Phase fitted and Amplification fitted Hybrid Methods for Solving Second-order Ordinary Differential Equations

F. Samat, F. Ismail and M. Suleiman

***Abstract*—Two fifth-order explicit hybrid methods are developed. Based on these methods, phase fitted and amplification fitted methods are constructed by vanishing both the phase-lag and the dissipation error. For the phase fitted and amplification fitted methods, computation of the output stage is dependent on the frequency of the problem being solved, thus the methods can only be applied when the frequency is known in advance. Numerical comparisons that have been carried out show the advantage of the new methods for solving several second-order ordinary differential equations with oscillating solutions.**

***Index Terms*—hybrid methods, second-order ordinary differential equations, zero dissipation error, zero phase-lag**

I. Introduction

IN this paper, we are interested in the research on numerical methods for solving second order ordinary differential equations of the form

$$y''(x) = f(x, y(x)),\, y(x_0) = y_0,\, y'(x_0) = y'_0$$

where the first derivative does not appear explicitly. These problems often arise in engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics, chemistry and electronics and can be solved by using Runge Kutta Nystrom methods (see for example Senu [1]) and multistep methods. Several authors such as Fatunla, et. al. [2], Chawla [3], Tsitouras [4] and Simos [5] proposed hybrid methods which are obtained from the idea underlying both the Runge Kutta and linear multistep methods.

In the developments of hybrid methods, it is important to increase the order of the methods to achieve higher accuracy. In addition, if the second order ordinary differential equations have oscillating solutions, then it is also essential to consider the phase-lag and the dissipation error that result from comparing the numerical solution with the analytical solution. These are actually two types of truncation errors. The first is the angle between the analytical solution and the numerical solution while the second is the distance from a standard cyclic solution. The study of phase-lag has been initiated by Brusa and Nigro [6]. The research of hybrid methods has been carried out by many authors paying attention to obtain methods with minimal phase-lag or with zero dissipation error (see [7] to [11]).

Consider the class of hybrid methods:

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{j=1}^{s} b_j\, f\left(x_n + c_j h, g_j\right) \quad (1)$$

with $g_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij}\, f\left(x_n + c_j h, g_j\right)$

This class of methods has been discussed in many papers (for example see [4,12,13]). By assuming $c_1 = 1$ and $c_2 = 0$, Tsitouras [4] derived an eight-order implicit hybrid method. Meanwhile, Franco [13] proposed a class of explicit hybrid methods by assuming $c_1 = -1$ and $c_2 = 0$. In [14], Fang et. al. derived one- frequency and two-frequency explicit hybrid methods based on the fifth-order hybrid method in [13]. The coefficients of the new methods in [14] are obtained by vanishing both the phase-lag and the dissipation error.

Here, inspired by Runge Kutta methods, we choose $c_1 = 0$ and $c_2 = 1$. The class of explicit hybrid methods with $c_1 = 0$ and $c_2 = 1$ can be represented by the Butcher tableau:

$$\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & a_{21} & 0 & 0 & \cdots & 0 \\ c_3 & a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} \quad = \quad \begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

The leading term associated with the local truncation error of a p-th order hybrid method is given as

$$e_{p+1}(t_i) = \frac{\alpha(t_i)}{(p+2)!}\left[1 + (-1)^{p+2} - \mathbf{b}^T \psi''(t_i)\right],\ t_i \in T_2,$$

$\rho(t_i) = p + 2$ where T_2, $\alpha(t_i)$ and $\psi''(t_i)$ are as defined in [12]. The quantity

$$E = \sqrt{\sum_{i=1}^{n_{p+2}} e_{p+1}^2(t_i)}$$

where n_{p+2} is the number of trees of order $p + 2$, is called the error constant for the p-th order method. Based on this class of methods, we derive fifth order explicit hybrid methods

Manuscript received April 22, 2013; revised June 20, 2013. This work was supported by Ministry of Higher Education Malaysia under Research Acculturation and Collaboration Effort (RACE) grant scheme.

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with four stages (s = 4). Then, based on these methods, we derive phase fitted and amplification fitted explicit hybrid methods. The phase fitted and amplification fitted methods are obtained by vanishing the phase-lag and the dissipation error. The implementation of the methods is investigated by comparing the accuracy of the methods with that of the base and other existing methods.

