# A Class of Control Variates for Pricing Asian Options under Stochastic Volatility Models

Kun Du, Guo Liu, and Guiding Gu

*Abstract*—**In this paper we present a strategy to form a class of control variates for pricing Asian options under the stochastic volatility models by the risk-neutral pricing formula. Our idea is employing a deterministic volatility function $\sigma(t)$ to replace the stochastic volatility $\sigma_t$. Under the Hull and White model[11] and the Heston model[10], the deterministic volatility function $\sigma(t)$ can be chosen with the same order moment as that of $\sigma_t$, and then a control variate can be derived. The numerical experiments report that our control variates work quite well by showing the standard deviation reduction ratio.**

*Index Terms*—**Asian Options pricing; Monte Carlo method; control variates.**

## I. Introduction

An Asian option is a kind of financial derivative whose payoff includes a time average of the underlying asset prices. The primary purpose for basing an option payoff on an average asset price is to make it more difficult for anyone to significantly affect the payoff by manipulation of the underlying asset price. So Asian options can be used to reduce the risk caused by unusual behaviors of the underlying asset price before expiry, and they are quite popular in risk management. According to different sampling types and strike price types, there are eight types of Asian options (in this paper, we do not distinguish call and put options), four fixed-strike options and four floating-strike options. The payoff functions of four fixed-strike options are:

1) fixed-strike continuous sampling arithmetic average Asian (call) option (1cAAO),

$$V_{1cAAO}|_{t=T} = \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+;$$

2) fixed-strike discrete sampling arithmetic average Asian (call) option(1dAAO),

$$V_{1dAAO}|_{t=T} = \left(\frac{1}{N}\sum_{i=1}^{N} S_i - K\right)^+;$$

3) fixed-strike continuous sampling geometric average Asian (call) option(1cGAO),

$$V_{1cGAO}|_{t=T} = \left(e^{\frac{1}{T}\int_0^T \log S_t dt} - K\right)^+;$$

4) fixed-strike discrete sampling geometric average Asian (call) option(1dGAO),

$$V_{1dGAO}|_{t=T} = \left(e^{\frac{1}{N}\sum_{i=1}^{N} logS_i} - K\right)^+,$$

where $K$ and $S_t$ are the fixed-strike price and the price of underlying asset at time $t$, respectively. $S_i \equiv S_{T_i}$ denotes the price of underlying asset at the $i$th observation date $T_i$ with $0 = T_0 < T_1 < T_2 < \cdots < T_N = T$. $[0,\ T]$ represents the valid period of the option.

Replacing $K$ with $S_T$ in $V_{1cAAO}|_{t=T}$ and $V_{1cGAO}|_{t=T}$, we can derive the payoff functions of the floating-strike continuous sampling arithmetic and geometric average Asian (put) options, denoted as $2cAAO$ and $2cGAO$,

$$V_{2cAAO}|_{t=T} = \left(\frac{1}{T}\int_0^T S_t dt - S_T\right)^+,$$

$$V_{2cGAO}|_{t=T} = \left(e^{\frac{1}{T}\int_0^T \log S_t dt} - S_T\right)^+.$$

Also, replacing $K$ with $S_N$ in $V_{1dAAO}|_{t=T}$ and $V_{1dGAO}|_{t=T}$, we have the payoff function of the floating-strike discrete sampling arithmetic and geometric average Asian (put) option, denoted as $2dAAO$ and $2dGAO$,

$$V_{2dAAO}|_{t=T} = \left(\frac{1}{N}\sum_{i=1}^{N} S_i - S_N\right)^+,$$

$$V_{2dGAO}|_{t=T} = \left(e^{\frac{1}{N}\sum_{i=1}^{N} \log S_i} - S_N\right)^+.$$

The Monte Carlo method is a numerical method based on probability and statistics, and is widely used in many fields, especially in the field of computational finance. One of the main advantages of the Monte Carlo method is that its convergence is independent on the number of state variables. It is usually used when the number of state variables is greater than three. However, the drawback of Monte Carlo method is that its convergence rate is slow. Let $V$ be a random variable(r.v. for short), and we want to calculate $\mu = E[V]$. By simulation, we get identically independent distributed ($i.i.d.$ for short) samples $\{V_i\}_{i=1}^n$ of $V$; Law of Larger Number guarantees $\overline{V}_n = \frac{1}{n}\sum_{i=1}^n V_i \xrightarrow{a.s.} \mu$; Central Limit Theorem guarantees that $\mu$ asymptotically falls in the confidence interval

$$\left[\overline{V}_n - \frac{\sigma}{\sqrt{n}} Z_{\frac{\delta}{2}}, \overline{V}_n + \frac{\sigma}{\sqrt{n}} Z_{\frac{\delta}{2}}\right]$$

with probability $1-\delta$, where $\sigma$ is the standard deviation of $V$, $n$ is the number of simulation paths, $\delta$ is the significance level and $Z_{\frac{\delta}{2}}$ is the quantile of standard normal distribution under $\frac{\delta}{2}$. It is clear that the convergence rate of the Monte Carlo method is $O(n^{-\frac{1}{2}})$, and a better way to improve

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the accuracy is reducing the standard deviation $\sigma$. We refer to Glasserman [8] for a summary of various techniques to reduce the variance.

The method of control variates is one of the most widely used variance reduction techniques. Suppose on each replication we can calculate another output $X_i$ along with $V_i$, that the pairs $\{(X_i,\ V_i)\}_{i=1}^n$ are $i.i.d.$ and that the expectation $E[X]$ of $X_i$ is known. We use $(X,\ V)$ to denote a generic pair of r.v.s with the same distribution as each $(X_i,\ V_i)$. Then for any fixed $b \in R$, we can calculate

$$V_i(b) = V_i - b(X_i - E[X]), \qquad i = 1, \cdots, n$$

through the $i$th replication and compute the sample mean

$$\overline{V}_n(b) = \overline{V}_n - b(\overline{X}_n - E[X]).$$

This is a control variate estimator. It is proved in Glasserman [8] that $\overline{V}_n(b)$ is a unbiased and consistent estimator of $\mu$. $V(b)$ has variance

$$Var(V(b)) = \sigma_V^2 - 2b\sigma_X\sigma_V\rho_{XV} + b^2\sigma_X^2. \tag{1}$$

The minimum point on $b$ is $b^* = \frac{\sigma_V}{\sigma_X}\rho_{XV}$. Substituting $b^*$ in (1), we have

$$\frac{Var(V(b^*))}{Var(V)} = 1 - \rho_{XV}^2. \tag{2}$$

We choose a control variate $X$ for $V$, if $X$ satisfies two conditions:

$i)$ the expectation $E[X]$ is known;
$ii)$ the correlation $\rho_{XV}^2$ is close to 1.

In practice, $b^*$ can't be derived exactly as $\sigma_V$ and $\rho_{XV}$ are generally unknown. We can use its sample counterpart yields the estimate

$$\widehat{b} = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)(V_i - \overline{V}_n)}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}$$

to approximate $b^*$. As mentioned in Glasserman [8], we may still get most of the benefit of a control variate using an estimate of $b^*$. Strictly speaking, to measure the efficiency of the Monte Carlo method, we need not only the variance reduction ratio but also expected computing time per replication. But in this paper, the computational effort per replication is roughly the same with and without a control variate, so we focus on the variance reduction ratio; see Ma and Xu [13].

