On Improving the Semilocal Convergence of Newton-Type Iterative Method for Ill-posed Hammerstein Type Operator Equations

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***Abstract*—George and Pareth(2012), presented a quartically convergent Two Step Newton type method for approximately solving an ill-posed operator equation in the finite dimensional setting of Hilbert spaces. In this paper we use the analogous Two Step Newton type method to approximate a solution of ill-posed Hammerstein type operator equation.**

***Index Terms*—Hammerstein operators, Quartic convergence, Newton Tikhonov method, monotone operator, ill-posed problems, adaptive method.**

I. Introduction

This paper deals with approximating a stable solution of ill-posed Hammerstein type operator equations. An equation of the form

$$KF(x) = f \tag{1}$$

where $F : D(F) \subseteq X \to X$ is nonlinear and $K : X \to Y$ is a bounded linear operator is called a (nonlinear) Hammerstein equation ([5], [8]). Here X and Y are Hilbert spaces with inner product $\langle .,. \rangle$ and norm $\|.\|$ respectively.

Equation (1) is ill-posed in the sense that its solution does not depend continuously on given data. It is assumed throughout that $f^\delta \in Y$ are the available noisy data with

$$\|f - f^\delta\| \leq \delta$$

and F possesses a uniformly bounded Fréchet derivative for each $x \in D(F)$, i.e.,

$$\|F'(x)\| \leq M, \qquad x \in D(F)$$

for some M(Here and below $F'(.)$ denotes the Fréchet derivative of F). The method of approximately solving an ill-posed equation is called regularization method. For various regularization techniques one can see [2], [3], [12] and [17], [6]. Observe that the solution x of (1) with f^δ in place of f can be obtained by first solving

$$Kz = f^\delta \tag{2}$$

for z and then solving the non-linear problem

$$F(x) = z. \tag{3}$$

The above formulation has been considered by authors in [5], [7] and [8]. The main purpose of the above formulation is that:

(a) We solve (2) and (3) separately, to obtain an approximate solution for (1). Here one can use any regularization method for linear ill-posed equation for solving (2) and any regularization method for solving (3). In fact in this paper we consider Tikhonov regularization for approximately solving (2) and we consider a modified two step Newton method for solving (3).

(b) The regularization parameter α is chosen according to the adaptive method considered by Pereverzev and Schock in [16] for the linear ill-posed operator equation (2) and the same parameter α is used for solving the non-linear operator equation (3), so the choice of the regularization parameter is not depending on the non-linear operator F.

In [5], George studied an iterative Newton-Tikhonov regularization (NTR) method for approximating (1), where z in (2) is approximated with z_α^δ;

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*f^\delta, \qquad \alpha > 0, \qquad \delta > 0,$$

and then solve (3) using the Newton type iteration

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta)$$

where $x_{0,\alpha}^\delta := x_0$. Here and in the following x_0 is the initial approximation to the solution $\hat{x}$ of (1). Local linear convergence was obtained in [5].

In [7], George and Kunhanandan used the iteration

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta)$$

where $x_{0,\alpha}^\delta := x_0$ and

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(f^\delta - KF(x_0)) + F(x_0) \tag{4}$$

for approximately solving (1). Local quadratic convergence was established in [7].

Motivated by Two Step Directional Newton Method of Argyros and Hilout (see [1], [9]) we propose, a Two Step Newton-Tikhonov Method (TSNTM) in this paper for solving (1). We consider two regularity classes of the operator F. In the first case it is assumed that $F'(u)^{-1}$ exists and is a bounded operator for all $u \in B_r(x_0)$ ($B_r(x_0)$ stands for the ball of radius r with center x_0); and in the second case it is assumed that F is a monotone operator and $F'(u)^{-1}$ does not exist.

Recall [15], [21], that an operator F is said to be monotone operator if $\langle F(x) - F(y), x - y \rangle \geq 0 \quad \forall \; x, y \in D(F)$.

In this paper we provide a semilocal convergence analysis of TSNTM for ill-posed Hammerstein operator equations with the advantage of quartic convergence over the work in [5] and [7].

As in [7] and [8], a solution $\hat{x}$ of (1) is called an x_0-minimum norm solution if it satisfies

$$\|F(\hat{x}) - F(x_0)\| \quad := \quad \min\{\|F(x) - F(x_0)\| : KF(x) = f, x \in D(F)\}. \tag{5}$$

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We assume throughout that the solution $\hat{x}$ of (1) satisfies (5).

The paper is organized as follows: In Section II, we give the preliminaries and adaptive scheme for choosing the regularization parameter α for Tikhonov regularization of (2). The proposed method and the error estimates are given in Section III. Section IV deals with the algorithm and a numerical example is given in Section V to confirm the efficiency of our approach. Finally we conclude the paper in Section VI.