II. Phase-lag Analysis

Let $H=\lambda h$ and $\mathbf{e}=\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}^T$. Applying the hybrid methods defined in (1) to equation $y''=-\lambda^2 y$, $\lambda>0$ yields the recursion

$$y_{n+1}-S\left(H^2\right)y_n+P\left(H^2\right)y_{n-1}=0 \qquad (2)$$

where

$$S\left(H^2\right)=2-H^2\mathbf{b}^T\left(\mathbf{I}+H^2\mathbf{A}\right)^{-1}(\mathbf{e}+\mathbf{c})$$

and $P\left(H^2\right)=1-H^2\mathbf{b}^T\left(\mathbf{I}+H^2\mathbf{A}\right)^{-1}\mathbf{c}$.

The characteristic equation associated with (2) is

$$\xi^2-S\left(H^2\right)\xi+P\left(H^2\right)=0 \qquad (3)$$

According to Houwen and Sommeijer [15], phase-lag is defined as the difference

$$t=H-\theta(H)$$

where H is the phase (or argument) of the exact solution of $y''=-\lambda^2 y$ and $\theta(H)$ is the phase of the principal root of (3). In case for explicit methods, the matrix **A** is nilpotent of degree s (that is $\mathbf{A}^s=\mathbf{O}$). Therefore,

$$\left(I+H^2A\right)^{-1}=I-H^2A+H^4A^2-H^6A^3+\ldots-H^{2s-2}A^{s-1}.$$

For the hybrid methods corresponding to the characteristic equation (3), the quantity

$$\phi(H)=H-\arccos\left(\frac{S\left(H^2\right)}{2\sqrt{P\left(H^2\right)}}\right)$$

is called phase-lag (or dispersion error) while the quantity

$$d(H)=1-\sqrt{P\left(H^2\right)}$$

is called dissipation (or amplification) error. A hybrid method corresponding to (3) is said to have the phase-lag of order n if $\phi(H)=O\left(H^{n+1}\right)$. If $\phi(H)=0$ then the method is said to be *phase fitted* or *zero dispersive*. If $P\left(H^2\right)=1$ then $d(H)=0$ and the method having this property is said to be *amplification fitted* or *zero dissipative*. If $P\left(H^2\right)\neq 1$ then $d(H)=O\left(H^{m+1}\right)$ and the method with this property is said to be dissipative of order m.

The interval $\left(0,H_p\right)$ is called the interval of periodicity of the method if

$$P\left(H^2\right)=1 \text{ and } \left|S\left(H^2\right)\right|<2 \text{ for all } H\in\left(0,H_p\right)$$

whereas the method is called P-stable if

$$P\left(H^2\right)=1 \text{ and } \left|S\left(H^2\right)\right|<2 \text{ for all } H\in(0,\infty).$$

The interval $\left(0,H_a\right)$ is called the interval of absolute stability if

$$\left|P\left(H^2\right)\right|<1 \text{ and } \left|S\left(H^2\right)\right|<1+P\left(H^2\right) \text{ for all } H\in\left(0,H_a\right).$$

III. Construction of Hybrid Methods

A. Construction of Fifth-order Methods

In this section, fifth-order explicit hybrid methods are constructed. The following are order conditions that have to be satisfied (see [12]).

$$\sum_{i=1}^{s}b_i=1$$

$$\sum_{i=1}^{s}b_ic_i=0$$

$$\sum_{i=1}^{s}b_ic_i^2=\frac{1}{6}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}b_ia_{ij}=\frac{1}{12}$$

$$\sum_{i=1}^{s}b_ic_i^3=0$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}b_ic_ia_{ij}=\frac{1}{12}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}b_ia_{ij}c_j=0$$

$$\sum_{i-1}^{s}b_ic_i^4=\frac{1}{15}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}b_ic_i^2a_{ij}=\frac{1}{30}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}b_ic_ia_{ij}c_j=-\frac{1}{60}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s}b_ia_{ij}a_{ik}=\frac{7}{120}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}b_ia_{ij}c_j^2=\frac{1}{180}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s}b_ia_{ij}a_{jk}=\frac{1}{360}$$