Kemma and Vorst [12] studied the valuation of arithmetic average Asian options by using the counterpart geometric average Asian options as control variates. This is one of the most successful applications of control variates in financial engineering. In the case of stochastic volatility models, a constant volatility can be chosen to replace the stochastic volatility in some conditions, and then this tractable dynamic process is used as an auxiliary process to form a control variate. How to choose this constant volatility is the key problem of the efficiency of control variates. The most intuitive way is to choose the initial value of the stochastic volatility as the constant volatility. Both Fouque and Han [5] and Han and Lai [9] use a method named as the Martingale Control Variate method to choose an *effective volatility* which is dependent on the initial value of the stochastic volatility as the constant volatility. This method has many advantages and can be used to other financial derivatives besides Asian options (see Fouque and Han [4, 6]). But the martingale control variate method also has a potential drawback. Calculating the *effective volatility* needs the invariant distribution function of stochastic volatility. If the stochastic volatility satisfies Ornstein-Uhlenbeck process under which the invariant distribution of stochastic volatility is easy to handle, the martingale control variate method is easy to implement, but if the stochastic volatility satisfies a process, which the invariant distribution of stochastic volatility is hard to handle such as Square-Root Diffusion, or the invariant distribution is unknown, the martingale control variate method is difficult to implement. There are many types of stochastic volatility models, such as those in Scott [14], Stein and Stein [16] and Ball and Roma [1]. We refer to Fouque *et al* [7] for a summary of various stochastic volatility models.

In this paper, we present a strategy to form a class of control variates for pricing Asian options under a stochastic volatility model. Our idea is employing a deterministic volatility function $\sigma(t)$ to replace the stochastic volatility $\sigma_t$. This deterministic volatility $\sigma(t)$ is not only dependent on the initial value of the stochastic volatility but also dependent on time $t$, so that $\sigma(t)$ can track down the stochastic volatility. Under the Hull and White model [11] and the Heston model [10], the deterministic volatility function $\sigma(t)$ can be chosen with the same order moment as that of $\sigma_t$, and then a control variate can be derived. The numerical experiments in our paper report that our control variates work quite well in terms of showing the standard deviation reduction ratio. It is worth noting that our control variate is a generalization of the control variate in [13] for pricing variance swap under the Hull and White model [11].

The rest of this paper is organized as follows. We introduce some basic settings for the model used in this paper in Section I and derive the idiographic control variates under the Hull and White model in Section II. In Section III we present an algorithm to estimate the standard deviation reduction ratio and then report some numerical results in terms of showing the standard deviation reduction ratios under the Hull and White model and the Heston model. Finally we give some conclusions in Section IV.

### A. *Basic Setting*

In this section we model the underlying asset price, but we do not give the concrete stochastic differential equation which the volatility satisfies. We get some general conclusions which will be useful in the following sections.

We begin with a probability space $(\Omega,\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathbf{P})$, here $\mathbf{P}$ is the risk-neutral measure. In this paper, all expectations are derived under the risk-neutral measure $\mathbf{P}$ unless there is a special statement. Suppose that the price of underlying asset $S_t$ follows the geometric Brownian motion

$$dS_t = rS_t dt + \sigma_t S_t dW_{1t}, \tag{3}$$

where $r$ is the risk-free interest rate which is a constant, $W_{1t}$ is the Winner process and $\sigma_t$ is the stochastic volatility which satisfies a diffusion process driving by another Winner process $W_{2t}$. $W_{1t}$ and $W_{2t}$ satisfy $cov(dW_{1t},\ dW_{2t}) = \rho dt$, so we have $W_{2t} = \rho W_{1t} + \sqrt{1-\rho^2}B_t$, in which $B_t$ is the Winner process and independent with $W_{1t}$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by the two-dimension Brownian motion $(W_{1t},\ B_t)$, so $S_t$ and $\sigma_t$ are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Suppose that $\sigma_t$ satisfies the square-integrability condition which is $E\int_0^t \sigma_s^2 ds < \infty$. It is known that by the risk-neutral pricing formula, the prices are

$$\begin{aligned} V_{1cAAO}|_{t=0} &= E[e^{-rT}(V_{1cAAO}|_{t=T})] \\ &= e^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \end{aligned}$$

for 1cAAO (fixed-strike discrete sampling arithmetic average Asian (call) option),

$$\begin{aligned} V_{1dAAO}|_{t=0} &= E[e^{-rT}(V_{1dAAO}|_{t=T})] \\ &= e^{-rT}E\left[\left(\frac{1}{N}\sum_{i=1}^N S_i - K\right)^+\right] \end{aligned}$$

for 1dAAO,

$$\begin{aligned} V_{1cGAO}|_{t=0} &= E[e^{-rT}(V_{1cGAO}|_{t=T})] \\ &= e^{-rT}E[(e^{\frac{1}{T}\int_0^T \log S_t dt} - K)^+] \end{aligned}$$

for 1cGAO, and

$$\begin{aligned} V_{1dGAO}|_{t=0} &= E[e^{-rT}(V_{1dGAO}|_{t=T})] \\ &= e^{-rT}E[(e^{\frac{1}{N}\sum_{i=1}^N \log S_i} - K)^+] \end{aligned}$$

for 1dGAO. Also for four floating-strike Asian options, the prices are

$$\begin{aligned} V_{2cAAO}|_{t=0} &= e^{-rT}E[(\frac{1}{T}\int_0^T S_t dt - S_T)^+], \\ V_{2dAAO}|_{t=0} &= e^{-rT}E[(\frac{1}{N}\sum_{i=1}^N S_i - S_N)^+], \\ V_{2cGAO}|_{t=0} &= e^{-rT}E[(e^{\frac{1}{T}\int_0^T \log S_t dt} - S_T)^+], \\ V_{2dGAO}|_{t=0} &= e^{-rT}E[(e^{\frac{1}{N}\sum_{i=1}^N \log S_i} - S_N)^+]. \end{aligned}$$

As said in Fouque and Han [5], when the volatility is randomly fluctuating, there is no analytic solution for GAO in general, neither for AAO. But if the volatility is a deterministic function(not necessarily constant), the prices of GAO have analytic solutions. In such case, these analytic solutions can be used as control variates for pricing corresponding Asian options with stochastic volatility.

For GAO with deterministic volatility, we have following theorems. $N(\cdot)$ is the standard normal distribution function in this paper.

**Theorem 1.** *Suppose that the stochastic volatility $\sigma_t$ in (3) is replaced by a deterministic square-integrable volatility $\sigma(t)$, there is an analytic solution for the fixed-strike continuous sampling geometric average Asian (call) option,*

$$\begin{aligned} X_{1cGAO}|_{t=0} &= E[e^{-rT}(X_{1cGAO}|_{t=T})] \\ &= e^{-rT}E\left[\left(e^{\frac{1}{T}\int_0^T \log S(t)dt} - K\right)^+\right] \\ &= e^{\frac{1}{2}\widehat{\sigma}^2 - rT + a}N(d_+) - Ke^{-rT}N(d_-), \end{aligned}$$

*where*

$$a = \log S_0 + \frac{1}{2}rT - \frac{1}{2T}\int_0^T [\int_0^t \sigma^2(s)ds]dt,$$

$$\widehat{\sigma}^2 = \lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^n [2(n-j)+1]\int_0^{j\frac{T}{n}} \sigma^2(s)ds,$$

*and* $d_- = \frac{a - logK}{\widehat{\sigma}},\ d_+ = d_- + \widehat{\sigma}$.