II. Preliminaries

This section deals with Tikhonov regularized solution z_α^δ of (2) and (an a priori and an a posteriori) error estimate for $\|F(\hat{x}) - z_\alpha^\delta\|$. The following assumption is used to obtain the error estimate.

Assumption 2.1: There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|K\|^2$ satisfying;

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$,
- $$\sup_{\lambda > 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha) \qquad \forall \lambda \in (0, a],$$
  and
- there exists $v \in X, \|v\| \leq 1$ such that
  $$F(\hat{x}) - F(x_0) = \varphi(K^*K)v.$$

THEOREM 2.2: (cf.[7], section 4) Let z_α^δ be as in (4) and Assumption 2.1 holds. Then

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{6}$$

A. A priori choice of the parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (6) is of optimal order for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$ and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$$

(Here φ^{-1} denotes the inverse of the function φ). So the relation (6) leads to $\|F(\hat{x}) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta)$.

B. An adaptive choice of the parameter

The above apriori choice of the parameter cannot be used in practice as the smoothness condition of the unknown solution $\hat{x}$ reflected in φ is generally not known. So, in practice we propose to choose the parameter α according to the balancing principle established by Pereverzev and Shock [16] for solving ill-posed problems. Let

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}$$

be the set of possible values of the parameter α.

The selection of numerical value k for the parameter α according to the balancing principle is performed using the following rule:

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} < N. \tag{7}$$

Let

$$k = \max\{i : \alpha_i \in D_N^+\} \tag{8}$$

where $D_N^+ = \{\alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\}$.

We will be using the following theorem from [7] for our error analysis.

THEOREM 2.3: (cf. [7], Theorem 4.3) Let l be as in (7), k be as in (8) and $z_{\alpha_k}^\delta$ be as in (4) with $\alpha = \alpha_k$. Then $l \leq k$ and

$$\|F(\hat{x}) - z_{\alpha_k}^\delta\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

III. Semilocal Convergence of TSNTM

In this paper we simply present the results without proofs. We refer though the reader to [9], [10] and [11] for the analogous proofs.

A. Case 1: $F'(.)$ is boundedly invertible in $B_r(x_0)$

Let $\|F'(u)^{-1}\| \leq \beta$, $\forall u \in B_r(x_0)$ and for some $\beta > 0$. In this case the ill-posedness of (1) is essentially due to the nonclosedness of the range of the linear operator K (see [17], page 26). Let $B_r(x)$ denote the ball of radius r centered at $x \in X$.

For an initial guess $x_0 \in X$ the TSNTM is defined as;

$$y_{n,\alpha_k}^\delta = x_{n,\alpha_k}^\delta - F'(x_{n,\alpha_k}^\delta)^{-1}(F(x_{n,\alpha_k}^\delta) - z_{\alpha_k}^\delta), \tag{9}$$

$$x_{n+1,\alpha_k}^\delta = y_{n,\alpha_k}^\delta - F'(y_{n,\alpha_k}^\delta)^{-1}(F(y_{n,\alpha_k}^\delta) - z_{\alpha_k}^\delta). \tag{10}$$

In order to establish the convergence of TSNTM and to obtain the error estimate $\|x_{\alpha_k}^\delta - \hat{x}\|$, we use the following

Assumption 3.1: (cf.[20], Assumption 3 (A3)) There exist a constant $k_0 \geq 0$ such that for every $x, u \in B_r(x_0) \cup B_r(\hat{x}) \subseteq D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$.

Let

$$e_{n,\alpha_k}^\delta := \|y_{n,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\|, \qquad \forall n \geq 0 \tag{11}$$

and for $0 < k_0 \leq 1$, let $g : (0, 1) \to (0, 1)$ be the function defined by

$$g(t) = \frac{27k_0^3}{8}t^3 \qquad \forall t \in (0, 1). \tag{12}$$

For convenience we will use the notation x_n, y_n and e_n for x_{n,α_k}^δ, y_{n,α_k}^δ and e_{n,α_k}^δ respectively.