Substituting $s=4,\ c_1=0,\ c_2=1,\ a_{ij}=0\ (j\geq i)$ into the above order conditions and solving the resulting equations using Maple software, we obtain

$$b_1=\frac{25c_3^2+7c_3-3}{6c_3\left(5c_3+2\right)},$$

$$c_4=-\frac{5c_3+2}{5\left(c_3+1\right)},\ b_2=\frac{5c_3^2-2}{6\left(7+10c_3\right)\left(-1+c_3\right)},$$

$$b_3=-\frac{1}{2c_3\left(-1+c_3\right)\left(10c_3+2+5c_3^2\right)},$$

$$b_4=\frac{125\left(c_3+1\right)^4}{6\left(7+10c_3\right)\left(5c_3+2\right)\left(10c_3+2+5c_3^2\right)},\quad a_{21}=1,$$

$$a_{31}=\frac{2}{3}c_3-\frac{1}{6}c_3^3+\frac{1}{2}c_3^2,\quad a_{32}=\frac{1}{6}c_3\left(-1+c_3\right)\left(c_3+1\right),$$

$$a_{41}=-\frac{\left(5c_3+2\right)\left(325c_3^3+570c_3^2+240c_3-14\right)}{3750c_3\left(c_3+1\right)^4},$$

$$a_{42}=\frac{\left(5c_3+2\right)\left(7+10c_3\right)\left(-17+10c_3^2-10c_3\right)}{3750\left(c_3+1\right)^4\left(-1+c_3\right)},$$

$$a_{43}=\frac{(7+10c_3)(5c_3+2)(10c_3+2+5c_3^2)}{3750c_3(c_3+1)^4(-1+c_3)}.$$

For the first method, the free parameter c_3 is chosen so that the error constant E, is as small as possible giving us

$$c_3=-\frac{69}{100},\, E=1.85\times 10^{-2}$$

The new method will be denoted by EHM5I. Coefficients of EHM5I method are displayed in Table I.

TABLE I
COEFFICIENTS OF EHM5I

0	0	0	0	0
1	1	0	0	0
$-\frac{69}{100}$	$-\frac{334397}{2000000}$	$\frac{120497}{2000000}$	0	0
$\frac{29}{31}$	$\frac{173865730}{191168847}$	$-\frac{6193240}{468225147}$	$\frac{292262000}{32307535143}$	0
	$\frac{905}{1334}$	$-\frac{761}{2028}$	$\frac{10000000}{58759779}$	$\frac{923521}{1753572}$

The phase-lag order and the dissipation order for this method are six and five respectively with the following quantities

$$\phi(H)=\frac{71}{1512000}H^7+O(H^9),$$
$$d(H)=\frac{31}{216000}H^6+O(H^{12}).$$

The interval of absolute stability is (0, 3.36).

For the second method, the free parameter c_3 is chosen so that the phase-lag order is eight. This gives us the values

$$c_3=-\frac{25}{28},\, E=7.09\cdot 10^{-2}$$

The new method is denoted by EHM5II. Coefficients of EHM5II method are shown in Table II.

TABLE II
COEFFICIENTS OF EHM5II

0	0	0	0	0
1	1	0	0	0
$-\frac{25}{28}$	$-\frac{3425}{43904}$	$\frac{1325}{43904}$	0	0
$\frac{23}{5}$	$\frac{454986}{15625}$	$\frac{16744}{33125}$	$-\frac{13866608}{828125}$	0
	$\frac{2791}{3450}$	$\frac{173}{1908}$	$\frac{307328}{3056775}$	$-\frac{125}{636732}$

This method has phase-lag of order eight and is dissipative of order five with the following quantities

$$\phi(H)=\frac{17}{7257600}H^9+O(H^{11})$$
$$d(H)=\frac{1}{20160}H^6+O(H^{12})$$

The interval of absolute stability is (0, 3.94).

B. Construction of Phase Fitted and Amplification Fitted Methods

Here, phase fitted and amplification fitted hybrid methods denoted by EHM5IPA and EHM5IIPA will be derived. The derivations of EHM5IPA and EHM5IIPA are being based on EHM5I and EHM5II methods respectively. Table III shows coefficients of EHM5IPA method.

TABLE III
COEFFICIENTS OF EHM5I PA

0	0	0	0	0
1	1	0	0	0
$-\frac{69}{100}$	$-\frac{334397}{2000000}$	$\frac{120497}{2000000}$	0	0
$\frac{29}{31}$	$\frac{173865730}{191168847}$	$-\frac{6193240}{468225147}$	$\frac{292262000}{32307535143}$	0
	b_1	b_2	$\frac{10000000}{58759779}$	$\frac{923521}{1753572}$

