Optimal Homotopy Asymptotic Method for Solving Integro-differential Equations

Yu Du Han and Jae Heon Yun

***Abstract*—Integro-differential equations arise in modeling various physical and engineering problems. Several numerical and analytical methods have been developed to solving such equations. We introduce the OHAM (Optimal Homotopy Asymptotic Method) for solving nonlinear integro-differential equations. Several examples for solving integro-differential equations are presented to illustrate the reliability and efficiency of the proposed OHAM.**

***Index Terms*—Optimal Homotopy Asymptotic Method, Integro-differential equations, Variational iteration method, Homotopy perturbation method.**

I. Introduction

INTEGRO-DIFFERENTIAL equations arise in modeling various physical and engineering problems. Solving such equations is very interesting, and several numerical and analytical methods have been developed [1], [5], [7], [8]. Numerical methods, such as finite difference and finite element methods, are computationally expensive and have less convergence speed and accuracy, which may produce inaccurate results. Therefore, science and engineering researchers attempt to propose new methods for solving functional equations. The various functional equations are generally difficult to solve and their exact solutions are difficult to obtain. Therefore, some various approximate methods have recently been developed such as Variational iteration method (VIM) [9], [13], [14], Adomain's decomposition method (ADM) [2], [3] and Homotopy perturbation method (HPM) [10], [11], [12], [18] to solve the various functional equations. In recent years, improvements in numerical techniques, together with the rapid advance in computer technology and computer algebra systems such as Maple and Mathematica, have meant that various functional equations arising in engineering and scientific applications, which were previously intractable, can now be routinely solved.

The present work is motivated by the desire to obtain analytic solutions of integro-differential equations using the OHAM (Optimal Homotopy Asymptotic Method), which is recently introduced by Marica et al. [19]. The advantage of OHAM is in the convergence criteria which is more flexible. In series of papers, authors [4], [15], [16], [17], [19], [20], [21] have applied this method successfully to obtain the solutions of currently important problems in science and also shown its effectiveness, generalization and reliability. Since many real-world modeling problems might lead to integro-differential equations, it is essential to have alternative approaches which can deal with such problems. In this paper, we consider the OHAM to obtain the approximate solutions of nonlinear integro-differential equations. The general nth-order integro-differential equations [6] are of the form

$$u^{(n)}(x) + \sum_{i=0}^{n-1} u^{(i)}(x) f_i(x) + \int_a^b w(x,t) u^{(m)}(t)\, dt = g(x), \;\; a < x < b$$

with initial conditions

$$u(a) = \alpha_0, \; u'(a) = \alpha_1, \; u''(a) = \alpha_2, \; \cdots, u^{(n-1)}(a) = \alpha_{n-1},$$

where α_i's are real constants, m and n are integers such that $m < n$, f_i's, g and w are given functions, and u is the solution to be determined.

This paper is organized as follows. In Section 2, we introduce the OHAM for solving nonlinear integro-differential equations. In Section 3, several examples for solving integro-differential equations are presented to illustrate the reliability and efficiency of the proposed OHAM. In Section 4, some conclusion are drawn.

II. Optimal Homotopy Asymptotic Method (OHAM)

In this section, we introduce the basic ideas of OHAM for solving the nonlinear integro-differential equation, which is of the form:

$$L(u(x)) + g(x) + N(u(x)) = 0, \;\; B\left(u, \frac{du}{dx}\right) = 0. \quad (1)$$

where L is a linear operator, $u(x)$ is unknown function and $g(x)$ is known function, N is an non-linear operator and B is boundary operator.