Hereafter we assume that $\delta \in (0, \delta_0]$ where $\delta_0 < \frac{\sqrt{\alpha_0}}{\beta}$. Let $\|\hat{x} - x_0\| \leq \rho$,

$$\rho < \frac{1}{M}(\frac{1}{\beta} - \frac{\delta_0}{\sqrt{\alpha_0}})$$

and

$$\gamma_\rho := \beta[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}].$$

THEOREM 3.2: Let e_n and $g(e_n)$ be as in equation (11) and (12) respectively, x_n and y_n be as in (10) and (9) respectively with $\delta \in (0, \delta_0]$. Then by Assumption 3.1 and Theorem 2.3, the following hold:

(a) $\|x_n - y_{n-1}\| \leq \frac{3k_0e_{n-1}}{2}\|y_{n-1} - x_{n-1}\|$;
(b) $\|x_n - x_{n-1}\| \leq (1 + \frac{3k_0e_{n-1}}{2})\|y_{n-1} - x_{n-1}\|$;
(c) $\|y_n - x_n\| \leq g(e_{n-1})\|y_{n-1} - x_{n-1}\|$;
(d) $g(e_n) \leq g(\gamma_\rho)^{4^n}, \qquad \forall n \geq 0$;
(e) $e_n \leq g(\gamma_\rho)^{(4^n-1)/2}\gamma_\rho \qquad \forall n \geq 0$.

THEOREM 3.3: Let $r = (\frac{1}{1-g(\gamma_\rho)} + \frac{3k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2})\gamma_\rho$ and let the hypothesis of Theorem 3.2 holds. Then $x_n, y_n \in B_r(x_0)$, for all $n \geq 0$.

The main result of subsection A of this Section is the following

THEOREM 3.4: Let y_n and x_n be as in (9) and (10) respectively, Assumptions of Theorem 3.3 hold and let $0 < g(\gamma_\rho) < 1$. Then (x_n) is a Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_{\alpha_k} \in \overline{B_r(x_0)}$. Further $F(x^\delta_{\alpha_k}) = z^\delta_{\alpha_k}$ and

$$\|x_n - x^\delta_{\alpha_k}\| \leq Ce^{-\gamma 4^n}$$

where $C = (\frac{1}{1-g(\gamma_\rho)^4} + \frac{3k_0\gamma_\rho}{2}\frac{1}{1-(g(\gamma_\rho)^2)^4} g(\gamma_\rho)^{4^n})\gamma_\rho$ and $\gamma = -\log g(\gamma_\rho)$.

REMARK 3.5: Note that $0 < g(\gamma_\rho) < 1$ and hence $\gamma > 0$. Hence the sequence (x_n) converges quartically to $x^\delta_{\alpha_k}$.

REMARK 3.6: Recall that a sequence (x_n) in X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals c_1, c_2, such that for all $n \in N$

$$\|x_n - x^*\| \leq c_1 e^{-c_2 p^n}.$$

If the sequence (x_n) has the property that $\|x_n - x^*\| \leq c_1 q^n$, $0 < q < 1$, then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate see Kelley [13]. Hereafter we assume that

$$\rho \leq r < \frac{1}{k_0}$$

REMARK 3.7: Note that the above assumption is satisfied if

$$k_0 \leq \min\{1, \frac{1-g(\gamma_\rho)^2}{3\gamma_\rho}[\frac{-1}{1-g(\gamma_\rho)} + \sqrt{\frac{1}{(1-g(\gamma_\rho))^2} + \frac{6}{1-g(\gamma_\rho)^2}}]\}.$$

THEOREM 3.8: Suppose that Assumption 2.1 and 3.1 hold. If in addition $k_0 r < 1$, then

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq \frac{\beta}{1-k_0 r}\|F(\hat{x}) - z^\delta_{\alpha_k}\|.$$

THEOREM 3.9: Let x_n be as in (10), assumptions in Theorem 3.4 and Theorem 3.8 hold. Then

$$\|\hat{x} - x_n\| \leq Ce^{-\gamma 4^n} + \frac{\beta}{1-k_0 r}\|F(\hat{x}) - z^\delta_{\alpha_k}\|$$

where C and γ are as in Theorem 3.4.

Now since $l \leq k$ and $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$ we have

$$\frac{\delta}{\sqrt{\alpha_k}} \leq \frac{\delta}{\sqrt{\alpha_l}} \leq \mu\frac{\delta}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This leads to the following theorem,

THEOREM 3.10: Let x_n be as in (10) and the assumptions of Theorems 2.3 and 3.9 hold. Let

$$n_k := \min\{n : e^{-\gamma 4^n} \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x} - x_{n_k}\| = O(\psi^{-1}(\delta)).$$

B. Case 2: F is a monotone operator and $F'(.)$ is non-invertible.

Let X be a real Hilbert space. In this situation, the ill-posedness of (1) is due to the ill-posedness of F as well as the nonclosedness of the range of the linear operator K.