It is noted that some of the values are taken from Table I. The coefficients b_1 and b_2 are obtained by vanishing the phase-lag and the dissipation error. The quantity $S(H^2)$ has to be equal to $2\cos(H)\sqrt{P(H^2)}$ in order to vanish the phase-lag. Solving the resulting equation, we get

$$\begin{aligned}b_1=\frac{1}{121740840000H^2}\Big[&-130516200000H^2+5582790000H^4\\&+121740840000H^4b_2(H)+3494413H^8\\&-37311313H^6+243481680000\\&-243481680000H^2b_2(H)-1040520\cos(H)(547560000\\&-547560000H^2b_2(H)-157170H^6-205470000H^2)^{1/2}\Big]\end{aligned}$$

For the dissipation error to vanish, we set $P(H^2)=1$ and then solve the resulting equation giving

$$b_2=-\frac{31}{108000}H^4-\frac{761}{2028}.$$

For small H, the coefficient b_1 is subject to heavy cancellations. Therefore, it is convenient to use the Taylor series expansion:

$$\begin{aligned}b_1=&\frac{905}{1334}+\frac{31}{108000}H^4-\frac{1}{20160}H^6+\frac{1}{1814400}H^8\\&-\frac{1}{239500800}H^{10}+\frac{1}{43589145600}H^{12}-\ldots\quad.\end{aligned}$$

Behaviour of the coefficients is given in Fig. 1.

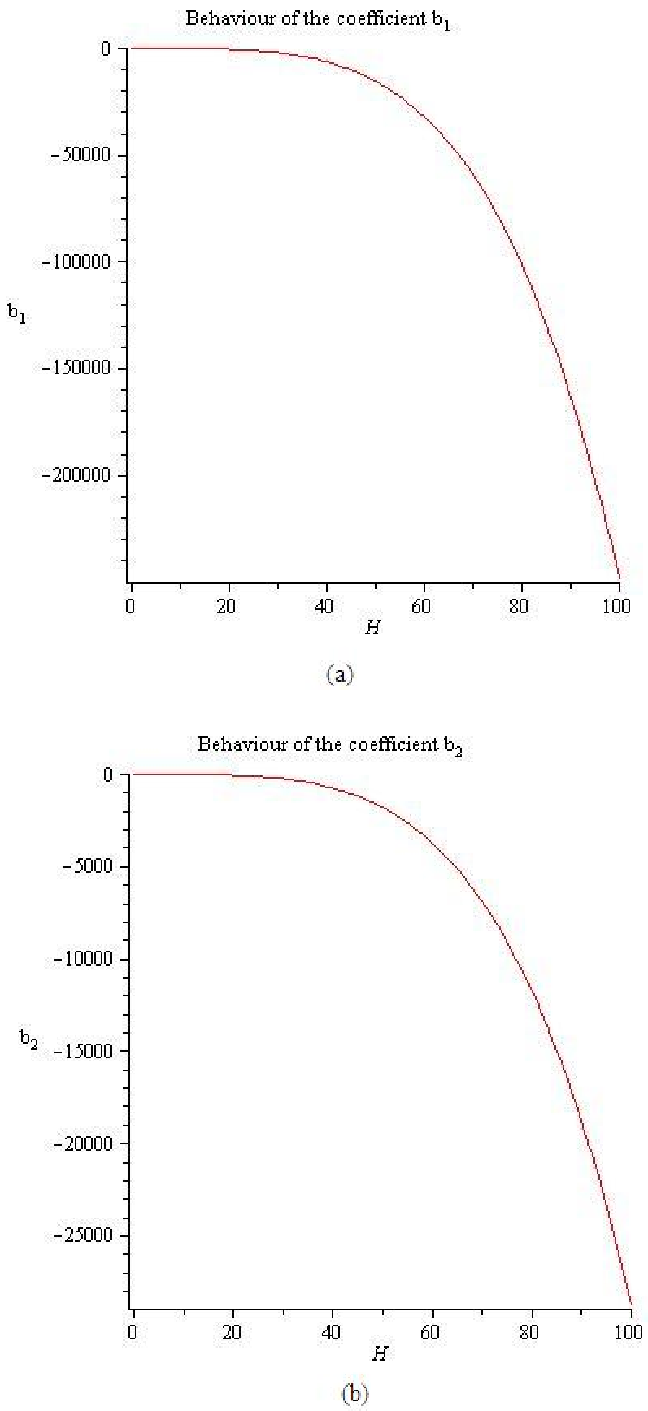


Fig. 1. Behaviour of the coefficients b_1 and b_2 of the new proposed method; EHM5IPA for several values of $H = \lambda h$.

Let us consider coefficients of EHM5IIPA given by Table IV.