**Proof.** By (3) and the assumptions, we have

$$\begin{aligned} \log S(t) &= \log S_0 + rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_{1s} \\ &\equiv a(t) + I(t). \end{aligned} \tag{4}$$

and

$$\frac{1}{T}\int_0^T \log S(t)dt = \frac{1}{T}\int_0^T a(t)dt + \frac{1}{T}\int_0^T I(t)dt.$$

By Theorem 4.4.9 in Shreve [14], we get

$$I(t) = \int_0^t \sigma(s)dW_{1s} \sim N(0,\ \int_0^t \sigma^2(s)ds).$$

It is easy to see

$$a \equiv \frac{1}{T}\int_0^T a(t)dt = \log S_0 + \frac{1}{2}rT - \frac{1}{2T}\int_0^T\int_0^t \sigma^2(s)dsdt.$$

Next, we focus on proving $\frac{1}{T}\int_0^T I(t)dt \sim N(0,\ \widehat{\sigma}^2)$. Let $0 = t_0 < t_1 < \cdots < t_n = T;\ \triangle t_i = t_i - t_{i-1} = \triangle t = \frac{T}{n},\ i = 1, 2, \ldots, n;\ t_i = i\triangle t,\ i = 0, 1, \ldots, n$. Denote $\Theta \equiv \frac{1}{T}\int_0^T I(t)dt$. Thus, we have

$$\begin{aligned} \Theta = \frac{1}{T}\int_0^T I(t)dt &= \lim_{n\to\infty}\sum_{i=1}^n \frac{1}{T}I(t_i)\triangle t_i \\ &= \lim_{n\to\infty}\sum_{i=1}^n \frac{1}{n}I(t_i) \equiv \lim_{n\to\infty}\Theta_n. \end{aligned}$$

Since it holds for any path, we have $\Theta_n \xrightarrow{a.s.} \Theta$($\xrightarrow{a.s.}$ means convergence in almost surely sense). By Theorem 5.3.1 and Theorem 5.5.1 in [17], we know that $\Theta_n \xrightarrow{a.s.} \Theta \Longrightarrow \Theta_n \xrightarrow{d} \Theta$. ($\xrightarrow{d}$ means convergence in distribution sense). Since

$$\begin{pmatrix} I(t_1) \\ I(t_2) \\ \vdots \\ I(t_n) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \Sigma\right),$$

where

$$\Sigma \equiv \begin{pmatrix} \int_0^{t_1}\sigma^2(s)ds & \int_0^{t_1}\sigma^2(s)ds & \cdots & \int_0^{t_1}\sigma^2(s)ds \\ \int_0^{t_1}\sigma^2(s)ds & \int_0^{t_2}\sigma^2(s)ds & \cdots & \int_0^{t_2}\sigma^2(s)ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^{t_1}\sigma^2(s)ds & \int_0^{t_2}\sigma^2(s)ds & \cdots & \int_0^{t_n}\sigma^2(s)ds \end{pmatrix}.$$

By setting $k = (1, 1, \ldots, 1)^T$, we have

$$\begin{aligned} \Theta_n = \sum_{i=1}^n \frac{1}{n}I(t_i) &= \frac{1}{n}(1, 1, \ldots, 1)\begin{pmatrix} I(t_1) \\ I(t_2) \\ \vdots \\ I(t_n) \end{pmatrix} \\ &\sim N(0,\ \frac{1}{n^2}k^T\Sigma k) = N(0,\ \sigma_n^2), \end{aligned}$$

and

$$\sigma_n^2 = \frac{1}{n^2}\sum_{j=1}^{n}[2(n-j)+1]\int_0^{t_j}\sigma^2(s)ds \longrightarrow \widehat{\sigma}^2.$$

Since $\Theta_n \sim N(0,\ \sigma_n^2)$ for any $n$, the characteristic function $\varphi_n(u)$ of $\Theta_n$ satisfies

$$\varphi_n(u) = e^{-\frac{1}{2}u^2\sigma_n^2} \longrightarrow e^{-\frac{1}{2}u^2\widehat{\sigma}^2} = \varphi(u).$$

It is easy to prove that in any interval $[U_1,\ U_2]$, $\varphi_n(u)$ uniformly converges to $\varphi(u)$ as $\varphi_n(u)$ and $\varphi(u)$ are both continuous functions. By Levi-Cramer Theorem([17], Theorem 5.4.1), we get $\delta_n \xrightarrow{d} N(0,\ \widehat{\sigma}^2)$. Thus as the uniqueness of limitation, we have $\Theta \sim N(0,\ \widehat{\sigma}^2)$ and

$$\xi \equiv \frac{1}{T}\int_0^T logS(t)dt = a + \Theta \sim N(a,\ \widehat{\sigma}^2).$$

By the risk-neutral pricing formula, it holds that

$$\begin{aligned} X_{1cGAO}|_{t=0} &= E[e^{-rT}(X_{1cGAO}|_{t=T})] \\ &= e^{-rT}E[(e^{\frac{1}{T}\int_0^T \log S(t)dt} - K)^+] \\ &= e^{-rT}E[(e^{\xi} - K)^+]. \end{aligned} \tag{5}$$

By setting $\xi = a - \widehat{\sigma}Z,\ Z \sim N(0,\ 1)$, we have

$$\begin{aligned} X_{1cGAO}|_{t=0} &= e^{-rT}E[(e^{a-\widehat{\sigma}Z} - K)^+] \\ &= e^{-rT}\int_{-\infty}^{+\infty}(e^{a-\widehat{\sigma}z} - K)^+\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \\ &= e^{-rT}\int_{-\infty}^{d_-}(e^{a-\widehat{\sigma}z} - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \\ &= e^{\frac{1}{2}\widehat{\sigma}^2 - rT + a}N(d_+) - Ke^{-rT}N(d_-), \end{aligned}$$

where $d_- = \dfrac{a - logK}{\widehat{\sigma}},\ d_+ = d_- + \widehat{\sigma}$.

**Theorem 2.** *Suppose that the stochastic volatility $\sigma_t$ in (3) is replaced by a deterministic square-integrable volatility $\sigma(t)$ , there is an analytic solution for the fixed-strike discrete sampling geometric average Asian (call) option,*

$$\begin{aligned} X_{1dGAO}|_{t=0} &= E[e^{-rT}(X_{1dGAO}|_{t=T})] \\ &= e^{-rT}E\left[\left(e^{\frac{1}{N}\sum_{i=1}^{N}\log S(T_i)} - K\right)^+\right] \\ &= e^{\frac{1}{2}\widehat{\sigma}^2 - rT + a}N(d_+) - Ke^{-rT}N(d_-), \end{aligned}$$

*where*

$$\begin{aligned} a &= \log S_0 + \frac{r}{N}\sum_{i=1}^{N}T_i - \frac{1}{2N}\sum_{i=1}^{N}\int_0^{T_i}\sigma^2(s)ds, \\ \widehat{\sigma}^2 &= \frac{1}{N^2}\sum_{j=1}^{N}[2(N-j)+1]\int_0^{T_j}\sigma^2(s)ds, \end{aligned}$$

*and* $d_- = \frac{a - \log K}{\widehat{\sigma}},\ d_+ = d_- + \widehat{\sigma}$.

