The General Solutions of the Normal Abel's Type Nonlinear ODE of the Second Kind

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***Abstract* — In this work we develop a new mathematical methodology about the general solutions of any Abel's second kind ordinary differential equation with integrable right hand side. This construction is established by expanding, improving and completing previous results, presented by Aleexeva, Zaitsev and Shvets, when the right hand side belongs to specific function classes.**

***Index Terms* — General Solutions, Abel's nonlinear ODEs, Second kind**

I. INTRODUCTION

Few physical phenomena in mathematical physics and nonlinear mechanics admit solutions in terms of known functions [5],[9-10] and [15]. It has been proved that an Abel differential equation ordinary ODE often appears after the reduction of many second and higher order ODEs. Because of this important role, many researchers investigated Abel's ODE deeply. Based on a nontrivial Lie symmetry, Schwarz [17] provided useful results about Abel's second kind ODE. Also, Cheb-Terrab [3] and Cheb-Terrab [4] et al presented a new three-parameters and a multi parameters non – constant - invariant solvable class, respectively. Recently, M. Güslu [6] et al found an approximate method, using shifted Chebyshev expansions of the unknown function and Khan et al [7] discovered a new mechanism for the solution of Abel's type singular equations using two steps of Laplace algorithm. Bougoffa [2] and Markakis [11] constructed the general solutions of a second order Abel ODE when the variable coefficients of this equation satisfies concrete functional restrictions. Exact particular solutions of a second order Abel ODE are constructed ([13] and [14]), including one arbitrary function.

This work deals with the construction of the general solutions of an Abel second kind ODE of the normal form with integrable right hand side. Since there are admissible functional transformations that reduce any Abel nonlinear ODE to the normal form [15], this mathematical methodology may be applied to any Abel equation of the second kind.

According to a proposition developed by Alexeeva, Zaitsev and Zhvets (AZS) ([1] and [15]), any Abel equation of the second kind of the normal form $y\,y'_x - y = F(x)$ admits the general solution $\prod_{k=1}^{n} |y - y_k|^{m_k} = C,$ where $y_k = y_k(x)$ $(k = 1, 2, ..., n)$ are n particular independent solutions and m are constant exponents. Here, a particular solution $y_k(x)$ corresponds to $C = 0$ $(if\ m_k > 0)$ and $C = \infty$ $(if\ m_k < 0)$; C is the integration constant. Extending, improving and completing this proposition we succeed in constructing the general solutions of an Abel nonlinear ODE of the second kind of the normal form with arbitrary smooth free integrable member $F(x)$.

II. SOME BASIC RESULTS

It is well known that a first order Abel nonlinear ODE of the second kind has the general form

$$(g_1\, y + g_0)\, y'_x = f_2\, y^2 + f_1\, y + f_0,\ g_1 \neq 0. \qquad (2.1)$$

This equation can be transformed to the normal form (canonical), namely

$$y\, y'_x - y = F; \qquad (2.2)$$

$F = F(x)$, arbitrary smooth function,

through a series of admissible functional transformations ([7] and [15]). These transformations are also presented extensively in [12]. Here the notation $y'_x = dy/dx$, $y''_{xx} = d^2y/dx^2, ...$ is used for the total derivatives.

Consider the Abel equation of the second kind of the normal form (2.2).

One can establish the following [12].

Theorem 2.1 Any Abel ODE of the second kind of the normal form (2.2) admits the particular solutions

$$y_k = \frac{1}{2}x\left[\overline{N_k} + \frac{1}{3}\right], \qquad (2.3)$$

where $\overline{N_k} = \overline{N_k}(x)$ are the roots of the cubic equation

$$\overline{N}^3 + p\overline{N} + q = 0, \qquad (2.4)$$

while

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$$p=-\frac{a^2}{3}+b,\qquad q=\frac{2}{27}a^3-\frac{1}{3}ab+c,\quad a=-4\;,$$
$$b=3+\frac{4\left[G+F\right]}{x},\quad c=-\frac{4\left[G+2F\right]}{x}\;. \tag{2.5}$$