TABLE IV
COEFFICIENTS OF EHM5IIPA

0	0	0	0	0
1	1	0	0	0
$-\frac{25}{28}$	$-\frac{3425}{43904}$	$\frac{1325}{43904}$	0	0
$\frac{23}{5}$	$\frac{454986}{15625}$	$\frac{16744}{33125}$	$-\frac{13866608}{828125}$	0
	b_1	b_2	$\frac{307328}{3056775}$	$-\frac{125}{636732}$

Some of the values in Table IV are taken from Table II. Using the similar procedure, by vanishing the phase-lag, we obtain

$$\begin{aligned} b_1 = \frac{1}{61437600H^2}\Big[& -594272H^2 - 450800H^4 \\ & + 61437600H^4 b_2 + 6095H^8 - 176755H^6 + 122875200 \\ & - 122875200H^2 b_2 - 2760\cos(H)\big(1982030400 \\ & - 1982030400H^2 b_2 - 196630H^6 + 179712400H^2\big)^{1/2}\Big] \end{aligned}$$

whereas by vanishing the dissipation error,
whereas by vanishing the dissipation error,

$$b_2 = -\frac{1}{10080}H^4 + \frac{173}{1908}.$$

The Taylor series expansion for b_1 is given by

$$\begin{aligned} b_1 = & \frac{2791}{3450} + \frac{1}{10080}H^4 - \frac{1}{20160}H^6 + \frac{1}{1814400}H^8 \\ & - \frac{1}{239500800}H^{10} + \frac{1}{43589145600}H^{12} - \ldots \end{aligned}$$

The behaviour of the coefficients is given in Fig. 2.

IV. NUMERICAL RESULTS

All new codes have been applied to some second-order problems to provide numerical comparisons with other competitive codes in the scientific literature. Codes that have been used for numerical comparisons are denoted by:

EHM5I : The first fifth-order explicit hybrid method with four stages derived in this paper. This method has phase-lag of order six and is dissipative of order five. The interval of absolute stability is (0, 3.36).

EHM5II : The second fifth-order explicit hybrid method with four stages derived in this paper. This method has phase-lag of order eight and is dissipative of order five. The interval of absolute stability is (0, 3.94)

EHM5IPA : The phase fitted and amplification fitted explicit hybrid method which is derived based on EHM5I in this paper.

EHM5IIPA : The phase fitted and amplification fitted explicit hybrid method which is derived based on EHM5II in this paper.

FETSH : The fifth-order explicit hybrid method with three stages derived by Franco [13]. This method has phase-lag of order eight and is dissipative of order five. The interval of absolute stability is (0, 2.84) whereas the formula for this method is given by

$$Y_1 = y_{n-1}, Y_2 = y_n$$

$$Y_3 = (1+c_3)y_n - c_3 y_{n-1} + h^2(a_{31} f_{n-1} + a_{32} f_n)$$

$$\begin{aligned} Y_4 = & (1+c_4)y_n - c_4 y_{n-1} + h^2[a_{41} f_{n-1} + a_{42} f_n \\ & + a_{43} f(x_n + c_3 h, Y_3)] \end{aligned}$$

$$\begin{aligned} y_{n+1} = & 2y_n - y_{n-1} + h^2[b_1 f_{n-1} + b_2 f_n \\ & + b_3 f(x_n + c_3 h, Y_3) + b_4 f(x_n + c_4 h, Y_4)] \end{aligned}$$

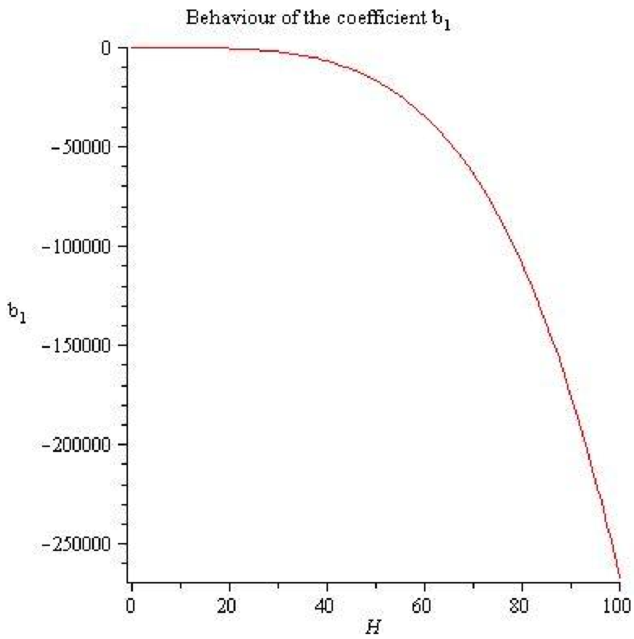
The coefficients of the method can be found in [13].