For an initial guess $x_0 \in X$, $0 < c < \alpha_k$ and for $R(x) := F'(x) + \frac{\alpha_k}{c}I$, the TSNTM in this case is defined as:

$$\tilde{y}^\delta_{n,\alpha_k} = \tilde{x}^\delta_{n,\alpha_k} - R(\tilde{x}^\delta_{n,\alpha_k})^{-1}[F(\tilde{x}^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}^\delta_{n,\alpha_k} - x_0)] \quad (13)$$

and

$$\tilde{x}^\delta_{n+1,\alpha_k} = \tilde{y}^\delta_{n,\alpha_k} - R(\tilde{y}^\delta_{n,\alpha_k})^{-1}[F(\tilde{y}^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{y}^\delta_{n,\alpha_k} - x_0)]. \quad (14)$$

where $\tilde{x}_{0,\alpha_k} := x_0$. Note that with the above notation

$$\|R(x)^{-1}F'(x)\| \leq 1.$$

First we consider $\tilde{x}^\delta_{n,\alpha_k}$ defined in (14) to approximate the zero x^δ_{c,α_k} of $F(x) + \frac{\alpha_k}{c}(x - x_0) = z^\delta_{\alpha_k}$ and then we show that x^δ_{c,α_k} is an approximation to the solution $\hat{x}$ of (1).

Let

$$\tilde{e}^\delta_{n,\alpha_k} := \|\tilde{y}^\delta_{n,\alpha_k} - \tilde{x}^\delta_{n,\alpha_k}\|, \qquad \forall n \geq 0. \quad (15)$$

Here also for convenience we use the notation $\tilde{x}_n$, $\tilde{y}_n$ and $\tilde{e}_n$ for $\tilde{x}^\delta_{n,\alpha_k}$, $\tilde{y}^\delta_{n,\alpha_k}$ and $\tilde{e}^\delta_{n,\alpha_k}$ respectively. Let Assumption 3.1 holds with $\tilde{r}$ in place of r and $\rho \leq \tilde{r} < \frac{1}{k_0}$. Let

$$\rho \leq \frac{1}{M}(1 - \frac{\delta_0}{\sqrt{\alpha_0}})$$

with $\delta_0 < \sqrt{\alpha_0}$ and

$$\tilde{\gamma}_\rho := M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}.$$

THEOREM 3.11: Let $\tilde{e}_n$ and g be as in equation (15) and (12) respectively, $\tilde{x}_n$ and $\tilde{y}_n$ be as in (14) and (13) respectively with $\delta \in (0, \delta_0]$ and $\alpha \in D_N$. If Assumption 3.1 and Theorem 2.3 are fulfilled, then the following hold:

(a) $\|\tilde{x}_n - \tilde{y}_{n-1}\| \leq \frac{3k_0\tilde{e}_{n-1}}{2}\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;
(b) $\|\tilde{x}_n - \tilde{x}_{n-1}\| \leq (1 + \frac{3k_0\tilde{e}_{n-1}}{2})\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;
(c) $\|\tilde{y}_n - \tilde{x}_n\| \leq g(\tilde{e}_{n-1})\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|$;
(d) $g(\tilde{e}_n) \leq g(\tilde{\gamma}_\rho)^{4^n}$, $\quad \forall n \geq 0$;
(e) $\tilde{e}_n \leq g(\tilde{\gamma}_\rho)^{(4^n-1)/2}\tilde{\gamma}_\rho \quad \forall n \geq 0$.

THEOREM 3.12: Let $\tilde{r} = (\frac{1}{1-g(\tilde{\gamma}_\rho)} + \frac{3k_0}{2}\frac{\tilde{\gamma}_\rho}{1-g(\tilde{\gamma}_\rho)^2})\tilde{\gamma}_\rho$ and the assumptions of Theorem 3.11 hold. Then $\tilde{x}_n, \tilde{y}_n \in B_{\tilde{r}}(x_0)$, for all $n \geq 0$.

THEOREM 3.13: Let $\tilde{y}_n$ and $\tilde{x}_n$ be as in (13) and (14) respectively and assumptions of Theorem 3.12 hold. Then $(\tilde{x}_n)$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and converges to $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$. Further $F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$ and

$$\|\tilde{x}_n - x^\delta_{c,\alpha_k}\| \leq \tilde{C}e^{-\gamma_1 4^n}$$

where $\tilde{C} = (\frac{1}{1-g(\tilde{\gamma}_\rho)^4} + \frac{3k_0\tilde{\gamma}_\rho}{2}\frac{1}{1-(g(\tilde{\gamma}_\rho)^2)^4} g(\tilde{\gamma}_\rho)^{4^n})\tilde{\gamma}_\rho$ and $\gamma_1 = -\log g(\tilde{\gamma}_\rho)$.

In order to obtain the error estimate $\|\hat{x} - x^\delta_{c,\alpha_k}\|$, we require the following assumption in addition to the previous assumptions of Section II and subsection A of Section III.