We first construct a family of homotopy equations [19]

$$\begin{aligned}(1-p)[L(u(x,p)) + g(x)] &= H(p)\big[L(u(x,p)) \\ &\quad + g(x) + N(u(x,p))\big], \\ B\left(u, \frac{du}{dx}\right) &= 0,\end{aligned} \quad (2)$$

where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, $u(x,p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$, it holds

$$u(x,0) = u_0(x), \;\; u(x,1) = u(x).$$

respectively. Thus, as p increases from 0 to 1, the solution $u(x,p)$ varies from $u_0(x)$ to the solution $u(x)$, where $u_0(x)$ is obtained from Eq.(2) for $p = 0$:

$$L(u_0(x)) + g(x) = 0, \;\; B\left(u_0, \frac{du_0}{dx}\right) = 0. \quad (3)$$

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We choose the auxiliary function $H(p)$ in the form

$$H(p) = c_1 p + c_2 p^2 + c_3 p^3 + \cdots ,$$

where $c_1, c_2, \ldots$ are constants, which can be determined later. Let us consider the solution of Eq.(2) in the form

$$u(x;p,c_i) = u_0(x) + \sum_{k\geq 1} u_k(x,c_i)p^k, \quad i = 1,2,\ldots, \qquad (4)$$

Now substituting Eq.(4) in Eq.(2) and equating the coefficients of the same powers of p, we obtain the governing equation of $u_0(x)$ given by Eq.(3) and the governing equations of $u_k(x)$ which are

$$\begin{aligned} &L(u_1(x)) = c_1 N_0(u_0(x)), \quad B\left(u_1, \frac{du_1}{dx}\right) = 0, \\ &L\left(u_k(x) - u_{k-1}(x)\right) = c_k N_0(u_0(x)) + \sum_{i=1}^{k-1} c_i \Big[L(u_{k-i}(x) \\ &\quad + N_{k-i}(u_0(x), u_1(x), \ldots, u_{k-1}(x)) \Big], \\ &B\left(u_k, \frac{du_k}{dx}\right) = 0, \quad k = 2,3,\ldots, \end{aligned} \qquad (5)$$

where $N_m(u_0(x), u_1(x), \ldots, u_m(x))$ is the coefficient of p^m, obtained by expanding $N(u(x;p,c_i))$ in series with respect to the embedding parameter p; For $i = 1,2,\ldots$,

$$N(u(x;p,c_i)) = N_0(u_0(x)) + \sum_{k\geq 1} N_m(u_0, u_1, \ldots, u_m)p^m, \qquad (6)$$

where $u(x;p,c_i)$ is given by Eq.(4). It should be emphasized that u_k for $k \geq 0$ are governed by the linear Eqs.(3) and (5) with the linear boundary conditions that come from original problem, which can be easily solved. The convergence of the series of Eq.(4) depends upon the auxiliary constants $c_1, c_2, \ldots$. If it is convergent at $p = 1$, one has

$$u(x,c_i) = u_0(x) + \sum_{k\geq 1} u_k(x,c_i).$$

Generally speaking, the solution of Eq.(1) can be determined approximately in the form:

$$u^m(x,c_i) = u_0(x) + \sum_{k=1}^{m} u_k(x,c_i), \quad i = 1,2,\ldots,m. \qquad (7)$$

Substituting Eq.(7) into Eq.(1), we obtain the following residual

$$\begin{aligned} R(x,c_i) = L\left(u^m(x,c_i)\right) + g(x) \\ + N\left(u^m(x,c_i)\right), \quad i = 1,2,\ldots,m. \end{aligned} \qquad (8)$$

If $R(x,c_i) = 0$, then $u^m(x,c_i)$ happens to be the exact solution. In general, such case will not arise for nonlinear problems, but we can minimize the following functional

$$J(c_i) = \int_a^b R^2(x,c_i)\, dx,$$

where a and b are two values, depending on the given problem. The unknown constants $c_i (i = 1,2,\ldots,m)$ can be optimally identified from the conditions

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_m} = 0.$$

With these known constants, the approximate solution (of order m) in Eq.(7) is well-determined.

III. Examples for the integro-differential equations

In this section, to illustrate the efficiency of the OHAM for solving the integro-differential equations, four examples are presented.