We omit the proof of Theorem 2 since it is similar to that of Theorem 1. For the floating-strike Asian options, we also have the following theorems.

**Theorem 3.** *Suppose that the stochastic volatility $\sigma_t$ in (3) is replaced by a deterministic square-integrable volatility $\sigma(t)$, there is an analytic solution for the floating-strike continuous sampling geometric average Asian (put) option,*

$$\begin{aligned} X_{2cGAO}|_{t=0} &= E\left[e^{-rT}(X_{2cGAO}|_{t=T})\right] \\ &= e^{-rT}E\left[\left(e^{\frac{1}{T}\int_0^T \log S(t)dt} - S(T)\right)^+\right] \\ &= S_0 e^{\frac{1}{2}b^2 + a}N(d_+) - S_0 N(d_-), \end{aligned}$$

*where*

$$\begin{aligned} a &= -\frac{1}{2}rT + \frac{1}{2T}\int_0^T\int_0^t \sigma^2(s)dsdt - \frac{1}{2}\int_0^T\sigma^2(s)ds, \\ b^2 &= \lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^{n}[2(n-j)+1]\int_0^{j\frac{T}{n}}\sigma^2(s)ds \\ &\quad - 2\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\int_0^{j\frac{T}{n}}\sigma^2(s)ds + \int_0^T\sigma^2(s)ds, \end{aligned} \tag{6}$$

*and* $d_- = \dfrac{a}{b},\ d_+ = d_- + b$.

**Proof.** Set $J(T) = e^{\frac{1}{T}\int_0^T \log S(t)dt}$. By the risk-neutral pricing formula, we have

$$\begin{aligned} X_{2cGAO}|_{t=0} &= E\left[e^{-rT}(X_{2cGAO}|_{t=T})\right] \\ &= e^{-rT}E\left[\left(e^{\frac{1}{T}\int_0^T \log S(t)dt} - S(T)\right)^+\right] \\ &= e^{-rT}E\left[(J(T) - S(T))^+\right] \\ &= e^{-rT}E\left[S(T)\left(\frac{J(T)}{S(T)} - 1\right)^+\right]. \end{aligned}$$

Set $Z(T) = e^{\int_0^T \sigma(s)dW_{1s} - \frac{1}{2}\int_0^T\sigma^2(s)ds}$ and $\widehat{\mathbf{P}}(A) = \int_A Z(T)d\mathbf{P},\ \forall A \in F$. By Girsanov's Theorem, $\widehat{W_{1s}} \equiv W_{1s} - \int_0^s \sigma(u)du$ is a Winner process under the new probability measure $\widehat{P}$. Then we have

$$\begin{aligned} X_{2cGAO}|_{t=0} &= e^{-rT}\widehat{E}\left[S(T)\left(\frac{J(T)}{S(T)} - 1\right)^+\frac{1}{Z(T)}\right] \\ &= S_0\widehat{E}\left[\left(\frac{J(T)}{S(T)} - 1\right)^+\right]. \end{aligned}$$

By (5), we have $\log\frac{J(T)}{S(T)} = a + \widehat{\Theta}$, where

$$\begin{aligned} a &= -\frac{1}{2}rT + \frac{1}{2T}\int_0^T\int_0^t\sigma^2(s)dsdt - \frac{1}{2}\int_0^T\sigma^2(s)ds, \\ \widehat{\Theta} &= \frac{1}{T}\int_0^T\widehat{I}(t)dt - \widehat{I}(T), \end{aligned}$$

and $\widehat{I}(t) = \int_0^t\sigma(s)d\widehat{W}_{1s}$. Under the new probability measure $\widehat{\mathbf{P}}$, similar to the proof of Theorem 1, we can prove $\widehat{\Theta} \sim N(0,\ b^2)$, and we omit it. Set $\widehat{\xi} \equiv \log\frac{J(T)}{S(T)}$. Then we have $\widehat{\xi} \sim N(a,\ b^2)$ under the measure $\widehat{\mathbf{P}}$. Thus it holds that

$$X_{2cGAO}|_{t=0} = S_0\widehat{E}\left[\left(\frac{J(T)}{S(T)} - 1\right)^+\right] = S_0\widehat{E}\left[(e^{\widehat{\xi}} - 1)^+\right].$$

Also similar to the proof of Theorem 1, we can get the conclusion of Theorem 3.

**Theorem 4.** *Suppose that the stochastic volatility $\sigma_t$ in (3) is replaced by a deterministic square-integrable volatility*

*$\sigma(t)$, there is an analytic solution for the floating-strike discrete sampling geometric average Asian (put) option,*

$$\begin{aligned} X_{2dGAO}|_{t=0} &= E\left[e^{-rT}(X_{2dGAO}|_{t=T})\right] \\ &= e^{-rT}E\left[\left(e^{\frac{1}{N}\sum_{i=1}^{N}\log S(T_i)} - S(T_N)\right)^+\right] \\ &= S_0 e^{\frac{1}{2}b^2+a}N(d_+) - S_0 N(d_-), \end{aligned}$$

*where*

$$\begin{aligned} a &= -\frac{r}{N}\left[(N-1)T_N - \sum_{i=1}^{N-1} T_i\right] \\ &\quad - \frac{1}{2N}\left[(N-1)\int_0^{T_N}\sigma^2(s)ds - \sum_{i=1}^{N-1}\int_0^{T_i}\sigma^2(s)ds\right], \\ b^2 &= \frac{1}{N^2}\sum_{j=1}^{N}[2(N-j)+1]\int_0^{T_j}\sigma^2(s)ds \\ &\quad - \frac{2}{N}\sum_{j=1}^{N}\int_0^{T_j}\sigma^2(s)ds + \int_0^{T}\sigma^2(s)ds, \end{aligned}$$

*and* $d_- = \dfrac{a}{b},\ d_+ = d_- + b.$

The proof is similar to that of Theorem 3.

Note that $\sigma(t)$ should be chosen such that the limitations in (4) and (6) both exist. By the call-put parity formula, for the fixed-strike GAO put option, the price formula is $Ke^{-rT}N(-d_-) - e^{\frac{1}{2}\widehat{\sigma}^2 rT+a}N(-d_+)$, and for the floating-strike GAO call option, the price formula is $S_0N(-d_-) - S_0e^{\frac{1}{2}b^2+a}N(-d_+)$.

## II. Control Variates under Two Models

The analytic solutions for GAO derived in Section I could be employed as control variates for valuing Asian options with stochastic volatility models in Section I. For example, we can employ $X_{1cGAO}$ as a control variate to get $V_{1cGAO}$ and $V_{1cAAO}$, and $X_{1dGAO}$ as a control variate to get $V_{1dGAO}$ and $V_{1dAAO}$, *et al*. However, by (2), it is important that how to choose the deterministic square integrable volatility $\sigma(t)$ to make $\rho^2_{XV}$ as large as possible. In this section, we show a strategy to choose an appropriate deterministic volatility $\sigma(t)$ under the Hull and White model [11] and the Heston model [10]. The idea is that $\sigma(t)$ is chosen with the same order moment as that of $\sigma_t$.