Here, $G=G(x)$ is a smooth subsidiary function which must be determined. Also, the type of these roots [12] and [13] is in accordance with the sign of the coefficient p of equation (2.4) as well as the sign of its discriminant

$$D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2\;. \tag{2.6}$$

In a few words, when $D=0$ there is one real root and another real root of double multiplicity, while if $D\neq 0$ there is one real root and two complex conjure $(D>0)$, or these real roots $(D<0)$. Thus, in order to elaborate the new mathematical methodology we investigate the following cases:

1. If $D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2<0$ and $p<0$, there are three real district roots $\overline{N_k}(x)$**,** that are

$$\overline{N}_1=2\sqrt{-\frac{p}{3}}\cos\frac{a}{3}\;,$$
$$\overline{N}_{2,3}=-2\sqrt{-\frac{p}{3}}\cos\frac{a\pm\pi}{3}\;;\quad \cos a=-\frac{q}{2\sqrt{-\left(\frac{p}{3}\right)^3}}, \tag{2.7}$$

2. If $D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2>0$ and $p>0$, there are one real root and two complex conjugate roots, that are

$$\overline{N_1}=2\sqrt{\frac{p}{3}}\cot(2a),\overline{N}_{2,3}=\sqrt{\frac{p}{3}}\left[\cot(2a)\pm i\frac{\sqrt{3}}{\sin(2a)}\right];$$
$$\tan a=\left(\tan\frac{\beta}{2}\right)^{1/3},\quad \tan\beta=\frac{2}{q}\left(\frac{p}{3}\right)^{3/2}, \tag{2.8}$$
$$|a|\le\frac{\pi}{4},\;\;|\beta|<\frac{\pi}{2},\;\; a\neq 0, \text{ and}$$

3. If $D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2>0$ and $p<0$, there are one real root and two complex conjugate roots, that are

$$\overline{N_1}=-2\sqrt{-\frac{p}{3}}\frac{1}{\sin(2a)},$$
$$\overline{N}_{2,3}=\sqrt{-\frac{p}{3}}\left[\frac{1}{\sin(2a)}\pm i\sqrt{3}\cot(2a)\right];$$
$$\tan a=\left(\tan\frac{\beta}{2}\right)^{1/3},\sin\beta=\frac{2}{q}\left(\frac{p}{3}\right)^{3/2}, \tag{2.9}$$
$$|a|\le\frac{\pi}{4},\;\;|\beta|<\frac{\pi}{2},\;\; a\neq 0.$$

4. Finally for $D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2=0$, there are one real root and another real root of double duplicity, namely

$$\overline{N}_1=2\sqrt[3]{-\frac{q}{2}}\;,\;\overline{N}_2=\overline{N}_3=-\sqrt[3]{-\frac{q}{2}}\;. \tag{2.10}$$

Consequently, the corresponding particular solutions of the Abel equation (2.2) in terms of the subsidiary function $G(x)$ are given by the combination of equations, (2.5) together with (2.3) and (2.7) to (2.10) . More details are presented in [12], [13] and [16].

We continue with the following

Theorem 2.2 *Consider an Abel nonlinear ODE of the normal form* (2.2). *According to the construction by (AZS), the general solution of this equation is given through the formulae*

$$\prod_{k=1}^{n}\left|y-y_k\right|^{m_k}=C;$$

where

$y_k=y_k(x)$ *are particular solutions of* (2.2), m_k *constant exponents and* C *is an integration constant.*

We postpone the proof of this theorem, as it is presented extensively in [1] and [15].

III. A NEW MATHEMATICAL METHODOLOGY

We are able to state the following new Proposition that extends, modifies and completes the above Theorem 2.2. This elaboration leads to the construction of the general solution of any Abel's ODE of the second kind of the normal form.

Proposition 3.1 : *The general solution (general integral) of any Abel equation of the second kind of the normal form* (2.2) *is given by the formula*

$$\prod_{k=1}^{3}\left|y-y_k\right|^{m_k(x)}=C;\;\;k=1,2,3\;, \tag{3.1}$$

where

$$y_k=y_k(x)=\frac{1}{2}x\left[\overline{N_k}(x)+\frac{1}{3}\right] \tag{3.2}$$

is a particular solution of equation (2.2), $\overline{N_k}=\overline{N_k}(x)$ *is any of the roots of the cubic equation* (2.4) *including the subsidiary function G=G(x),* $m_k(x)$ are suitable *exponents that are functions of the independent variable x and that are to be determined*.

