)) - I^{\alpha} f(t, u_0(t) + u_1(t))$

$u_3(t) = I^{\alpha} f(t, I^{\alpha} f(t, \frac{t^{\alpha}}{\Gamma(\alpha+1)})) - I^{\alpha} f(t, \frac{t^{\alpha}}{\Gamma(\alpha+1)})$
...
Setting $\alpha = 1$ in equation (31), the exact solution for the integer order Riccati is

$$
u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.
$$

Expanding $u(t)$ using Taylor expansion about $t = 0$ gives

$$
\begin{aligned}
u(t) \quad = \quad & t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835} - \frac{1382t^{11}}{155925} \\
&+ \frac{21844t^{13}}{6081075} - \frac{929569t^{15}}{638512875} + ..
\end{aligned}
$$

Applying ADM [[1], [7]] to solve (31), starting from $t = 0$ with five terms, we get

$$
\Phi_{4_{ADM}}(t) = t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835}.
$$

Applying FTDM to solve (31), starting from $t = 0$ with five terms, we get

$$
\begin{aligned}
\Phi_{4_{FTDM}}(t) \quad = \quad & t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835} - \frac{1142t^{11}}{155925} \\
&+ \frac{13324t^{13}}{6081075} - \frac{377017t^{15}}{638512875} + \ldots
\end{aligned}
$$

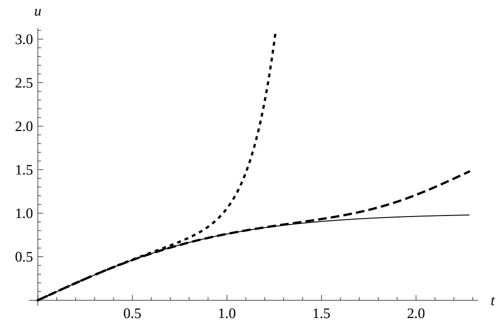

Fig. 2. Approximation solution for equation (31) with α=1, using ADM: Dotted line, FTDM: Dashed line, and exact solution: Solid line.

As we see in these examples, the solution given by the FTDM method is more efficient than the one given by ADM. The alone problem is that the approximate solutions are generally, not valid for large t.

VI. Multi-stage fractional telescoping decomposition method

Generally finding more terms of the sequence (u_n) is difficult or impossible. Thus, an accurate approximation is provided near the initial condition only. In fact for a $t_1 > 0$ the approximate solution is accurate only in $t_0 \leq t \leq t_1$. The reason of this phenomenon is the behavior of the u_n. As u_n has been found by successive integrations, it contains terms with high powers of t. Therefore, even if u converges to infinity by the growth of t, u_n converges to infinity faster than u. More precisely, we can rewrite the sequence (25) of FTDM method as

$U_n(t) := \sum_{k=0}^{n} u_k(t)$
$U_n(t) := \sum_{k=0}^{m-1} \frac{u^{(k)}(0^+)}{k!} t^k + \int_0^t \frac{(t-s)^{\alpha-1} f(s, \sum_{k=0}^{n-1} u_k(s))}{\Gamma(\alpha)} ds$
$U_n(t) := \sum_{k=0}^{m-1} \frac{u^{(k)}(0^+)}{k!} t^k + \int_0^t \frac{(t-s)^{\alpha-1} f(s, U_{n-1}(s))}{\Gamma(\alpha)} ds.$

In addition, the differential equation can be written as
$u(t) = \sum_{k=0}^{m-1} \frac{u^{(k)}(0^+)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds.$

Therefore, we have

Lemma 6.1: Let M be a bound for $|f(t, u)|$ on $\mathcal{W}$ and $\delta = min(T, \sqrt[\alpha]{\frac{\Gamma(\alpha+1)}{M} l})$. Then, all solution $u = u(t)$ on $[0, \delta]$ verifies $\|u - u_0\| \leq l$.