### A. Hull and White Model

Hull and White [10] introduced the concept of stochastic volatility. Suppose that square of the stochastic volatility $Y_t(\sigma_t = \sqrt{Y_t})$ satisfies the following equation

$$dY_t = \mu Y_t dt + \sigma Y_t dW_{2t}, \tag{7}$$

where $\mu,\ \sigma$ are constants. It is hold that

$$Y_t = \sigma_t^2 = Y_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_{2t}} = \sigma_0^2 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_{2t}}. \tag{8}$$

We choose $\sigma(t)$ such that $\sigma(t)$ and $\sigma_t$ have the same $m$th order moment, that is

$$[Y(t)]^{\frac{m}{2}} = [\sigma(t)]^m = E[\sigma_t^m] = E[Y_t^{\frac{m}{2}}]. \tag{9}$$

By (8) and the property of lognormal distribution, we have

$$\sigma(t) = \sigma_0 e^{\frac{1}{2}a_m t}, \tag{10}$$

where $a_m = \mu + \frac{1}{4}(m-2)\sigma^2$ and $m$ is any real number. Substituting $\sigma(t)$ in Theorem 1 – 4, we can solve the parameters $(a, \widehat{\sigma})$ of the analytic solutions in Theorem 1 – 4.

**Theorem 5** *Suppose that $\sigma(t)$ is defined by (10). Then the parameters $a$, $\widehat{\sigma}^2$ and $b^2$ in Theorem 1 – 4 have the expressions*

*(i) in Theorem 1,*

$$a = \begin{cases} \log S_0 + \frac{1}{2}rT - \frac{1}{4}\sigma_0^2 T, & \text{if } a_m = 0 \\ \log S_0 + \frac{1}{2}rT - \\ \frac{\sigma_0^2}{2Ta_m}[\frac{1}{a_m}(e^{a_m T}-1) - T], & \text{if } a_m \neq 0 \end{cases}$$

$$\widehat{\sigma}^2 = \begin{cases} \frac{1}{3}\sigma_0^2 T, & \text{if } a_m = 0 \\ \frac{2\sigma_0^2}{T^2 a_m^3}(e^{a_m T}-1) - \frac{2\sigma_0^2}{Ta_m^2} - \frac{\sigma_0^2}{a_m}, & \text{if } a_m \neq 0 \end{cases}$$

*(ii) in Theorem 2,*

$$a = \begin{cases} \log S_0 + \frac{r}{N}\sum_{i=1}^N T_i - \frac{\sigma_0^2}{2N}\sum_{i=1}^N T_i, & \text{if } a_m = 0 \\ \log S_0 + \frac{r}{N}\sum_{i=1}^N T_i - \\ \frac{\sigma_0^2}{2N}\sum_{i=1}^N \frac{1}{a_m}e^{a_m T_i}, & \text{if } a_m \neq 0 \end{cases}$$

$$\widehat{\sigma}^2 = \begin{cases} \frac{\sigma_0^2}{N^2}\sum_{j=1}^N [2(N-j)+1]T_j, & \text{if } a_m = 0 \\ \frac{\sigma_0^2}{a_m N^2}\sum_{j=1}^N [2(N-j)+1][e^{a_m T_j}-1], & \text{if } a_m \neq 0 \end{cases}$$

*(iii) in Theorem 3,*

$$a = \begin{cases} -\frac{1}{2}(r+\frac{1}{2}\sigma_0^2)T, & \text{if } a_m = 0 \\ \frac{\sigma_0^2}{2Ta_m^2}(e^{a_m T}-1) - \frac{\sigma_0^2}{2a_m}e^{Ta_m} - \frac{1}{2}rT, & \text{if } a_m \neq 0 \end{cases}$$

$$b^2 = \begin{cases} \frac{1}{3}\sigma_0^2 T, & \text{if } a_m = 0 \\ \frac{\sigma_0^2}{a_m}[(1-\frac{2}{Ta_m})e^{a_m T} + \frac{2(e^{a_m T}-1)}{T^2 a_m^2}], & \text{if } a_m \neq 0 \end{cases}$$

*(iv) in Theorem 4,*

$$a = \begin{cases} -\frac{r}{N}[(N-1)T - \sum_{i=1}^{N-1}T_i] - \\ \frac{\sigma_0^2}{2N}[(N-1)T - \sum_{i=1}^{N-1}T_i], & \text{if } a_m = 0 \\ -\frac{r}{N}[(N-1)T - \sum_{i=1}^{N-1}T_i] - \frac{\sigma_0^2}{2Na_m}[(N-1) \\ (e^{a_m T}-1) - \sum_{i=1}^{N-1}(e^{a_m T_i}-1)], & \text{if } a_m \neq 0 \end{cases}$$

$$b^2 = \begin{cases} \frac{\sigma_0^2}{N^2}\sum_{i=1}^N [2(N-j)+1]T_j - \\ \frac{2\sigma_0^2}{N}\sum_{i=1}^N T_j + \sigma_0^2 T, & \text{if } a_m = 0 \\ \frac{\sigma_0^2}{N^2 a_m}\sum_{j=1}^N [2(N-j)+1](e^{a_m T_j}-1) \\ -\frac{2\sigma_0^2}{Na_m}\sum_{j=1}^N (e^{a_m T_j}-1) \\ +\frac{\sigma_0^2}{a_m}(e^{a_m T}-1), & \text{if } a_m \neq 0 \end{cases}$$

The proof of this theorem is computational process and we omit it. The only one point is that when solving the limitations in (4) and (6), we should use the Taylor expansion $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ and the concept of the same order infinitesimal.

Thus we can obtain a control variate $X$ to an option $V$ since the expectation of $X$ can be solved analytically by the theorems.

### B. Heston Model

The Hull and White model is the earliest stochastic volatility model and because of its tractable in mathematics, it's applied very widely. But in the long run, it is unreasonable in financial sense. If the volatility $Y_t$ satisfies (7), by (9) and (10), we have $E[\sigma_t] = \sigma_0 e^{\frac{1}{2}(\mu - \frac{1}{4}\sigma^2)t}$ which illustrates that the volatility mean grows exponentially. This is not likely to be true. Heston [10] supposed that square of the volatility satisfies the mean-reversion process

$$dY_t = (\alpha - \beta Y_t)dt + \sigma\sqrt{Y_t}dW_{2t}, \quad (11)$$

where $\alpha > 0,\ \beta > 0,\ \sigma > 0$. The process in (11) is a square-root diffusion process, which was first studied by Cox, Ingersoll and Ross [3]. This model guarantees that $Y_t$ converges to its long run mean $\alpha/\beta$ and $Y_t$ is nonnegative. In financial point of view, the Heston model is more reasonable than the Hull-White model, but the Heston model is less tractable in mathematics. Unlike (7), (11) doesn't have a closed-form solution, but we can easily solve its expectation ([10],pp.142,ex4.4.11)

$$E[\sigma_t^2] = E[Y_t] = e^{-\beta t}Y_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}).$$

We now choose $\sigma(t)$ such that $\sigma(t)$ and $\sigma_t$ have the same order moment

$$Y(t) = \sigma^2(t) = E[\sigma_t^2] = e^{-\beta t}Y_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}). \quad (12)$$

Thus, $\sigma(t)$ can be used in Theorem 1 – 4, and then $X$ can be employed as a control variate to a option $V$ under the Heston model.