Assumption 3.14: There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \to (0, \infty)$ with $b \geq \|F'(x_0)\|$ satisfying;

- $\lim_{\lambda\to 0}\varphi_1(\lambda)=0,$
- $$\sup_{\lambda>0}\frac{\alpha\varphi_1(\lambda)}{\lambda+\alpha}\leq\varphi_1(\alpha)\qquad\forall\lambda\in(0,b],$$
  and
- there exists $v\in X$ with $\|v\|\leq 1$ (cf.[14]) such that
$$x_0-\hat{x}=\varphi_1(F'(x_0))v.$$
- for each $x\in B_{\tilde{r}}(x_0)$ there exists a bounded linear operator $G(x,x_0)$ (cf.[18]) such that
$$F'(x)=F'(x_0))G(x,x_0)$$
  with $\|G(x,x_0)\|\leq k_1$.

Assume that $k_1<\frac{1-k_0\tilde{r}}{1-c}$ and for the sake of simplicity assume that $\varphi_1(\alpha)\leq\varphi(\alpha)$ for $\alpha>0$.

THEOREM 3.15: (cf. [11], Theorem 3.14) Suppose x^δ_{c,α_k} is the solution of

$$F(x)+\frac{\alpha_k}{c}(x-x_0)=z^\delta_{\alpha_k}$$

and Assumptions 3.1 and 3.14 holds. Then

$$\|\hat{x}-x^\delta_{c,\alpha_k}\|\leq\frac{\varphi_1(\alpha_k)+(2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1-k_0\tilde{r}}.$$

Proof. Note that $c(F(x^\delta_{c,\alpha_k})-z^\delta_{\alpha_k})+\alpha_k(x^\delta_{c,\alpha_k}-x_0)=0$, so

$$\begin{aligned}\|x^\delta_{c,\alpha_k}-\hat{x}\| &\leq \|\alpha_k(F'(x_0)+\alpha_kI)^{-1}(x_0-\hat{x})\|\\ &\quad+\|(F'(x_0)+\alpha_kI)^{-1}c(F(\hat{x})-z^\delta_{\alpha_k})\|\\ &\quad+\|(F'(x_0)+\alpha_kI)^{-1}[F'(x_0)(x^\delta_{c,\alpha_k}-\hat{x})\\ &\quad-c(F(x^\delta_{c,\alpha_k})-F(\hat{x}))]\|\\ &\leq \|\alpha_k(F'(x_0)+\alpha_kI)^{-1}(x_0-\hat{x})\| \qquad (16)\\ &\quad+\|F(\hat{x})-z^\delta_{\alpha_k}\|+\Gamma\end{aligned}$$

where $\Gamma:=\|(F'(x_0)+\alpha_kI)^{-1}\int_0^1[F'(x_0)-cF'(\hat{x}+t(x^\delta_{c,\alpha_k}-\hat{x})](x^\delta_{c,\alpha_k}-\hat{x})dt\|$. So by Assumption 3.14, we obtain

$$\begin{aligned}\Gamma &\leq \|(F'(x_0)+\alpha_kI)^{-1}s_1\|\\ &\quad+(1-c)\|(F'(x_0)+\alpha_kI)^{-1}s_2\|\\ &\leq k_0\tilde{r}\|x^\delta_{c,\alpha_k}-\hat{x}\|+(1-c)k_1\|x^\delta_{c,\alpha_k}-\hat{x}\| \qquad (17)\end{aligned}$$

where

$$s_1:=\int_0^1[F'(x_0)-F'(\hat{x}+t(x^\delta_{c,\alpha_k}-\hat{x}))](x^\delta_{c,\alpha_k}-\hat{x})dt,$$

$$s_2:=F'(x_0)\int_0^1 G(\hat{x}+t(x^\delta_{c,\alpha_k}-\hat{x}),x_0)(x^\delta_{c,\alpha_k}-\hat{x})dt$$

and hence by (16) and (17) we have

$$\begin{aligned}\|x^\delta_{c,\alpha_k}-\hat{x}\| &\leq \frac{\tau_x}{1-(1-c)k_1-k_0\tilde{r}}\\ &\leq \frac{\varphi_1(\alpha_k)+(2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1-k_0\tilde{r}},\end{aligned}$$

where

$$\tau_x:=\|\alpha_k(F'(x_0)+\alpha_kI)^{-1}(x_0-\hat{x})\|+\|F(\hat{x})-z^\delta_{\alpha_k}\|.$$

This completes the proof of the theorem.

The following Theorem is a consequence of Theorem 3.13 and Theorem 3.15.

THEOREM 3.16: Let $\tilde{x}_n$ be defined as in (14). If assumptions of the Theorem 3.13 and 3.15 are fulfilled, then

$$\|\hat{x}-\tilde{x}_n\|\leq\tilde{C}e^{-\gamma_1 4^n}+O(\psi^{-1}(\delta))$$

where $\tilde{C}$ and γ_1 are as in Theorem 3.13.

