IAENG International Journal of Applied Mathematics, 43:2, IJAM_43_2_05

The Canonical Form of Multi-player Combinatorial Games

Alessandro Cincotti

***Abstract*—In combinatorial games, few results are known about the overall structure of multi-player games. In particular, multi-player games born by day d form a completely distributive lattices with respect to every partial order relation under an arbitrary coalition of players. In this paper, we introduce the canonical form of a multi-player game and we redefine the upper and lower bounds of the lattices of multi-player games born by day d.**

***Index Terms*—canonical form, combinatorial game, multi-player game.**

I. Introduction

COMBINATORIAL game theory [2][7] is a branch of mathematics devoted to studying the optimal strategy in two-player perfect information games under *normal play* which declares as loser the first player unable to make a legal move. Such a theory is based on a straightforward and intuitive recursive definition of games, which yields a quite rich algebraic structure. Games can be added and subtracted in a natural way, forming a commutative group with a partial order.

The ordered structure of the set of combinatorial games lasting at most n moves, also known as the games born by day n was investigated in [3], where it was proved that:

Theorem 1 (Calistrate et al.): The set of games born by day n is a distributive lattice.

Subsequently, [8], [16] and [1] extended and refined this result.

When combinatorial game theory is generalized to n-player games, the problem of coalition arises. A coalition makes it hard to have a simple game value in any additive algebraic structure. To circumvent the coalition problem in n-player games, different approaches have been proposed [10][14][11][9] with various restrictive assumptions about the rationality of one's opponents and the formation and behavior of coalitions. Alternatively, Propp [12] and Cincotti [4] adopt in their work an agnostic attitude toward such issues, and seek only to understand in what circumstances one player has a winning strategy against the combined forces of the others.

In general, the algebraic structure of n-player games strongly depends on the rules of the games and, in particular, the winning condition. In this paper, we will consider the following scenario. Players take turns making legal moves in a cyclic fashion:

$$(i,\ (i+1) \bmod n,\ \dots,\ (i+n-1) \bmod n,\ i,\ (i+1) \bmod n,\ \dots)$$

where player i, $i \in \{1, \dots, n\}$ makes the first move. A group of players C will form the first coalition, the other players will form the second coalition. The coalition of the first player that is unable to make a legal move, loses.

The author is with the School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa 923-1292 Japan (phone: +81-761-51-1293; e-mail: cincotti@jaist.ac.jp).

In two previous works [5][6], it was proved that multi-player games born by day d form a completely distributive lattice with respect to every partial order relation $\leq_C$, where C is an arbitrary coalition of players. In this work, we introduce the canonical form of a multi-player game and we redefine the upper and lower bound of $\mathsf{G_n}[d]$.

The article is organized as follows. In Section 2, we recall the basic definitions concerning multi-player games. In Section 3, we recall the main results about the mathematical structure of multi-player games born by day d. Section 4 shows how to simplify a multi-player game removing the dominated options, i.e., the moves that are not necessary to take into account during the analysis of the game. Moreover, the canonical form of a multi-player game is introduced. Section 5 redefines the upper and lower bound of $\mathsf{G_n}[d]$.

II. Multi-player Games

For the sake of self-containment, we recall in this section the main definitions concerning multi-player games.

Definition 1: We define n-player games born by day d, which we will denote by $\mathsf{G_n}[d]$, recursively as

$$\begin{aligned} \mathsf{G_n}[0] &= \{0\} \\ \mathsf{G_n}[d] &= \{\{G_1|\dots|G_n\} : G_1, \dots, G_n \subseteq \mathsf{G_n}[d-1]\} \end{aligned}$$

The sets G_1, …, G_n are called respectively the sets of options of the 1st, 2nd, …, nth player.

Definition 2: Let

$$x = \{X_1|\dots|X_n\}$$

and

$$y = \{Y_1|\dots|Y_n\}$$

be two games. We define the sum of two games as follows

$$x + y = \{X_1 + y, x + Y_1|\dots|X_n + y, x + Y_n\}$$

The previous definition introduces a couple of abuses of notation requiring explanation. x and y are games but X_1, Y_1, …, X_n, and Y_n are sets of games. We define the addition of a single game x, to a set of games, G, as the set of games obtained by adding x to each element of G:

$$x + G = \{x + g\}_{g \in G}$$

The other abuse of notation is the use of the comma between two sets of games to indicate set union.