Proof: The requirement $\delta \leq T$ is necessary. On the other hand, the requirement $\delta \leq \sqrt[\alpha]{\frac{\Gamma(\alpha+1)}{M}} l$ is dictated by the fact that if $u = u(t)$ is solution of (17)–(18) on $[0, \delta]$, then

$$\|u-u_0\| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds\right|$$
$$\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} M ds\right| \leq \frac{M}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha} \leq l. \qquad \blacksquare$$

Consequently, one can give an evaluation of the error as:

$$\|u-\Phi_n\| := \left\|\int_0^t \frac{(t-s)^{\alpha-1}(f(s,u(s))-f(s,\Phi_{n-1}(s)))}{\Gamma(\alpha)} ds\right\|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|(f(s,u(s))-f(s,\Phi_{n-1}(s)))\right| ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\left|u(s)-\Phi_{n-1}(s)\right| ds$$
$$\leq L\|u-\Phi_{n-1}\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds$$
$$\leq L\|u-\Phi_{n-1}\|\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$\leq L\frac{t^\alpha}{\Gamma(\alpha+1)}\|u-\Phi_{n-1}\| \;\leq L\frac{t^\alpha}{\Gamma(\alpha+1)}L\frac{t^\alpha}{\Gamma(\alpha+1)}\|u-\Phi_{n-2}\|$$
$$\leq ... \leq (\frac{Lt^\alpha}{\Gamma(\alpha+1)})^n\|u-\Phi_0\| \leq \;(\frac{Lt^\alpha}{\Gamma(\alpha+1)})^n\|u-u_0\|$$
$$\leq \;(\frac{Lt^\alpha}{\Gamma(\alpha+1)})^n l,$$

which results that the maximum error increases by the growth of t. In fact, the approximate solution is not accurate in the larger regions. That is what we saw in the previous examples and it is due to the fact that all methods (ADM, VIM,..., etc) used until there introduce noise terms, i.e. the same term appears in different iterations, in the series solution for $u(t)$. This phenomenon cause problems in determining the convergence of the technique. In addition, since the decomposition method provides a series solution, it is, in general, very difficult to establish its convergence or the number of terms required to achieve a specified accuracy. On the other hand, the convergence tolerance is related to the product Lt^α where L is the Lipschitz constant and t denotes the interval of the independent variable. This suggests that one may accelerate the convergence of the latter by determining the solution in sufficiently small intervals of t, i.e., $[t_i, t_{i+1}]$, so that $L(\Delta t_i)^\alpha$ is sufficiently small, where $\Delta t_i = t_{i+1} - t_i$. The time interval $[0, T]$ can be divided into a sequence of subintervals $[t_0, t_1]$, $[t_1, t_2]$,..., $[t_{N-1}, t_N]$, in which $t_0 = 0$, $t_N = T$, and $\bigcup_{i=0}^{N-1}\Omega_i = [0, T]$, where $\Omega_i = [t_i, t_{i+1}]$. Without loss of generality, the subintervals can be chosen as the same length Δt , i.e. $\Delta t_i = \Delta t$, $i = 0, ..., N-1$. Furthermore, the equations in (1) can be solved by the FTDM in every sequential interval Ω_i, $(i = 0, 1, ..., N-1)$.

As we want to apply the method on every subinterval, we will need an initial value in each of these subinterval. The only one known until there is at the point $t_0 = 0$ of the first subinterval $[t_0, t_1]$. For the other initial values, they will be calculated from the previous N_0-term approximate solutions approximations $\Phi_{i-1,N_0}(t)$, $(i = 1, 2, ..N-1)$ at the point t_i the left-end of the subinterval $\Omega_i, (i = 1, 2, ..., N-1)$. Therefore, one will have the new sequence as follow:
in $\Omega_0 = [t_0, t_1]$ one puts

$$\begin{cases} u_{0,0}(t) := \sum_{k=0}^{m-1}\frac{u^{(k)}(0^+)}{k!}t^k \quad ; \\ u_{0,1}(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_{0,0}(s))ds; \\ u_{0,k}(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s,\sum_{j=0}^{k-1}u_{0,j}(s)) \\ -f(s,\sum_{j=0}^{k-2}u_{0,j}(s))]ds, k\geq 2 \\ \Phi_{0,N_0}(t) := \sum_{j=0}^{N_0}u_{0,j}(t), N_0 \in N-\{0\}, \end{cases} \tag{32}$$

TABLE I
TABLE OF COMPARISON BETWEEN THE EXACT SOLUTION OF EQUATION (31) AND THE RESULT FROM THE DIFFERENT METHODS.