THEOREM 3.17: Let $\tilde{x}_n$ be defined as in (14) and assumptions of Theorem 2.3 and 3.16 hold. Let

$$n_k:=\min\{n:e^{-\gamma_1 4^n}\leq\frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x}-\tilde{x}_{n_k}\|=O(\psi^{-1}(\delta)).$$

IV. Algorithm

Note that for $i,j\in\{0,1,2,\cdots,N\}$

$$\begin{aligned}z^\delta_{\alpha_i}-z^\delta_{\alpha_j} &= (\alpha_j-\alpha_i)(K^*K+\alpha_jI)^{-1}\\ &\quad\times(K^*K+\alpha_iI)^{-1}[K^*(f^\delta-KF(x_0))].\end{aligned}$$

Therefore the balancing principle algorithm associated with the choice of the parameter specified in Section II involves the following steps.

- $\alpha_0=\mu^2\delta^2,\mu>\max\{1,\beta\}$ for Case 1 and $\mu>1$ for Case 2.
- $\alpha_i=\mu^{2i}\alpha_0$;
- solve for w_i : $(K^*K+\alpha_iI)w_i=K^*(f^\delta-KF(x_0))$;
- solve for $j<i$, z_{ij} : $(K^*K+\alpha_jI)z_{ij}=(\alpha_j-\alpha_i)w_i$;
- if $\|z_{ij}\|>\frac{4}{\mu^{j+1}}$, then take $k=i-1$;
- otherwise, repeat with $i+1$ in place of i.
- choose $n_k=\min\{n:e^{-\gamma 4^n}\leq\frac{\delta}{\sqrt{\alpha_k}}\}$ for Case 1 and $n_k=\min\{n:e^{-\gamma_1 4^n}\leq\frac{\delta}{\sqrt{\alpha_k}}\}$ in Case 2,
- solve x_{n_k} using the iteration (10) or $\tilde{x}_{n_k}$ using the iteration (14).

V. Numerical Examples

In this section we give an example for Case 2 (subsection B of Section III) for illustrating the algorithm considered in the above section. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with dim $V_n=n+1$. Precisely we choose V_n as the space of linear splines in a uniform grid of $n+1$ points in $[0,1]$.

EXAMPLE 5.1: We consider the same example of nonlinear integral operator as in [20], section 4.3. To illustrate the method for Case 2, we consider the operator $KF:L^2(0,1)\longrightarrow L^2(0,1)$ where $K:L^2(0,1)\longrightarrow L^2(0,1)$ defined by

$$K(x)(t)=\int_0^1 k(t,s)x(s)ds$$

and $F:D(F)\subseteq H^1(0,1)\longrightarrow L^2(0,1)$ defined by

$$F(u):=\int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s)=\begin{cases}(1-t)s, 0\leq s\leq t\leq 1\\ (1-s)t, 0\leq t\leq s\leq 1\end{cases}.$$

Then for all $x(t),y(t):x(t)>y(t)$:

$$\begin{aligned}\langle F(x)-F(y),x-y\rangle &= \int_0^1\left[\int_0^1 k(t,s)(x^3-y^3)(s)ds\right]\\ &\quad(x-y)(t)dt\geq 0.\end{aligned}$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

So for any $u \in B_r(x_0), {x_0}^2(s) \geq k_3 > 0, \forall s \in (0,1)$, we have

$$F'(u)w = F'(x_0)G(u,x_0)w,$$

where $G(u,x_0) = (\frac{u}{x_0})^2$.

Further observe that

$$\begin{aligned}[F'(v) - F'(u)]w(s) &= 3\int_0^1 k(t,s)(v^2(s) - u^2(s)) \\ &\quad \times w(s)ds \\ &:= F'(u)\Phi(u,v,w),\end{aligned}$$

where $\Phi(u,v,w) = [\frac{v^2}{u^2} - 1]w$.

Thus Φ satisfies the Assumption 3.1 (cf. [19], Example 2.7).

In our computation, we take

$$\begin{aligned}f(t) &= (\frac{1}{18\pi^2})(1-t)(14t - 7 + cos^3(\pi t) \\ &\quad +6cos(\pi t))t^2 - (\frac{1}{18\pi^2})t(14t - 7 + cos^3(\pi t) \\ &\quad +6cos(\pi t))(1-t^2) + (\frac{1}{9\pi^2})t(1-t)(14t - 7 \\ &\quad +cos^3(\pi t) + 6cos(\pi t))\end{aligned}$$