Definition 3: Let

$$x = \{X_1|\dots|X_n\}$$

and

$$y = \{Y_1|\dots|Y_n\}$$

(Advance online publication: 21 May 2013)

be two games. We say that $x \leq_C y$ if and only if the following two conditions are satisfied

$$(\forall i \in C)(\forall x_i \in X_i)(\exists y_i \in Y_i)(x_i \leq_C y_i) \quad (1)$$
$$(\forall i \notin C)(\forall y_i \in Y_i)(\exists x_i \in X_i)(x_i \leq_C y_i) \quad (2)$$

where $C \subset \{1, \ldots, n\}, C \neq \emptyset$. Moreover, we say that $x =_C y$ if and only if $(x \leq_C y)$ and $(y \leq_C x)$.
The previous definition formalizes the preference between two games for the coalition C. In term of games, the coalition C will never receive any disadvantage substituting the game x with the game y as shown in the following theorem.

Theorem 2: If $x \leq_C y$ then for any game g, the coalition C has a winning strategy in $y + g$ when player i moves first whenever the coalition C has a winning strategy in $x + g$ when player i moves first.
Games are partially ordered with respect to $\leq_C$, but every coalition produces a different order.

Theorem 3: The set of multi-player games born by day d forms a distributive lattice with respect to every partial order relation $\leq_C$, where C is an arbitrary coalition of players.
For further details, please refer to [5].

III. The Lattice Structure of $\mathsf{G_n}[d]$

In this section, we recall the main definitions and results concerning the mathematical structure of $\mathsf{G_n}[d]$. First, we briefly recall the definition of complete lattice.

Definition 4: A complete lattice

$$\left(L, \bigvee, \bigwedge\right)$$

is a partially ordered set $(L, \leq)$ with the additional property that every subset $A \subseteq L$ has a least upper bound or join denoted by

$$\bigvee A$$

and a greatest lower bound or meet denoted by

$$\bigwedge A$$

Formally,

$$\left(\forall x \in A\right)\left(x \leq \bigvee A\right)$$

and if there exists $y \in L$ such that

$$\left(\forall x \in A\right)\left(x \leq y\right)$$

then

$$\left(\bigvee A \leq y\right)$$

Symmetrically,

$$\left(\forall x \in A\right)\left(\bigwedge A \leq x\right)$$

and if there exists $y \in L$ such that

$$\left(\forall x \in A\right)\left(y \leq x\right)$$

then

$$\left(y \leq \bigwedge A\right)$$

Definition 5: Let $G \subseteq \mathsf{G_n}[d]$ be a set of games. We define floor and ceiling functions relative to $\mathsf{G_n}[d]$ as follows:

$$\begin{aligned} \lceil G \rceil &= \{h \in \mathsf{G_n}[d] : g \leq_C h, \text{ for some } g \in G\} \\ \lfloor G \rfloor &= \{h \in \mathsf{G_n}[d] : h \leq_C g, \text{ for some } g \in G\} \end{aligned}$$

Definition 6: Let

$$G = \{g^1, \ldots, g^m\} \subseteq \mathsf{G_n}[d]$$

be a set of games where

$$\begin{aligned} g^1 &= \{G_1^1 | \ldots | G_n^1\} \\ &\vdots \\ g^m &= \{G_1^m | \ldots | G_n^m\} \end{aligned}$$

We define the join and meet operations over $\mathsf{G_n}[d]$ for a given coalition C by

$$\begin{aligned} \bigvee^C G &= \{J_1 | \ldots | J_n\} \\ \bigwedge^C G &= \{M_1 | \ldots | M_n\} \end{aligned}$$

where

$$J_i = \begin{cases} G_i^1 \cup \ldots \cup G_i^m & \text{if } i \in C \\ \lceil G_i^1 \rceil \cap \ldots \cap \lceil G_i^m \rceil & \text{if } i \notin C \end{cases}$$

and

$$M_i = \begin{cases} \lfloor G_i^1 \rfloor \cap \ldots \cap \lfloor G_i^m \rfloor & \text{if } i \in C \\ G_i^1 \cup \ldots \cup G_i^m & \text{if } i \notin C \end{cases}$$

Theorem 4:

$$\left(\mathsf{G_n}[d], \bigvee^C, \bigwedge^C\right)$$

is a complete lattice.