## III. Numerical Experiment

By (2), the efficiency of a control variate $X$ to an option $V$ can be shown by the correlation $\rho^2_{XV}$, or by the standard deviation reduction ratio

$$R = \sqrt{\frac{1}{1 - \rho^2_{XV}}}. \quad (13)$$

A larger $R$ means that a control variate $X$ has more efficiency to an option $V$. In this section, we first present a algorithm to estimate $R$, then perform some numerical experiments to report the efficiency of our control variates by showing the estimation of $R$.

Following the way of Ma and Xu [12], we present the following numerical algorithm to estimate $R$ for the control variate $X_{1dGAO}$ to the option $V_{1dGAO}$ under the Hull-White model.

**Algorithm 1.** Estimate $R$ for $X_{1dGAO}$ to $V_{1dGAO}$ under the Hull-White model.

1) Divide [0, $T$] into $n$ intervals with mesh size $\triangle t = T/n = t_{k+1} - t_k$, and make sure that the set of time discrimination points $\{t_k\}_{k=1}^n$ covers the set of observation dates $\{T_i\}_{i=1}^N$.
2) After putting $\sigma(t)$ into (3), we can generate $S(t_{k+1})$ from $S(t_k)$ (also see (5)) by

$$S(t_{k+1}) = S(t_k)\exp\left\{r\triangle t - \frac{1}{2}\int_{t_k}^{t_{k+1}}\sigma^2(s)ds + \int_{t_k}^{t_{k+1}}\sigma(s)dW_{1s}\right\}.$$

As $\int_{t_k}^{t_{k+1}}\sigma(s)dW_{1t} \sim N(0,\ \int_{t_k}^{t_{k+1}}\sigma^2(s)ds)$, we generate standard normal random number $Z_k^{1,j}$ and get

$$S^j(t_{k+1}) = S^j(t_k)\exp\left\{r\triangle t - \frac{1}{2}\int_{t_k}^{t_{k+1}}\sigma^2(s)ds + \sqrt{\int_{t_k}^{t_{k+1}}\sigma^2(s)ds}Z_k^{1,j}\right\}, (j = 1, \cdots, p)$$

where $S^j(t_0) = S_0$ and $p$ is the number of the replication simulation. Thus a replication $j$ of the underlying asset price $S(t)$ is derived.
3) By the contract of the option, set the value of control variate

$$X^j_{1dGAO} = \left(e^{\frac{1}{N}\sum_{i=1}^N \log S^j(T_i)} - K\right)^+. \quad (14)$$

4) Similarly, we generate $S_{t_{k+1}}$ from $S_{t_k}$ by

$$S^j_{t_{k+1}} = S^j_{t_k}\exp\left\{(r - \frac{(\sigma^j_{t_k})^2}{2})\triangle t + \sigma^j_{t_k}\sqrt{\triangle t}Z_k^{1,j}\right\},$$

with $S^j_{t_0} = S_0$, where $\sigma^i_{t_k} = \sqrt{Y_k^j}$, and $Y_{t_{k+1}}$ from $Y_{t_k}$ by

$$Y^j_{t_{k+1}} = Y^j_{t_k}\exp\left[(\mu - \frac{1}{2}\sigma^2)\triangle t + \sigma\sqrt{\triangle t}Z_k^{2,j}\right], \quad (15)$$

where $Z_k^{2,j}$ is the standard normal random number with the correlative coefficient $\rho$ with $Z_k^{1,j}$. Thus a replication $j$ of the underlying asset prices $S_t$ following processes (3) and (13) is simulated.
5) By the clause of the option, set the value of the option

$$V^j_{1dGAO} = \left(e^{\frac{1}{N}\sum_{i=1}^N \log S_i^j} - K\right)^+. \quad (16)$$

6) Let $\overline{X}_p = \frac{1}{p}\sum_{j=1}^p X_j$, $\overline{V}_p = \frac{1}{p}\sum_{j=1}^p V_j$, then

$$\widehat{\rho}_{XV} = \frac{\sum_{j=1}^p (X_j - \overline{X}_p)(V_j - \overline{V}_p)}{\sqrt{\sum_{j=1}^p (X_j - \overline{X}_p)^2}\sqrt{\sum_{j=1}^p (V_j - \overline{V}_p)^2}},$$

and

$$\widehat{R} = \sqrt{\frac{1}{1 - (\widehat{\rho}_{XV})^2}}.$$

**Remark:**

1) For other control variate $X$ to other option $V$, it is only need to modify (14) and (16).

2) For the Heston model, it is only need to modify (15) by (11).

### A. Hull-White Model

Based on the algorithm, we perform some numerical experiments to report the efficiency of our control variates by showing the standard deviation reduction ratio $\widehat{R}$ under the Hull-White model . We report our numerical results of with a Matlab 7.0 implementation of the algorithm.

Following Ma and Xu [13], we set the parameters $T = 1, n = 100, N = 50, r = 0.05, \mu = 0.05, S_0 = 100, \sigma = 0.01, Y_0 = \sigma_0^2 = 0.15^2, p = 10000$. We test serval groups of the other parameters $m, \rho, K$. Note that if $m = 2 - \frac{4\mu}{\sigma^2}$, $Y(t) = \sigma^2(t) = \sigma_0^2 = Y_0$ is constant. The data in all the tables are the standard deviation reduction ratio $\widehat{R}$, rather than the variance reduction ratio $\widehat{R}^2$.

**Experiment 1**. In this experiment, we report the efficiency of the control variate $X_{1dGAO}$ to the option $V_{1dGAO}$ by showing the standard deviation reduction ratio $\widehat{R}$ in Table I. We test serval groups of the parameters $m, \rho, K$. The data in Table I show us that:

1) when $m = 2 - \frac{4\mu}{\sigma^2}$ at the last column, $\sigma(t) = \sigma_0$ in (10) is a constant, so $\sigma(t)$ can't track down $\sigma_t$. In such

TABLE I
$X_{1dGAO}$ TO $V_{1dGAO}$

| | | m=-50 | m=0 | m=1 | m=2 | m=100 | m=2 $-\frac{4\mu}{\sigma^2}$ |
|---|---|---|---|---|---|---|---|
| | K=90 | 409.3333 | 418.7455 | 408.6419 | 414.3889 | 407.7352 | 162.6211 |
| $\rho = 0.1$ | K=100 | 379.4101 | 360.4291 | 377.2734 | 377.3868 | 370.9005 | 150.3187 |
| | K=110 | 222.9084 | 223.0590 | 211.9047 | 224.6605 | 236.2057 | 111.6436 |
| | K=90 | 465.8595 | 488.0159 | 474.7598 | 484.4832 | 482.1097 | 156.7584 |
| $\rho = 0.9$ | K=100 | 413.7979 | 411.2028 | 417.6350 | 425.4420 | 428.6535 | 143.0066 |
| | K=110 | 372.6171 | 379.2108 | 358.8228 | 369.4438 | 356.1442 | 117.0795 |

case, the efficiency of the control variate $X_{1dGAO}$ to the option $V_{1dGAO}$ is small. For the other $m$, the difference of the efficiency is not significant;

2) there is some influence for different $\rho$. The larger $\rho$ is, the larger $\widehat{R}$ is;

3) when the option is in-the-money (i.e., $K < 100$), the control variate works better. This is because when the option is out-of-the-money (i.e., $K > 100$), there are many paths giving zero payoff.