t	Exact-adm	Exact-ftdm	Exact-mftdm1	Exact-mftdm2
0.	0.	0.	0.	0.
0.5	0.0000527184	0.000258835	0.000258835	0.000014335
1.	4.11266	0.0225328	0.00392702	0.00019862
1.5	10326.1	0.211146	0.0062468	0.00023658
2.	$5.68459*10^6$	0.604226	0.00493945	0.00021436
2.5	$1.03131*10^9$	0.362234	0.00285711	0.00060917
3.	$8.13081*10^{10}$	9.30934	0.00141448	0.000064131
3.5	$3.43045*10^{12}$	43.8865	0.00524324	0.00019049
4.	$8.97823*10^{13}$	141.863	0.0213876	0.0022787
4.5	$1.6177*10^{15}$	373.963	0.0587671	0.0099358
5.	$2.16245*10^{16}$	861.079	0.126514	0.027505
5.5	$2.26564*10^{17}$	1796.48	0.23312	0.059500
6.	$1.93904*10^{18}$	3473.63	0.385885	0.11034

and in each interval $\Omega_i = [t_i, t_{i+1}]$, $(i = 1, 2, ..N-1)$ one puts

$$\begin{cases} u_{i,0}(t) := \Phi_{i-1,N_0}(t_i) \quad ; \\ u_{i,1}(t) := \frac{1}{\Gamma(\alpha)}\int_{t_i}^t (t-s)^{\alpha-1} f(s,u_{i,0}(s))ds \\ u_{i,k}(t) := \frac{1}{\Gamma(\alpha)}\int_{t_i}^t (t-s)^{\alpha-1}[f(s,\sum_{j=0}^{k-1}u_{i,j}(s)) \\ -f(s,\sum_{j=0}^{k-2}u_{i,j}(s))]ds, k\geq 2 \\ \Phi_{i,N_0}(t) := \sum_{j=0}^{N_0}u_{i,j}(t), \\ N_0 \in N-\{0\}, i = 1, 2, ..., N-1. \end{cases} \tag{33}$$

Now the solution of the problem is provided in the piecewise form as

$$\Phi_{N_0}(t) = \begin{cases} \Phi_{0,N_0}(t) := \sum_{j=0}^{N_0}u_{0,j}(t), 0\leq t\leq t_1 \\ \Phi_{i,N_0}(t) := \sum_{j=0}^{N_0}u_{i,j}(t), \\ t_i \leq t \leq t_{i+1}, i = 1, 2, .., N-1. \end{cases} \tag{34}$$

Thus, if one takes the previous example in Eq. (31) and applying the new method, fixing $N_0 = 3$ and applying the MFTDM for respectively $\Delta t = 0.5$ (MFTDM1) and $\Delta t = 0.2$ (MFTDM2), one obtains the result described in table **??** and the corresponding graphic in Fig. 3.

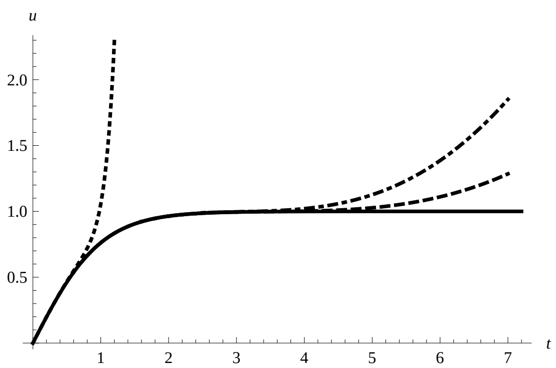

Fig. 3. Approximation solution for equation (31), using ADM: Dotted line, using MFTDM1: Dash-dotted line, using MFTDM2: Dashed line, and exact solution: solid line.

The method just described is a piecewise fractional telescoping decomposition method that provides series solutions in each Ω_i which are continuous at t_i, i.e., at the end points of the subintervals Ω_i. These solutions are expected to converge faster than the one that would result from applying the fractional telescoping decomposition method to (the larger domain) $(0, T]$.

VII. Conclusion

In the present paper, we have proposed the Multistage Fractional Telescoping Decomposition Method for solving nonlinear fractional differential equations. The approximate solutions obtained by this algorithm are compared with the exact solutions. The results show that the MFTDM method is a powerful mathematical tool for solving nonlinear fractional differential equations. This model of fractional differential equations has been solved by Adomian method and the advantage is to overcome the difficulty arising in calculating Adomian's polynomials. Another advantage of this method over the other analytical methods, such as ADM ,VIM and HPM, in this method we can choose a proper value for the Δt the length of the subintervals $[t_i, t_{i+1}]$ and therefore the initial guess $u_{i,0}$ to adjust and control convergence region of the series solutions. It should be noted that in our work we use the MATHEMATICA Package to calculate the series obtained from the ADM and the FTDM method.

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