TSI7: The seventh-order explicit hybrid method with four stages derived in [16]. This method has the form

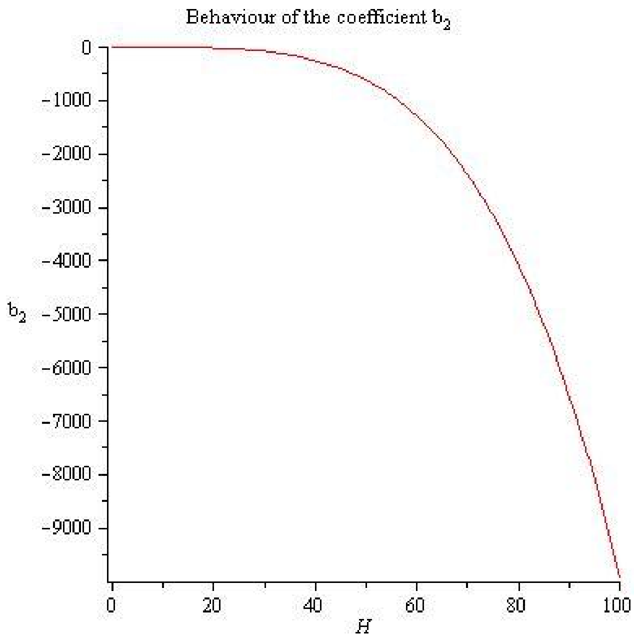
$$f_n = f(x_n, y_n),$$

$$y_a = c_1 y_{n-1} + (1-c_1) y_n + h^2 (d_{11} f_{n-1} + d_{12} f_n),$$
$$f_a = f(x_n - c_1 h, y_a),$$
$$y_b = c_2 y_{n-1} + (1-c_2) y_n + h^2 (d_{21} f_{n-1} + d_{22} f_n + g_{21} f_a),$$
$$f_b = f(x_n - c_2 h, y_b),$$
$$y_c = c_3 y_{n-1} + (1-c_3) y_n + h^2 (d_{31} f_{n-1} + d_{32} f_n + g_{31} f_a + g_{32} f_b),$$
$$f_c = f(x_n - c_3 h, y_c),$$
$$y_{n+1} = -y_{n-1} + 2 y_n + h^2 (w_1 f_{n-1} + w_2 f_n + b_1 f_a + b_2 f_b + b_3 f_c).$$

The coefficients can be found in [16]. According to Tsitouras [16], the coefficients of this method have been selected so that the local truncation error is minimized.



(a)



(b)

Fig. 2. Behaviour of the coefficients b_1 and b_2 of the new proposed method; EHM5IIPA for several values of $H = \lambda h$.

Criterion used as a measure for accuracy is the maximum global error given by the formula

$$\text{MAXGE} = \max \| y(x_n) - y_n \|$$

where $y(x_n)$ is the exact solution and y_n is the computed solution. The numerical results are given in Fig. 3 to 6. Test problems that have been used are as follows.

Problem 1 (non-homogeneous problem)

$$y'' = -100y + 99\sin(x),\ y(0) = 1,\ y'(0) = 11, \quad 0 \le x \le 100$$

Solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$.

We choose $H = 10h$ for EHM5IPA and EHM5IIPA codes.

Problem 2 (homogeneous problem)

$$y'' = -2500y, \quad y(0) = 0,\ y'(0) = 1, \quad 0 \le x \le 100$$

Solution: $y(x) = \frac{1}{50}\sin(50x)$.

We choose $H = 50h$ for EHM5IPA and EHM5IIPA codes.

Problem 3 (almost periodic orbit problem)

$$z''(x) + z(x) = \frac{1}{1000} e^{ix},\ z(0) = 1,\ z'(0) = 0.9995i,\ z \in \mathbf{C},$$
$0 \le x \le 100$.

The theoretical solution is $z(x) = (1 - 0.0005ix)e^{ix}$.