Example 1: Consider the first-order integro-differential equation:

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xt\, u(t)dt, \quad u(0) = 0. \qquad (9)$$

which has the exact solution $u(x) = x$. Eq.(9) can be written as

$$u(x) - x + \frac{1}{6}x^2 - \int_0^x \int_0^1 \tau t\, u(t)dt\; d\tau = 0. \qquad (10)$$

The OHAM formulation of Eq.(10) is

$$\begin{aligned} &L(u(x;p)) = u(x), \quad N(u(x;p)) = -\int_0^x \int_0^1 \tau t\, u(t)dt\; d\tau, \\ &\quad g(x) = -x + \frac{1}{6}x^2, \end{aligned}$$

which satisfies

$$\begin{aligned} &(1-p)\left[\left(u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right) - x + \frac{1}{6}x^2\right] \\ &= \left(c_1 p + c_2 p^2 + \cdots\right)\Bigg[\left(u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right) \\ &\qquad - x + \frac{1}{6}x^2 - \int_0^x \int_0^1 \tau t\, u(t)dt\; d\tau\Bigg]. \end{aligned}$$

By equating the coefficients of the same powers of p, we obtain

$$\begin{aligned} &p^0 : u_0(x) = x - \frac{1}{6}x^2, \\ &p^1 : u_1(x,c_1) = -c_1 \int_0^x \int_0^1 \tau t\, u_0(t)dt\; d\tau = -\frac{7c_1}{48}x^2, \\ &p^2 : u_2(x,c_1,c_2) = (1+c_1)u_1(x,c_1) \\ &\quad - c_1 \int_0^x \int_0^1 \tau t\, u_1(t,c_1)dt\; d\tau - c_2 \int_0^x \int_0^1 \tau t\, u_0(t)dt\; d\tau. \\ &\quad = -\frac{7\left(8c_1 + 7c_1^2 + 8c_2\right)}{384}x^2. \end{aligned}$$

Thus we have the second order approximate solution of Eq.(9)

$$u^2(x,c_1,c_2) = x - \frac{x^2}{384}\left(64 + 112c_1 + 49c_1^2 + 56c_2\right). \qquad (11)$$

Substituting Eq.(11) into Eq.(10), one obtains the following residual

$$\begin{aligned} R(x,c_1,c_2) &= u^2(x,c_1,c_2) - x + \frac{1}{6}x^2 - \int_0^x \int_0^1 st\, u^2(t,c_1,c_2)dtds \\ &= -\frac{7(64 + 112c_1 + 49c_1^2 + 56c_2)}{3072}x^2. \end{aligned}$$

Next we consider the following functional

$$J(c_1,c_2) = \int_0^1 R^2(x,c_1,c_2)dx = \frac{49(64 + 112c_1 + 49c_1^2 + 56c_2)^2}{47185920}.$$

TABLE I
ABSOLUTE ERRORS OF DIFFERENT METHODS FOR EXAMPLE 1

x	VIM $U_5(x)$	HPM $\sum_{k=0}^{5} v_k(x)$	OHAM $u^2(x)$
0.1	4.06901×10^{-7}	4.06901×10^{-7}	0
0.2	1.62760×10^{-6}	1.62760×10^{-6}	0
0.3	3.66211×10^{-6}	3.66211×10^{-6}	0
0.4	6.51042×10^{-6}	6.51042×10^{-6}	0
0.5	1.01725×10^{-5}	1.01725×10^{-5}	0
0.6	1.46484×10^{-5}	1.46484×10^{-5}	0
0.7	1.99382×10^{-5}	1.99382×10^{-5}	0
0.8	2.60417×10^{-5}	2.60417×10^{-5}	0
0.9	0.32959×10^{-4}	0.32959×10^{-4}	0
1.0	4.06901×10^{-5}	4.06901×10^{-5}	0

From the conditions $\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = 0$, we obtain

$$c_1 = -\frac{8}{7}, \;\; c_2 = 0. \tag{12}$$

Substituting Eq.(12) into Eq.(11), we obtain second order approximate solution $u^2(x, c_1, c_2) = x$ which is the exact solution of Eq.(9). Comparison results with other methods are presented in Table I, which contains absolute errors of the approximate solutions obtained from 3 different methods.