Definition 7: A lattice, L, is *completely distributive* [13] if:

$$\bigwedge_{j \in J} \bigvee_{k \in K_j} x_{j,k} = \bigvee_{f \in F} \bigwedge_{j \in J} x_{j,f(j)}$$

for all doubly indexed families $\{x_{j,k} : j \in J, k \in K_j\} \subseteq L$, where F is the set of all choice functions from J to $\cup_{j \in J} K_j$. As in the case for ordinary distributivity, it turns out that this condition is self-dual, that is, that it implies the alternative with $\bigwedge$ and $\bigvee$ interchanged. Another, more obviously symmetric, form of the definition can be found in [15].

Theorem 5: The lattice

$$\left(\mathsf{G_n}[d], \bigvee^C, \bigwedge^C\right)$$

is completely distributive.
For further details, please refer to [6].

IV. Simplifying Multi-player Games

In this section we show how to simplify a multi-player game removing the dominated options, i.e., the moves that are not necessary to take into account during the analysis of the game.

Theorem 6: Let

$$x = \{X_1 | \ldots | X_n\}$$

be a game and let

$$C \subset \{1, \ldots, n\}, C \neq \emptyset$$

be an arbitrary coalition of players.

1) If $a \leq_C b$ with $a, b \in X_i$ and $i \in C$, then

$$x =_C \{X_1 | \ldots | X_i - \{a\} | \ldots | X_n\}.$$

2) If $a \leq_C b$ with $a, b \in X_i$ and $i \notin C$, then

$$x =_C \{X_1 | \ldots | X_i - \{b\} | \ldots | X_n\}.$$

Proof:

1) It is sufficient to observe that

$$(\forall x_i \in X_i)(\exists \overline{x}_i \in X_i - \{a\})(x_i \leq_C \overline{x}_i)$$
$$(\forall x_i \in X_i - \{a\})(\exists \overline{x}_i \in X_i)(x_i \leq_C \overline{x}_i)$$

are always satisfied when we choose

$$\overline{x}_i = \begin{cases} x & \text{if } x_i \in X_i - \{a\} \\ b & \text{if } x_i = a \end{cases}$$

2) It is sufficient to observe that

$$(\forall x_i \in X_i - \{b\})(\exists \overline{x}_i \in X_i)(\overline{x}_i \leq_C x_i)$$
$$(\forall x_i \in X_i)(\exists \overline{x}_i \in X_i - \{b\})(\overline{x_i} \leq_C x_i)$$

are always satisfied when we choose

$$\overline{x}_i = \begin{cases} x & \text{if } x_i \in X_i - \{b\} \\ a & \text{if } x_i = b \end{cases}$$

■

Definition 8: We say that two games x and y are identical ($x \cong y$) if their sets are identical, that is, if X_i is identical to Y_i, $\forall i \in \{1, \ldots, n\}$.

We say that x is in canonical form if X and all of X's options have no dominated options. The following theorem justifies the term canonical:

Theorem 7: If x and y are in canonical form and $x =_C y$, then $x \cong y$.

Proof: Let x and y be two games in canonical form. By hypothesis

$$(\forall i \in C)(\forall x_i \in X_i)(\exists \overline{y}_i \in Y_i)(x_i \leq_C \overline{y}_i)$$
$$(\forall i \in C)(\forall y_i \in Y_i)(\exists \overline{x}_i \in X_i)(y_i \leq_C \overline{x}_i)$$

We observe that $x_i \not<_C \overline{y}_i \not<_C \overline{x}_i$ because otherwise x_i should be dominated by $\overline{x}_i$. By inductive hypothesis, $(\forall x_i \in X_i)(\exists \overline{y}_i \in Y_i)(x_i \cong \overline{y}_i)$, i.e. $X_i \subseteq Y_i$. By symmetric arguments, $Y_i \subseteq X_i$, so $(\forall i \in C)(X_i = Y_i)$.

By hypothesis

$$(\forall i \notin C)(\forall y_i \in Y_i)(\exists \overline{x}_i \in X_i)(\overline{x}_i \leq_C y_i)$$
$$(\forall i \notin C)(\forall x_i \in X_i)(\exists \overline{y}_i \in Y_i)(\overline{y}_i \leq_C x_i)$$

We observe that $y_i \not<_C \overline{x}_i \not<_C \overline{y}_i$ because otherwise y_i should be dominated by $\overline{y}_i$. By inductive hypothesis, $(\forall y_i \in Y_i)(\exists \overline{x}_i \in X_i)(y_i \cong \overline{x}_i)$, i.e. $Y_i \subseteq X_i$. By symmetric arguments, $X_i \subseteq Y_i$, so $(\forall i \notin C)(X_i = Y_i)$.