and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = cos\pi t.$$

We use

$$\begin{aligned}x_0(t) &= cos(\pi t) + 3[\frac{-1}{4\pi^2}(1 - t + 2\pi t^2 cos(\pi t) \\ &\quad \times sin(\pi t) + \pi^2 t^3 + tcos^2(\pi t) - 2\pi tcos(\pi t) \\ &\quad \times sin(\pi t) - \pi^2 t^2 - cos^2(\pi t)) + \frac{1}{4\pi^2}t \\ &\quad \times(-2cos(\pi t)sin(\pi t)\pi - 2\pi^2 t + 2\pi tcos(\pi t) \\ &\quad \times sin(\pi t) + \pi^2 t^2 + cos^2(\pi t) + \pi^2 - cos^2(\pi t))]\end{aligned}$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

$$x_0 - \hat{x} = \varphi_1(F'(x_0))1$$

where $\varphi_1(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O(\delta^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.3)\delta^2, \mu = 1.3,\ \delta = 0.1 = c,\ \rho = 0.19,\ \tilde{\gamma}_\rho = 0.8173$ and $g(\tilde{\gamma}_\rho) = 0.54$ approximately. For all n the number of iteration $n_k = 1$. The results of the computation are presented in Table 1. The plots of the exact and the approximate solution obtained are given in Fig.1 to Fig.8.

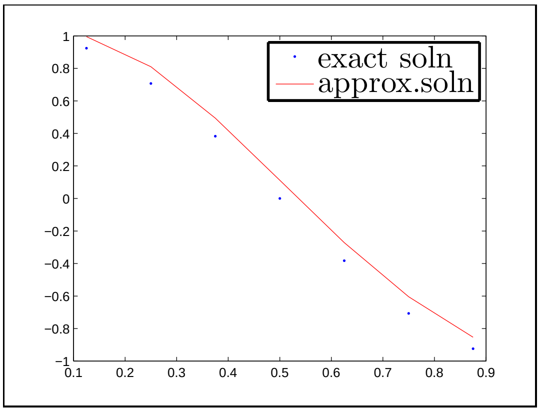

Fig. 1. Curves of the exact and approximate solutions for n=8

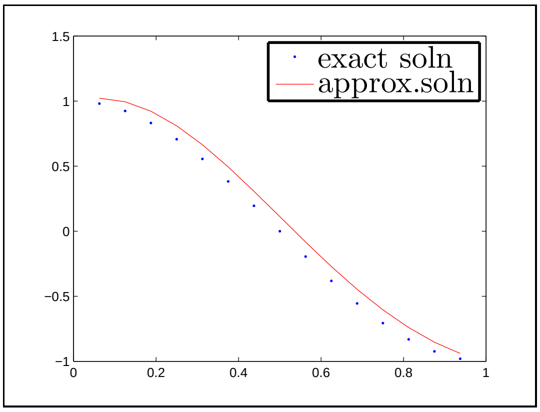

Fig. 2. Curves of the exact and approximate solutions for n=16

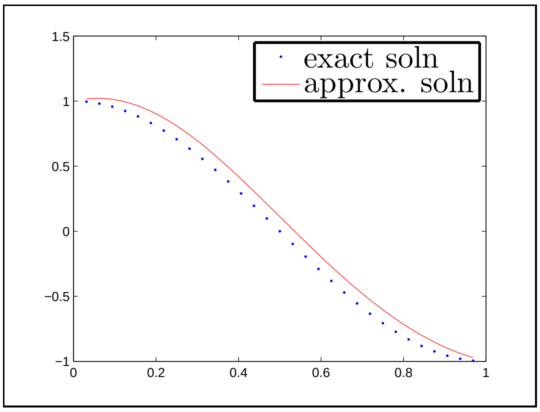

Fig. 3. Curves of the exact and approximate solutions for n=32

VI. Conclusion

A Two Step Newton-Tikhonov Methods (TSNTM) for obtaining an approximate solution for a nonlinear ill-posed Hammerstein type operator equation $KF(x) = f$, with the available noisy data f^δ in place of the exact data f has been considered. Two implementations are considered, in the first case it is assumed that the Fréchet derivative $F'(.)$ of the nonlinear operator F has a bounded inverse in a neighbourhood of the initial guess x_0 of the actual solution $\hat{x}$. And in the second case it is assumed that the nonlinear operator F is monotone but $F'(.)$ is non-invertible. The derived error estimate using an a priori and adaptive scheme([16]) in both situations are of optimal order with respect to a general source condition. Also in both the cases we obtained local quartic convergence compared to the local linear convergence obtained by NTR method considered in [5] and local quadratic convergence obtained in [7].