Note that, expanding the general solutions' forms which was introduced by Alexeeva, Zaitsev and Shvets (AZS) [1] and [15] the basic difference is that in the present case the exponents m_i are not constants, but functions of the independent variable x which will be determined. In addition, the new types of the particular solutions are of different (formulae (3.2)) and general forms.

Proof: Supposing, without loss of generality, that the three roots satisfy the inequality $y > y_k(x)$ (k =1,2 or k= 1,2,3) we deduce from equation (3.1) that $C > 0$. Then, according to Proposition 3.1, the general solution of the Abel ODE (2.2) is written as follows:

a Case a: *k*=2

$$(y-y_1)^{m_1(x)}(y-y_2)^{m_2(x)} = C, \qquad (3.3)$$

b Case b: *k*=3

$$(y-y_1)^{m_1(x)}(y-y_2)^{m_2(x)}(y-y_3)^{m_3(x)} = C, \qquad (3.4)$$

where C is a positive integration constant and $y(x)$ as in (3.2).

a. Solution in the case k=2 (Case a)

The logarithmization of (3.3), followed by the differentiation of the resulting expression and rearrangement, leads to the equation

$$\begin{aligned}&(m_1+m_2)y'_x\, y + \left[m'_{1_x}\ln(y-y_1)+m'_{2_x}\ln(y-y_2)\right]y^2 - \\ &\left[(y_2+y_1)m'_{1_x}\ln(y-y_1)+m_1y'_{1_x} + \right. \qquad (3.5)\\ &\left.+(y_2+y_1)m'_{2_x}\ln(y-y_2)+m_2y'_{2_x}\right]y + (-m_1y_2-m_2y_1)y'_x + \\ &\left[y_1y_2m'_{1_x}\ln(y-y_1)+m_1y'_{1_x}y_2+y_1y_2m'_{2_x}\ln(y-y_2)+m_2y'_{2_x}y_1\right]=0.\end{aligned}$$

We require that equation (3.5) be equivalent to the Abel equation (2.2) and thus we extract a system of nonlinear differential (algebraic) equations, that is

$$\begin{aligned}&m_1+m_2 = M(x), && \text{(a)}\\ &m'_{1_x}\ln(y-y_1)+m'_{2_x}\ln(y-y_2)=0, && \text{(b)}\\ &(y_2+y_1)m'_{1_x}\ln(y-y_1)+m_1y'_{1_x} + && \\ &+(y_2+y_1)m'_{2_x}\ln(y-y_2)+m_2y'_{2_x} = M(x), && \text{(c)} \quad (3.6)\\ &m_1y_2+m_2y_1=0, && \text{(d)}\\ &y_1y_2m'_{1_x}\ln(y-y_1)+m_1y'_{1_x}y_2 + && \\ &+y_1y_2m'_{2_x}\ln(y-y_2)+m_2y'_{2_x}y_1 = -M(x)F, && \text{(e)}\end{aligned}$$

where $M(x) \neq 0$ is a new subsidiary differentiable function that is to be determined. Since, $y_1(x)$ and $y_2(x)$ are two particular solutions of the initial equation (2.2) after some algebra one concludes that equations (3.6c) and (3.6e) are always true. Therefore, system (3.6) includes only three independent equations (3.6a,b,d) and we have to determine the functions $m_1(x)$ and $m_2(x)$ as well as the subsidiaries functions $G(x)$ and $M(x)$. Consequently, we need one more equation that results from the cubic (2.4) and refers to the kind of roots of this equation. Thus, according to (2.10) we get

$$\overline{N_i} = -2\overline{N}_2. \qquad (3.7)$$

The decoupling of the system (3.6a,b,d) after tedious and cumbersome algebra, various complicated manipulations and rearrangements, introduction and elimination of another subsidiary intermediate functions [12], as well as fruitful substitutions and differentions, leads to the following for $\overline{N}_2(x)$ unique equation

$$-\frac{1}{4}x^2\,\overline{N}_2(x)-\frac{9}{2}xF(x)\overline{N}_2(x)-\frac{3}{4}x^2\overline{N}_2^2(x)=0. \qquad (3.8)$$