Hence, $x \cong y$. ■

V. Upper and Lower Bound of $\mathsf{G_n}[d]$

In this section we redefine the upper and lower bound of $\mathsf{G_n}[d]$ in canonical form.

Theorem 8: The lattice

$$\left(\mathsf{G_n}[d], \bigvee^{C}, \bigwedge^{C}\right)$$

is bounded.

Proof: We define the upper bound of the lattice as

$$u_d = \begin{cases} 0 & \text{if } d = 0 \\ \{U_1 | \ldots | U_n\} & \text{if } d \geq 1 \end{cases}$$

where

$$U_i = \begin{cases} \{u_{d-1}\} & \text{if } i \in C \\ \emptyset & \text{if } i \notin C \end{cases}$$

We observe that $\forall x \in \mathsf{G_n}[\mathsf{d}]$,

$$\bigvee^{C}(x \cup u_d) =_C u_d$$

and

$$\bigwedge^{C}(x \cup u_d) =_C x.$$

We define the lower bound of the lattice as

$$l_d = \begin{cases} 0 & \text{if } d = 0 \\ \{L_1 | \ldots | L_n\} & \text{if } d \geq 1 \end{cases}$$

where

$$L_i = \begin{cases} \emptyset & \text{if } i \in C \\ \{l_{d-1}\} & \text{if } i \notin C \end{cases}$$

We observe that $\forall x \in \mathsf{G_n}[\mathsf{d}]$,

$$\bigvee^{C}(x \cup l_d) =_C x$$

and

$$\bigwedge^{C}(x \cup l_d) =_C l_d.$$

■

References

[1] M. H. Albert and R. J. Nowakowski, "Lattices of games," *Order*, vol. 29, no. 1, pp. 75–84, 2012.

[2] E. R. Berlekamp, J. H. Conway and R.K. Guy, *Winning ways for your mathematical plays.* A K Peters, 2001.

[3] D. Calistrate, M. Paulhus and D. Wolfe, "On the lattice structure of finite games," in *More Games of No Chance*, R. J. Nowakowski, Ed. Cambridge University Press, 2002, pp. 25–30.

[4] A. Cincotti, "N-player partizan games," *Theoretical Computer Science*, vol. 411, no. 34-36, pp. 3224–3234, 2010.

[5] A. Cincotti, "The lattice structure of n-player games," *Theoretical Computer Science*, vol. 459, pp. 113–119, 2012.

[6] A. Cincotti, "The Structure of N-Player Games Born by Day d," *Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2013, IMECS 2013*, 13-15 March, 2013, Hong Kong, pp. 1113–1116.

[7] J. H. Conway, *On numbers and games.* A K Peters, 2001.

[8] W. Fraser, S. Hirshberg and D. Wolfe, "The structure of the distributive lattice of games born by day n," *INTEGERS: Electronic Journal of Combinatorial Number Theory*, vol. 5, no. 2, #A06, 2005.

[9] W. O. Krawec, "Analyzing n-player impartial games," *International Journal of Game Theory*, vol. 41, no. 2, pp. 345–367, 2012.

[10] S. Y. R Li, "N-person Nim and N-person Moore's games," *International Journal of Game Theory*, vol. 7, no. 1, pp. 31–36, 1978.

[11] D. E. Loeb, "Stable Winning coalitions," in *Games of No Chance*, R. J. Nowakowski, Ed. Cambridge University Press, 1996, pp. 451–471.

[12] J. Propp, "Three-player impartial games," *Theoretical Computer Science*, vol. 233, no. 1-2, pp. 263–278, 2000.

[13] G. N. Raney, “Completely distributive complete lattices,” *Proceedings of the American Mathematical Society*, vol. 3, no. 5, pp. 677–680, 1952.
[14] P. D. Straffin Jr., “Three-person winner-take-all games with McCarthy’s revenge rule,” *The College Mathematics Journal*, vol. 16, no. 5, pp. 386–394, 1985.
[15] W. R. Tunnicliffe, “On defining “completely distributive”,” *Algebra Universalis*, vol. 19, no. 3, pp. 397–398, 1984.
[16] D. Wolfe and W. Fraser, “Counting the number of games,” *Theoretical Computer Science*, vol. 313, no. 3, pp. 527–532, 2004.