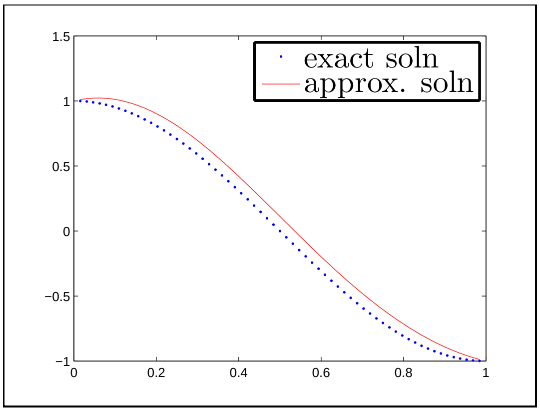

Fig. 4. Curves of the exact and approximate solutions for n=64

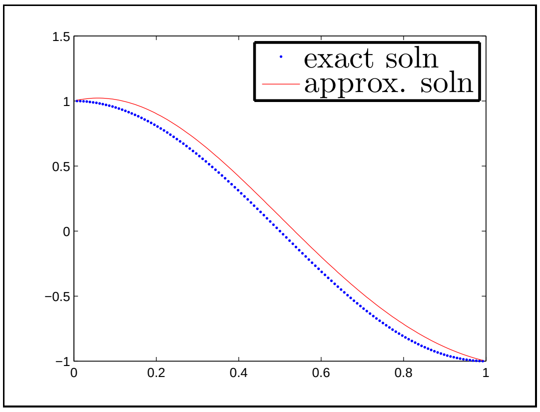

Fig. 5. Curves of the exact(lower curve) and approximate(upper curve) solutions for n=128

TABLE I
ITERATIONS AND CORRESPONDING ERROR ESTIMATES

n	k	δ	α	$\|\tilde{x}_k - \hat{x}\|$	$\frac{\|\tilde{x}_k - \hat{x}\|}{(\delta)^{1/2}}$
8	4	0.1016	0.1094	0.3652	1.1458
16	4	0.1004	0.1069	0.2664	0.8408
32	4	0.1001	0.1063	0.1994	0.6303
64	4	0.1000	0.1061	0.1554	0.4914
128	4	0.1000	0.1061	0.1278	0.4042
256	4	0.1000	0.1060	0.1115	0.3526
512	4	0.1000	0.1060	0.1024	0.3238
1024	4	0.1000	0.1060	0.0975	0.3083

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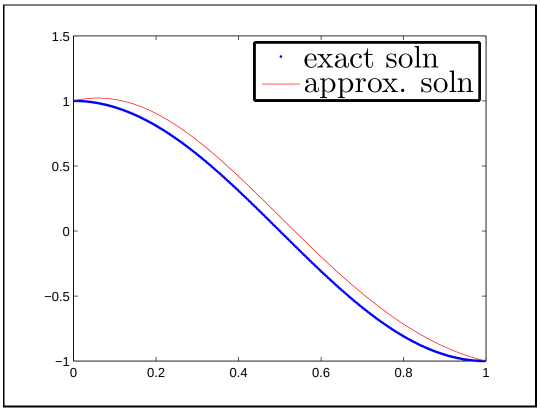

Fig. 6. Curves of the exact(lower curve) and approximate(upper curve) solutions for n=256

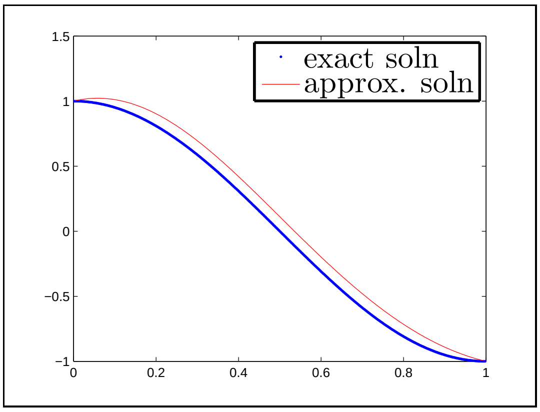

Fig. 7. Curves of the exact(lower curve) and approximate(upper curve) solutions for n=512

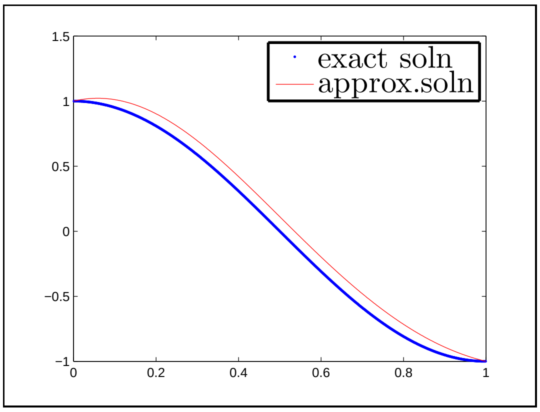

Fig. 8. Curves of the exact(lower curve) and approximate(upper curve) solutions for n=1024

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