Assuming $x \neq 0$ and $\overline{N}_2 \neq 0$, equation (3.8) provides $\overline{N}_2(x)$ in terms of x and the free member $F(x)$ of the Abel nonlinear ODE (2.2), that is

$$\overline{N}_2 = -\frac{x+18F}{3x}. \qquad (3.9)$$

From now on, the first subsidiary function $G(x)$ is estimated by the fact that the discriminant D of the cubic equation (2.4) equals to zero [12] and (2.4). This observation guides to the result

$$G(x)=\frac{1}{36}\left[72F(x)+20x+54x\left(\overline{N}_2\right)^3\right], \qquad (3.10)$$

or to the equivalent result

$$G(x) = -F(x)-\frac{324F(x)^3}{x^2}-\frac{54F(x)^2}{x}+\frac{x}{2}, \quad x\neq 0. \qquad (3.11)$$

Following the inverse procedure, the other subsidiary function $M(x)$ is defined by an intermediate equation furnishing the integral

$$\begin{aligned}&\frac{1}{M(x)}=\\ &=\frac{1}{\ln C}\int\frac{4[x+15F(x)]}{3[x+12F(x)][x+18F(x)]}\ln\left|4+\frac{x}{3F(x)}\right|dx\,; \quad (3.12)\\ &C>0.\end{aligned}$$

The above results complete the proof of Proposition 3.1 in case $k=2$, because the assertion (3.3) comes true. □

b Solutions in the case when k=3 (Case b)

In this case according to the previous developed, the general solution of the Abel ODE (2.2) is written as

$$(y-y_1)^{m_1(x)}(y-y_2)^{m_2(x)}(y-y_3)^{m_3(x)} = C, \qquad (3.13)$$

where C is a positive integration constant. Here, formula (3.5) becomes

$$\begin{aligned}&A\left[y^3-(y_1+y_2+y_3)y^2+(y_1y_2+y_2y_3+y_1y_3)y-y_1y_2y_3\right]+\\ &+(m_1+m_2+m_3)y^2\,y'_x+\\ &+(-m_1y'_{1_x}-m_2y'_{2_x}-m_3y'_{3_x})y^2+\end{aligned}$$

$$+\left(-m_1 y_3 - m_1 y_2 - m_2 y_3 - m_2 y_1 - m_3 y_3 - m_3 y_1\right) y\, y'_x + \quad (3.14)$$
$$+\left(m_1 y_2 y_3 + m_2 y_1 y_3 + m_3 y_1 y_2\right) y'_x +$$
$$+\left(m_1 y_2 y'_{1_x} + m_2 y_1 y'_{2_x} + m_3 y_1 y'_{3_x} + m_3 y_2 y'_{3_x} + m_2 y_3 y'_{2_x} + m_1 y_3 y'_{1_x}\right) y +$$
$$+ m_1 y_2 y_3 y'_{1_x} + m_2 y_1 y_3 y'_{2_x} + m_3 y_1 y_2 y'_{3_x} = 0;$$
$$A = m'_{1_x} \ln\left(y - y_1\right) + m'_{2_x} \ln\left(y - y_2\right) + m'_{3_x} \ln\left(y - y_3\right).$$