To overcome this drawback, we can use the call-put parity formula,

$$\begin{aligned} V_{1dGAO}|_{t=0} =& E\left[e^{-rT}\left(e^{\frac{1}{N}\sum_{i=1}^{N}\log S_i} - K\right)^+\right] \\ =& E\left[e^{-rT}\left(K - e^{\frac{1}{N}\sum_{i=1}^{N}\log S_i}\right)^+\right] + \\ & E\left[e^{-rT}\left(e^{\frac{1}{N}\sum_{i=1}^{N}\log S_i} - 0\right)^+\right] - e^{-rT}K. \end{aligned}$$

It is clear that if $(e^{\frac{1}{N}\sum_{i=1}^{N}\log S_i} - K)^+$ is (deep) out-of-the-money, $(K - e^{\frac{1}{N}\sum_{i=1}^{N}\log S_i})^+$ is (deep) in-the-money. Thus we can use the Monte Carlo method with our control variate to simulate the (deep) in-the-money option $V_{1dGAO}|_{t=0}$.

**Experiment 2**. In this experiment, we report the efficiency of the control variate $X_{1dGAO}$ to the option $V_{1dAAO}$ by showing the standard deviation reduction ratio $\widehat{R}$ in Table II. In such case, we replace $V^j_{1dGAO}$ in (16) by

$$V^j_{1dAAO} = \left(\frac{1}{N}\sum_{i=1}^{N} S^j_i - K\right)^+.$$

We also test the same group of the parameters $m, \rho, K$ as that in the experiment 1.

The data in Table II show us that:

1) the efficiency of the control variate $X_{1dGAO}$ to the option $V_{1dAAO}$ is much lower than that to the option $V_{1dGAO}$. This is reasonable since the difference between $V_{1dAAO}$ and $X_{1dGAO}$ lies not only in the volatility, but also in the payoff structure. Even so, the variance reduce ratio is about 2000($\approx 45^2$), which means the correlation coefficient between $V_{1dAAO}$ and $X_{1dGAO}$ is about 0.9998;

2) the efficiency of the control variate with the constant $\sigma_0$ (*i.e.* when $m = 2 - \frac{4\mu}{\sigma^2}$ at the last column) is still lower than others $m$, but that is not much;

3) the effect of $K$ is the same as that in the experiment 1;

4) there is some affect for different $\rho$, but not very clear.

**Experiment 3**. In this experiment, we report the efficiency of the control variate $X_{2dGAO}$ to the options $V_{2dGAO}$ and $V_{2dAAO}$ by showing $\widehat{R}$ in Table III. In such cases, we replace $X^j_{1dGAO}$ in (14) by

$$X^j_{2dGAO} = \left(e^{\frac{1}{N}\sum_{i=1}^{N}\log S^j(T_i)} - S^j(T)\right)^+;$$

also, $V_{1dGAO}$ in (4.4) should be replaced by

$$V^j_{2dGAO} = \left(e^{\frac{1}{N}\sum_{i=1}^{N}\log S^j_i} - S^j_T\right)^+$$

and by

$$V^j_{2dAAO} = \left(\frac{1}{N}\sum_{i=1}^{N} S^j_i - S^j_T\right)^+$$

respectively. We test several groups of the parameters $m$ and $\rho$.

The data in Table III show us that:

1) just like the results of the experiment 1 and the experiment 2, the efficiency of the control variate $X_{2dGAO}$ to the option $V_{2dAAO}$ is much lower than that to the option $V_{2dGAO}$;

2) the efficiency of the control variate with the constant $\sigma_0$ (*i.e.* when $m = 2 - \frac{4\mu}{\sigma^2}$ at the last column) is still lower than others $m$;

3) there is some affect for different $\rho$, and basically, the smaller $|\rho|$ is, the smaller $\widehat{R}$ is.

Next two experiments are about the continuous sampling Asian options.

**Experiment 4**. We report the efficiency of the control variate $X_{1cGAO}$ to the options $V_{1cGAO}$ and $V_{1cAAO}$ by showing $\widehat{R}$ in Table IV. In such cases, we replace $X^j_{1dGAO}$ in (14) by

$$\begin{aligned} X^j_{1cGAO} &= \left(e^{\frac{1}{T}\int_0^T \log S^j(t)dt} - K\right)^+ \\ &\approx \left(e^{\frac{1}{T}\sum_{k=1}^{n}\log S^j(t_k)\triangle t} - K\right)^+; \end{aligned}$$

also, $V_{1dGAO}$ in (16) is replaced by

$$\begin{aligned} V^j_{1cGAO} &= \left(e^{\frac{1}{T}\int_0^T \log S^j_t dt} - K\right)^+ \\ &\approx \left(e^{\frac{1}{T}\sum_{k=1}^{n}\log S^j_{t_k}\triangle t} - K\right)^+ \end{aligned}$$

and by

$$\begin{aligned} V^j_{1cAAO} &= \left(\frac{1}{T}\int_0^T S^j_t dt - K\right)^+ \\ &\approx \left(\frac{1}{T}\sum_{k=1}^{n} S^j_{t_k} \triangle t - K\right)^+ \end{aligned}$$

TABLE II
$X_{1dGAO}$ TO $V_{1dAAO}$

| | | m=-50 | m=0 | m=1 | m=2 | m=100 | m=2 $-\frac{4\mu}{\sigma^2}$ |
|---|---|---|---|---|---|---|---|
| | K=90 | 51.2470 | 52.4568 | 51.5750 | 51.5767 | 52.5609 | 44.0307 |
| $\rho=0.1$ | K=100 | 46.9175 | 45.6624 | 48.0509 | 47.3113 | 47.4938 | 38.6150 |
| | K=110 | 25.7714 | 26.3152 | 26.8835 | 25.5878 | 27.3461 | 22.3833 |
| | K=90 | 49.4656 | 50.3513 | 49.6439 | 46.7907 | 51.2375 | 42.4617 |
| $\rho=0.9$ | K=100 | 44.7613 | 44.2899 | 44.5601 | 45.2786 | 44.5297 | 38.1805 |
| | K=110 | 25.6553 | 27.1686 | 25.9317 | 26.5944 | 26.3661 | 22.1021 |

TABLE III
$X_{2dGAO}$ TO $V_{2dGAO}$ AND TO $V_{2dAAO}$

| to | | m=-50 | m=0 | m=1 | m=2 | m=100 | m=2 $-\frac{4\mu}{\sigma^2}$ |
|---|---|---|---|---|---|---|---|
| | $\rho=-0.9$ | 204.5360 | 196.3468 | 200.3387 | 201.6838 | 189.5257 | 143.5079 |
| | $\rho=-0.5$ | 172.5371 | 172.84889 | 165.8014 | 172.5469 | 168.5937 | 111.4999 |
| $V_{2dGAO}$ | $\rho=0$ | 164.5061 | 166.2659 | 164.9895 | 165.5065 | 164.4224 | 92.5353 |
| | $\rho=0.5$ | 166.4444 | 170.4592 | 171.1020 | 173.4244 | 174.8499 | 83.8239 |
| | $\rho=0.9$ | 194.5422 | 201.2479 | 200.7856 | 202.9127 | 212.1137 | 77.9394 |
| | $\rho=-0.9$ | 47.7312 | 49.8563 | 49.7212 | 52.1304 | 50.0864 | 49.4327 |
| | $\rho=-0.5$ | 48.2151 | 48.24820 | 49.1194 | 47.4271 | 49.5690 | 48.8880 |
| $V_{2dAAO}$ | $\rho=0$ | 48.3288 | 48.3719 | 48.3538 | 49.4576 | 46.8591 | 46.9129 |
| | $\rho=0.5$ | 49.0753 | 49.2757 | 49.6113 | 50.1640 | 48.3352 | 45.8860 |
| | $\rho=0.9$ | 48.7109 | 49.3867 | 50.8092 | 50.7244 | 48.4695 | 42.7990 |