If $z(x) = y_1(x) + iy_2(x),\ y_1, y_2 \in \mathbf{R}$, then the problem can be transformed into the equivalent form

$$y_1'' = -y_1 + \frac{1}{1000}\cos(x),\ y_1(0) = 1,\ y_1'(0) = 0$$
$$y_2'' = -y_2 + \frac{1}{1000}\sin(x),\ y_2(0) = 0,\ y_2'(0) = 0.9995$$

with the theoretical solution:

$$y_1(x) = \cos(x) + 0.0005x\sin(x),$$
$$y_2(x) = \sin(x) - 0.0005x\cos(x).$$

We choose $H = h$ for EHM5IPA and EHM5IIPA codes.

Problem 4 (perturbed system)

$$y_1'' + 100y_1 + \frac{2y_1 y_2}{y_1^2 + y_2^2} = f_1(x),\ y_1(0) = 1,\ y_1'(0) = \varepsilon$$
$$y_2'' + 25y_2 + \frac{y_1^2 - y_2^2}{y_1^2 + y_2^2} = f_2(x),\ y_2(0) = -\varepsilon,\ y_2'(0) = 5$$

with $\varepsilon = 10^{-3}$ and

$$f_1(x) = \big[(2\cos(10x)\sin(5x) + 2\varepsilon(\sin(5x)\sin(x) - \cos(10x)\cos(x)) - \varepsilon^2\sin(2x)) / (\cos^2(10x) + \sin^2(5x) + 2\varepsilon(\sin(x)\cos(10x) - \cos(x)\sin(5x)) + \varepsilon^2)\big] + 99\varepsilon\sin(x),$$

$$f_2(x) = \big[(\cos^2(10x) - \sin^2(5x) + 2\varepsilon(\sin(x)\cos(10x) + \cos(x)\sin(5x)) - \varepsilon^2\cos(2x)) / (\cos^2(10x) + \sin^2(5x) + 2\varepsilon(\sin(x)\cos(10x) - \cos(x)\sin(5x)) + \varepsilon^2)\big] - 24\varepsilon\cos(x).$$

Solution:
$y_1(x) = \cos(10x) + \varepsilon \sin(x),\ y_2(x) = \sin(5x) - \varepsilon \cos(x),$
$0 \le x \le 10.$
For EHM5IPA and EHM5IIPA codes, we choose H = 10h for the first component and H = 5h for the second component.

From the numerical results, it is observed that EHM5I is almost as accurate as FETSH method. In addition, the maximum global error for EHM5II is of the same order as that for FETSH method with advantage for EHM5II as EHM5II is more stable for solving Problem 2. For Problem 3, EHM5IPA is the most accurate followed by TSI7. TSI7 is unstable when it was used to solve Problem 2 and Problem 4 for big step-size. Of all methods, EHM5IPA and EHM5IIPA are the most accurate for solving most problems considered. This is because EHM5IPA and EHM5IIPA are being phase fitted and amplification fitted compared to other methods.