and the corresponding to (3.6) system of equations results in

$$m'_{1_x} \ln\left(y - y_1\right) + m'_{2_x} \ln\left(y - y_2\right) + m'_{3_x} \ln\left(y - y_3\right) = 0, \quad \text{(a)}$$
$$m_1 + m_2 + m_3 = 0, \quad \text{(b)}$$
$$m_1 y'_{1_x} + m_2 y'_{2_x} + m_3 y'_{3_x} = 0, \quad \text{(c)}$$
$$m_1 y_3 + m_1 y_2 + m_2 y_3 + m_2 y_1 + m_3 y_3 + m_3 y_1 = -M\left(x\right), \quad \text{(d)} \quad (3.15)$$
$$m_1 y_2 y_3 + m_2 y_1 y_3 + m_3 y_1 y_2 = 0, \quad \text{(e)}$$
$$m_1 y_2 y'_{1_x} + m_2 y_1 y'_{2_x} + m_3 y_1 y'_{3_x} + m_3 y_2 y'_{3_x} + m_2 y_3 y'_{2_x} + m_1 y_3 y'_{1_x} = -M\left(x\right), \quad \text{(f)}$$
$$m_1 y_2 y_3 y'_{1_x} + m_2 y_1 y_3 y'_{2_x} + m_3 y_1 y_2 y'_{3_x} = M\left(x\right) F\left(x\right), \quad \text{(g)}$$

The form $y_k\left(x\right)\ \left(k=1,2,3\right)$ depends heavily on the sign of the coefficient p of the cubic (2.4) and the sign of its discriminant. Also, $y_k\left(x\right)\ \left(k=1,...,3\right)$ includes the first auxiliary function $G(x)$. Moreover, $M = M\left(x\right) \neq 0$ is a second subsidiary smooth function that must be determined. Since, $y_1\left(x\right)$, $y_2\left(x\right)$ and $y_3\left(x\right)$ are particular solutions of the initial equation (2.2) after some algebra one concludes that equations (3.15c,f,g) are always true. Consequently, system (3.15) is restricted only to equations (3.15a,b,d). Tedious and cumbersome algebra, (as it was developed in case a, [12]), guides to the following functional relation among subsidiaries functions $M\left(x\right)$ and $G(x)$

$$\frac{\ln C}{M\left(x\right)} = \frac{y_2}{\left(y_2 - y_1\right)\left(y_2 - y_3\right)} \times \quad (3.16)$$

$$\left[\frac{y_1\left(y_3 - y_2\right) \ln\left|\dfrac{y_1\left(y_3 - y_2\right)}{2y_1 y_2 - \left(y_1 + y_2\right) y_3}\right|}{y_2\left(y_1 - y_3\right)} + \ln\left|\frac{y_1\left(y_3 - y_2\right)}{2y_1 y_2 - \left(y_1 + y_2\right) y_3}\right|\right];$$

$$C > 0,$$

or executive convenient algebra

$$M\left(x\right) = \left[C - \frac{y_1\left(y_2 - y_3\right)}{2y_1 y_2 + \left(y_1 + y_2\right) y_3}\right]^{\left(y_1 - y_2\right)}. \quad (3.17)$$

Since the cubic equation (2.4) admits three distinct roots $\overline{N_k}\left(x\right)$, three particular solutions $y_1\left(x\right); y_2\left(x\right); y_3\left(x\right)$ of the Abel ODE (2.2) exist and there is no need a similar equation of (3.10). As $M\left(x\right)$ or $G\left(x\right)$ may be any random smooth functions, we can assume without loss of generality, that

$$M\left(x\right) = G\left(x\right). \quad (3.18)$$

Both equations (3.16) and (3.18) are sufficient for the evaluations of the subsidiaries functions $M\left(x\right)$ and $G(x)$ in terms of the roots $y_k\left(x\right)$.

Remark: The main differences between the Theorem 2.2 state by (AZS) and the Proposition 3.1 are: i) The exponents in the present proposition are considered to be smooth functions of x and they are defined accordingly, and ii) The independent particular solutions $y_k\left(x\right)$ have concreted number, roots of the cubic (2.4). Both these above extensions together with the ascertainment that they are sufficient and necessary, establish the obtained general solution.

The above analysis concerning *case b*, together with the results of *case a* completes the proof of the Proposition 3.1.