TABLE IV
$X_{1cGAO}$ TO $V_{1cGAO}$ AND TO $V_{1cAAO}$

| to | | m=-50 | m=0 | m=1 | m=2 | m=100 | m=2 $-\frac{4\mu}{\sigma^2}$ |
|---|---|---|---|---|---|---|---|
| $V_{1cGAO}$ | K=90 | 458.5034 | 475.1777 | 481.9701 | 490.3650 | 493.4915 | 160.8114 |
| | K=100 | 411.9112 | 422.7765 | 409.4287 | 411.8081 | 433.6935 | 144.0880 |
| $V_{1cAAO}$ | K=90 | 48.8511 | 49.7269 | 48.7308 | 48.9061 | 49.2664 | 42.9104 |
| | K=100 | 45.3342 | 45.3444 | 45.6680 | 46.2055 | 43.8524 | 39.6856 |

respectively. We set the parameter $\rho = 0.9$, and test several groups of the parameters $m$ and $K$.

**Experiment 5**. We report the efficiency of the control variate $X_{2cGAO}$ to the options $V_{2cGAO}$ and $V_{2cAAO}$ by showing $\widehat{R}$ in Table V. In such cases, we replace $X^j_{1dGAO}$ in (14) by

$$X^j_{2cGAO} = \left(e^{\frac{1}{T}\int_0^T \log S^j(t)dt} - S^j(T)\right)^+$$
$$\approx \left(e^{\frac{1}{T}\sum_{k=1}^n \log S^j(t_k)\triangle t} - S^j(T)\right)^+;$$

also, $V_{1dGAO}$ in (16) should be replaced by

$$V^j_{2cGAO} = \left(e^{\frac{1}{T}\int_0^T \log S^j_t dt} - S^j_T\right)^+$$
$$\approx \left(e^{\frac{1}{T}\sum_{k=1}^n \log S^j_{t_k}\triangle t} - S^j_T\right)^+$$

and by

$$V^j_{2cAAO} = \left(\frac{1}{T}\int_0^T S^j_t dt - S^j_T\right)^+$$
$$\approx \left(\frac{1}{T}\sum_{k=1}^n S^j_{t_k}\triangle t - S^j_T\right)^+$$

respectively. We test several groups of the parameters $m$ and $\rho$.

The numerical results of two experiments above for the control variates to the continuous sampling Asian options show the similar efficiency like those to the discrete sampling Asian options.

### *B. Heston Model*

**Experiment 6**. In this experiment, we report the efficiency of the control variate $X_{1dGAO}$ to the option $V_{1dGAO}$ under the Heston model by showing $\widehat{R}$ in Table VI. In such case, we replace (15) by

$$Y^j_{t_{k+1}} = Y^j_{t_k} + (\alpha - \beta Y^j_{t_k})\triangle t + \sigma\sqrt{Y^j_{t_k}}Z^{2,j}_k.$$

We set the parameters by $n = 100,\ r = 0.1,\ \alpha = 0.25,\ \beta = 5,\ S_0 = 100,\ \sigma = 0.01,\ T = 1,\ Y_0 = \sigma_0^2 = 0.04,\ p = 10000,\ N = 10,\ K = 100$. We test several parameters $\rho$ and two kind forms of the control variates $X_{1dGAO}$; one is based on the deterministic volatility function (12), and the other is based on the constant volatility $Y(t) = Y_0$.

The numerical results show that our control variate also works well under the Heston model.

## IV. CONCLUSION

In this paper, we present a strategy to form a class of control variates for pricing Asian options under the stochastic volatility models. Our idea is using a deterministic volatility $\sigma(t)$ to replace the stochastic volatility $\sigma_t$ by choosing $\sigma(t)$ with the same order moment as that of $\sigma_t$ under the Hull-White model and the Heston model. Numerical experiments report that our control variates work quite well by showing the standard deviation reduction ratio $\widehat{R}$ and the efficiency is obviously better than one formed by the constant volatility $\sigma_0$, the initial value of the stochastic volatility. Our strategy can also be extend to other stochastic volatility models, as long as their order moment can be obtained in the closed-form. This is much easier than to calculate the distribution

TABLE V
$X_{2cGAO}$ TO $V_{2cGAO}$ AND TO $V_{2cAAO}$

| to | | m=-50 | m=0 | m=1 | m=2 | m=100 | m=2 $-\frac{4\mu}{\sigma^2}$ |
|---|---|---|---|---|---|---|---|
| | $\rho = -0.9$ | 204.5256 | 197.8080 | 201.6697 | 199.6693 | 186.1615 | 143.2022 |
| | $\rho = -0.5$ | 170.8696 | 175.7991 | 172.4018 | 175.2517 | 167.4408 | 112.2702 |
| $V_{2cGAO}$ | $\rho = 0$ | 160.9377 | 165.5935 | 162.0692 | 163.5159 | 160.9531 | 91.7656 |
| | $\rho = 0.5$ | 169.7676 | 175.2745 | 171.4413 | 169.7005 | 169.2797 | 80.7796 |
| | $\rho = 0.9$ | 193.8323 | 199.2215 | 202.1001 | 202.0420 | 210.4669 | 76.8739 |
| | $\rho = -0.9$ | 49.5630 | 48.8867 | 47.8118 | 48.9376 | 48.4962 | 50.3260 |
| | $\rho = -0.5$ | 47.2847 | 47.5158 | 48.5479 | 48.8120 | 48.7709 | 46.6673 |
| $V_{2cAAO}$ | $\rho = 0$ | 47.7823 | 49.2654 | 48.5468 | 48.9377 | 48.9082 | 45.3849 |
| | $\rho = 0.5$ | 49.3646 | 48.9727 | 49.1666 | 47.9563 | 48.1193 | 44.4376 |
| | $\rho = 0.9$ | 48.8006 | 48.9003 | 48.9083 | 48.9110 | 49.0476 | 41.2342 |

TABLE VI
$X_{1dGAO}$ CONTROL $V_{1dGAO}$ BASED ON TWO $Y(t)$

| $Y(t)$ | $\rho = -0.9$ | $\rho = -0.5$ | $\rho = 0$ | $\rho = 0.5$ | $\rho = 0.9$ |
|---|---|---|---|---|---|
| $e^{-\beta t}Y_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ | 148.5916 | 146.1014 | 136.5857 | 141.0351 | 151.1588 |
| $Y_0$ | 24.8804 | 24.9325 | 23.8939 | 22.9603 | 23.0285 |

function of the stochastic volatility such as in the Heston model. In addition, our strategy can be extend to pricing other financial derivatives under stochastic volatility models.

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