Example 2: Consider the following nonlinear integro-differential equation:

$$u'(x) = -e^{-x} - \left(1 - 2e^{-1}\right) x^2 + \int_0^1 x^2 t\, u(t)dt, \;\; u(0) = 1. \tag{13}$$

which has the exact solution $u(x) = e^{-x}$. Eq.(13) can be written as

$$u(x) - e^{-x} + \frac{1}{3}\left(1 - 2e^{-1}\right) x^3 - \int_0^x \int_0^1 \tau^2 t\, u(t)dt\; d\tau = 0. \tag{14}$$

The OHAM formulation of Eq.(14) is

$$L(u(x;p)) = u(x), \;\; N(u(x;p)) = -\int_0^x \int_0^1 \tau^2 t\, u(t)dt\; d\tau,$$
$$g(x) = -e^{-x} + \frac{1}{3}\left(1 - 2e^{-1}\right) x^3,$$

which satisfies

$$\begin{aligned}(1-p)\Big[&\left\{u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right\} - e^{-x} \\ &+ \frac{1}{3}\left(1 - 2e^{-1}\right)x^3\Big] \\ = \left(c_1 p + c_2 p^2 + \cdots\right)&\Big[\left\{u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right\} \\ &- e^{-x} + \frac{1}{3}\left(1 - 2e^{-1}\right)x^3 - \int_0^x \int_0^1 \tau^2 t\, u(t)dt\; d\tau\Big].\end{aligned}$$

By equating the coefficients of the same powers of p, we obtain

$$p^0: u_0(x) = e^{-x} - \frac{(-2+e)}{3e}x^3,$$
$$p^1: u_1(x, c_1) = -c_1 \int_0^x \int_0^1 \tau^2 t\, u_0(t)dt\; d\tau = -\frac{14c_1(-2+e)}{45e}x^3,$$
$$\begin{aligned}p^2: u_2(x, c_1, c_2) &= (1 + c_1)u_1(x, c_1) - c_1 \int_0^x \int_0^1 \tau^2 t\, u_1(t, c_1)dt\; d\tau \\ &- c_2 \int_0^x \int_0^1 \tau^2 t\, u_0(t)dt\; d\tau. \\ &= -\frac{14\left(15c_1 + 14c_1^2 + 15c_2\right)(-2+e)}{675e}x^3.\end{aligned}$$

Thus we have the second order approximate solution of Eq.(13)

$$u^2(x, c_1, c_2) = e^{-x} - \frac{(225 + 420c_1 + 196c_1^2 + 210c_2)(-2+e)}{675e}x^3. \tag{15}$$

Substituting Eq.(15) into Eq.(14), one obtains the following residual

$$\begin{aligned}R(x, c_1, c_2) &= u^2(x, c_1, c_2) - e^{-x} + \frac{1}{3}\left(1 - 2e^{-1}\right)x^3 \\ &- \int_0^x \int_0^1 \tau^2 t\, u^2(t, c_1, c_2)dt\; d\tau \\ &= -\frac{14(225 + 420c_1 + 196c_1^2 + 210c_2)(-2+e)}{10125e}x^3.\end{aligned}$$

Next we consider the following functional

$$\begin{aligned}J(c_1, c_2) &= \int_0^1 R^2(x, c_1, c_2)dx \\ &= \frac{28(225 + 420c_1 + 196c_1^2 + 210c_2)^2(-2+e)^2}{102515625e^2}.\end{aligned}$$

From the conditions $\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = 0$, we obtain

$$c_1 = -\frac{15}{14}, \;\; c_2 = 0. \tag{16}$$

Substituting Eq.(16) into Eq.(15), we obtain the second order approximate solution $u^2(x, c_1, c_2) = e^{-x}$ which is the exact solution of Eq.(13). Comparison results with other methods are presented in Table II, which contains absolute errors of the approximate solutions obtained from 3 different methods.