□

IV. Final Results

Summarizing, we are able to state

Theorem 4.1. *If in a second kind Abel nonlinear ODE of the normal form*

$$y\, y'_x - y = F\left(x\right); \quad (4.1)$$

the discriminant D of the cubic equation $\overline{N}^3 + p\overline{N} + q = 0,$ equals zero $\left(D = 0\right)$, *then we dispose two independent particular solutions and its general solutions results as follows*

$$\left[y\left(x\right) - y_1\left(x\right)\right]^{m_1(x)} \left[y\left(x\right) - y_2\left(x\right)\right]^{m_2(x)} = C;$$

$$y_1\left(x\right) = \frac{1}{2} x\left[2\sqrt[3]{-\frac{q\left(x\right)}{2}} + \frac{1}{3}\right],\ y_2\left(x\right) = \frac{1}{2} x\left[-\sqrt[3]{-\frac{q\left(x\right)}{2}} + \frac{1}{3}\right];$$

$$m_1\left(x\right) = \frac{x + 12F\left(x\right)}{x + 18F\left(x\right)} M\left(x\right),$$

$$m_2\left(x\right) = \frac{6F\left(x\right)}{x + 18F\left(x\right)} M\left(x\right); \quad (4.2)$$

$$M\left(x\right) = \frac{\ln\left|C\right|}{\displaystyle\int \frac{4\left[x + 15F\left(x\right)\right]}{3\left[x + 12F\left(x\right)\right]\left[x + 18F\left(x\right)\right]} \ln\left|4 + \frac{x}{3F\left(x\right)}\right| dx};$$

$C =$ constant of integration;

$F\left(x\right)$ = *smooth free member of the given Abel equation.*

Theorem 4.2. *If in a second kind Abel nonlinear ODE of the normal form*

$$y\, y'_x - y = F\left(x\right); \quad (4.3)$$

the discriminant D of the cubic equation $\overline{N}^3 + p\overline{N} + q = 0,$ *is positive or negative* $\left(D < or > 0\right)$ *then we dispose three independent particular solutions and its then general solution results as follows*

$$\left[y(x)-y_1(x)\right]^{m_1(x)}\left[y(x)-y_2(x)\right]^{m_2(x)}\left[y(x)-y_3(x)\right]^{m_3(x)}=C;$$

$$m_1(x)=\frac{y_1(x)}{\left[y_1(x)-y_2(x)\right]\left[y_1(x)-y_3(x)\right]}G(x),$$

$$m_2(x)=\frac{y_2(x)}{\left[-y_1(x)+y_2(x)\right]\left[y_2(x)-y_3(x)\right]}G(x),$$

$$m_3(x)=-\left[m_1(x)+m_2(x)\right]. \qquad (4.4)$$

$$G(x)=M(x);$$

$$M(x)=\left[C-\frac{y_1(y_2-y_3)}{2y_1y_2+(y_1+y_2)y_3}\right]^{(y_1-y_2)};$$

C = positive integration constant;

$F(x)$ = *smooth free member of the given Abel equation;*

In this case the three independent particular solutions according (2.7) - (2.9) *are given for*

$$\text{i) } D>0;\ p<0 \Rightarrow y_1(x)=\frac{1}{2}x\left[2\sqrt{\frac{p}{3}}\cos\frac{a}{3}+\frac{1}{3}\right],$$

$$y_{2,3}(x)=\frac{1}{2}x\left[2\sqrt{-\frac{p}{3}}\cos\left(\frac{a\pm\pi}{3}\right)+\frac{1}{3}\right],$$

$$\cos a=\frac{q}{2\sqrt{-\left(\frac{p}{3}\right)^3}};$$

$$\text{ii) } D>0;\ p>0 \Rightarrow y_1(x)=\frac{1}{2}x\left[2\sqrt{\frac{p}{3}}\cot(2a)+\frac{1}{3}\right],$$

$$y_{2,3}(x)=\frac{1}{2}x\left\{\frac{p}{3}\left[\cot(2a)\pm i\frac{\sqrt{3}}{\sin(2a)}+\frac{1}{3}\right]\right\},$$

$$\tan a=\left(\tan\frac{\beta}{2}\right)^{1/3},\ \tan\beta=\frac{2}{q}\left(\frac{p}{3}\right)^{3/2},$$

$$|a|\le\frac{\pi}{4},\ |\beta|\le\frac{\pi}{2},\ \alpha\ne 0;$$

$$\text{iii) } D>0;\ p<0 \Rightarrow y_1(x)=\frac{1}{2}x\left[-2\sqrt{\frac{p}{3}}\frac{1}{\sin(2a)}+\frac{1}{3}\right],$$