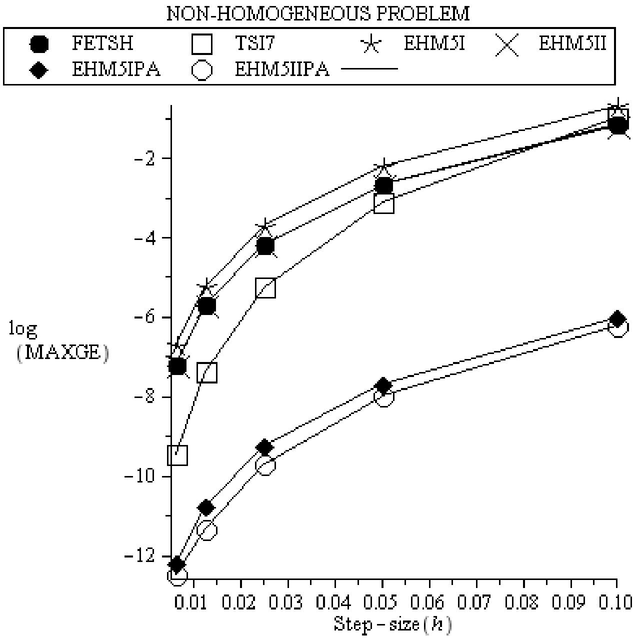


Fig. 3. Numerical results for Problem 1

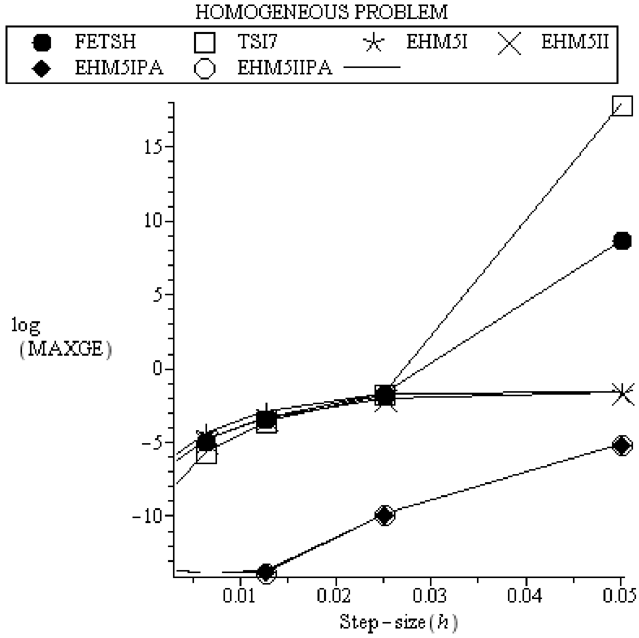


Fig. 4. Numerical results for Problem 2

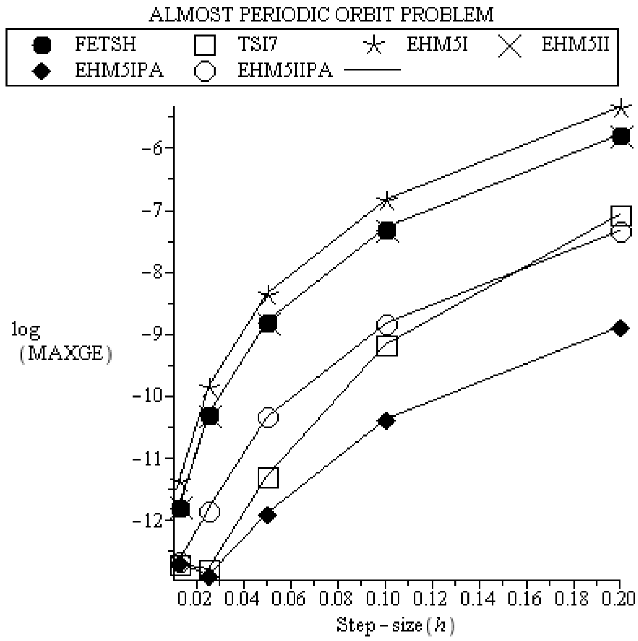


Fig. 5. Numerical results for Problem 3

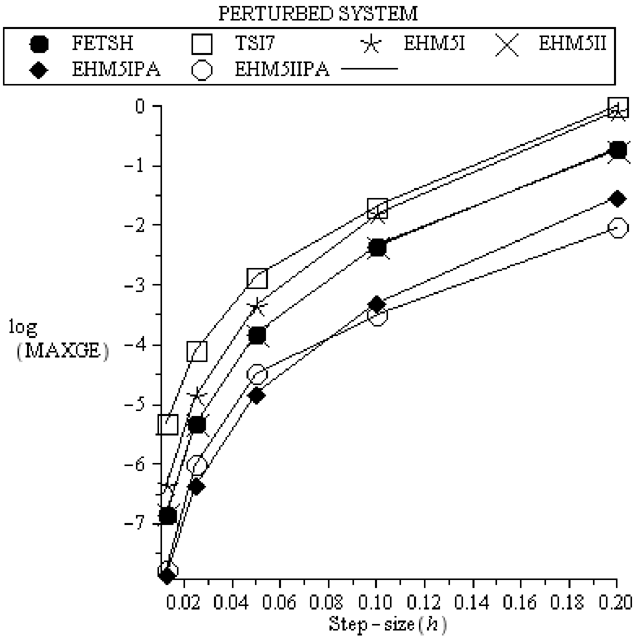


Fig. 6. Numerical results for Problem 4

V. Conclusion

In this paper, we investigate the implementation of phase fitted and amplification fitted explicit hybrid methods for solving second order ordinary differential equations. From numerical observations, we conclude that the phase fitted and amplification fitted explicit hybrid methods are very accurate for solving second-order ordinary differential equations having oscillating solutions. The results also indicate that phase fitting and amplification fitting gives us methods with better accuracy compared to the base methods. Moreover, all of the new methods are capable to solve any physical problems whose solutions are in the oscillatory form. All codes are designed using Microsoft Visual C++ version 6.0 software in HP computer with specification Intel(R)Core(TM)2DuoCPU P8600@2.40GHz.

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