$$y_{2,3}(x)=\frac{1}{2}x\left\{\frac{p}{3}\left[\frac{1}{\sin(2a)}\pm i\cot(2a)\right]+\frac{1}{3}\right\},$$

$$\tan a=\left(\tan\frac{\beta}{2}\right)^{1/3},\ \tan\beta=\frac{2}{q}\left(\frac{p}{3}\right)^{3/2},$$

$$|a|\le\frac{\pi}{4},\ |\beta|\le\frac{\pi}{2},\ \alpha\ne 0.$$

□

The above final results expressed by Theorems 4.1 and 4.2 as they were previously proved and Proposition 3.1, complete the solution of the problem under consideration, that is the construction of the general solutions of the normal form Abel's equations of the second kind.

REFERENCES

[1] T.A Alexeeva., V.F. Zaitsev and T.B. Shvets, On Discrete Symmetries of the Abel Equation of the 2nd Kind (in Russian), *Applied Mechanics and Mathematics, MIDT*, Moscow, 1992, pp. 4-11.

[2] L. Bougoffa, New exact general solutions of Abel equation of the second kind, *Applied Mathematics and Computation,* 216, 2010, pp. 689-691.

[3] E. S. Cheb-Terrab, A connection between Abel and hypergeometric differential equations, *European Journal of Applied Mathematics*, 16, 2005, pp. 53-63.

[4] E. S. Cheb-Terrab and A. D. Roche, Abel ODEs: Equivalence and integrable classes, *Computer Physics Communications,* 130, 2000, pp. 204-231.

[5] M. Danish, S. Kumar and S. Kumar, Exact Solutions of Three Nonlinear Heat Transfer Problems, *Engineering Letters*, 19:3, 2011, EL 19 3 15.

[6] M. Güslu, Y. Öztürk, M. Sezer, On the solution of the Abel equation of the second kind by the shifted Chebyshev polynomials, *Applied mathematics and computations,* 217, 2011, pp. 4827-4833.

[7] E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen,* Vol.1, B.G. Teubeur, Stuttgart, 1977.

[8] M. Khan and M. A. Gondal, A reliable treatment of Abel's second kind integral equations, *Applied Mathematics Letters,* 25, 2012, pp.1666-1670.

[9] M. R. Islam and M. A. Akbar, A New Asymptotic Solution for Third Order More Critically Damped Nonlinear Systems, *IAENG International Journal of Applied Mathematics*, 39 1, 2009, IJAM 39 1 02.

[10] T. Hasuike, Exact and Explicit Solution Algorithm for Linear Programming Problem with a Second-Order Cone, *IAENG International Journal of Applied Mathematics,* 41 3, 2011.

[11] M. P. Markakis, Closed-form solutions of certain Abel equations of the first kind, *Applied Mathematics Letters*, vol. 22, no. 9, 2009, pp. 1401–1405.

[12] D.E Panayotounakos and Th. Zarmpoutis, General Solutions of the non-linear ODEs Abel Riccati, and Emden-Fowler. Applications in non linear Mechanics and Mathematical Physics, *Korfiatis Publications,* Athens, 2010.

[13] D. E. Panayotounakos, Exact analytic solutions of unsolvable classes of first and second order nonlinear ODEs (Part I: Abel's equations), *Applied Mathematics Letters* 18, 2005, pp. 155-162.

[14] D. E. Panayotounakos, and D. C. Kravvaritis, Exact analytic solutions of the Abel, Emden-Fowler and generalized Emden-Fowler nonlinear ODEs, *Nonlinear Analysis - Real World Applications,* 7, 2006, pp. 634-650.

[15] A. D. Polyanin and V. F. Zaitsev *Handbook of Exact Solutions for Ordinary Differential Equations*, Chapman &Hall/CRC Press, New York, Second edition, 2003.

[16] A. D. Polyanin, and A. V. Manzhirov, *Handbook of Mathematics for Engineers and Scientist*, Chapman &Hall/CRC Press, New York, 2007.

[17] F. F. Schwarz, Symmetry analysis of Abel's equation, *Studies in Applied Mathematics* 100 (3) , 1998, pp. 269-294.