TABLE II
ABSOLUTE ERRORS OF DIFFERENT METHODS FOR EXAMPLE 2

x	VIM $U_5(x)$	HPM $\sum_{k=0}^{5} v_k(x)$	OHAM $u^2(x)$
0.1	1.55232×10^{-9}	1.55232×10^{-9}	0
0.2	1.24186×10^{-8}	1.24186×10^{-8}	0
0.3	4.19127×10^{-8}	4.19127×10^{-8}	0
0.4	9.93486×10^{-8}	9.93486×10^{-8}	0
0.5	1.94040×10^{-7}	1.94040×10^{-7}	0
0.6	3.35302×10^{-7}	3.35302×10^{-7}	0
0.7	5.32446×10^{-7}	5.32446×10^{-7}	0
0.8	7.94789×10^{-7}	7.94789×10^{-7}	0
0.9	1.13164×10^{-6}	1.13164×10^{-6}	0
1.0	1.55232×10^{-6}	1.55232×10^{-6}	0

Example 3: Consider the second-order integro-differential equation:

$$u''(x) = e^x - x + \int_0^1 xt\, u(t)dt, \;\; u(0) = 1, \;\; u'(0) = 1. \tag{17}$$

which has the exact solution $u(x) = e^x$. Eq.(17) can be written as

$$u(x) - e^x + \frac{1}{6}x^3 - \int_0^x \int_0^\tau \int_0^1 st\, u(t)dt\ ds\ d\tau = 0. \quad (18)$$

The OHAM formulation of Eq.(18) is

$$L(u(x;p)) = u(x), \quad N(u(x;p)) = -\int_0^x \int_0^\tau \int_0^1 st\, u(t)dt\ ds\ d\tau,$$
$$g(x) = -e^x + \frac{1}{6}x^3,$$

which satisfies

$$(1-p)\left[\left(u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right) - e^x + \frac{1}{6}x^3\right]$$
$$= \left(c_1 p + c_2 p^2 + \cdots\right)\Big[\left(u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right)$$
$$- e^x + \frac{1}{6}x^3 - \int_0^x \int_0^\tau \int_0^1 st\, u(t)dt\ ds\ d\tau\Big].$$

By equating the coefficients of the same powers of p, we obtain

$$p^0 : u_0(x) = e^x - \frac{1}{6}x^3,$$
$$p^1 : u_1(x, c_1) = -c_1 \int_0^x \int_0^\tau \int_0^1 st\, u_0(t)dt\ ds\ d\tau$$
$$= -\frac{29c_1}{180}x^3,$$
$$p^2 : u_2(x, c_1, c_2) = (1 + c_1)u_1(x, c_1)$$
$$- c_1 \int_0^x \int_0^\tau \int_0^1 st\, u_1(t, c_1)dt\ ds\ d\tau$$
$$- c_2 \int_0^x \int_0^\tau \int_0^1 st\, u_0(t)dt\ ds\ d\tau.$$
$$= -\frac{29\left(30c_1 + 29c_1^2 + 30c_2\right)}{5400}x^3$$

Thus we have the second order approximate solution of Eq.(17)

$$u^2(x; c_1, c_2) = e^x - \frac{900 + 1740c_1 + 841c_1^2 + 870c_2}{5400}x^3. \quad (19)$$

Substituting Eq.(19) into Eq.(18), one obtains the following residual

$$R(x, c_1, c_2) = u^2(x, c_1, c_2) - e^x + \frac{1}{6}x^3$$
$$- \int_0^x \int_0^\tau \int_0^1 st\ u^2(t, c_1, c_2)dtdsd\tau$$
$$= -\frac{29(900 + 1740c_1 + 841c_1^2 + 870c_2)}{162000}x^3.$$

Next we consider the following functional

$$J(c_1, c_2) = \int_0^1 R^2(x, c_1, c_2)dx$$
$$= \frac{841(900 + 1740c_1 + 841c_1^2 + 870c_2)^2}{183708000000}.$$

From the conditions $\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = 0$, we obtain

$$c_1 = -\frac{30}{29}, \quad c_2 = 0. \quad (20)$$

Substituting Eq.(20) into Eq.(19), we obtain the second order approximate solution $u^2(x, c_1, c_2) = e^x$ which is the exact solution of Eq.(17). Comparison results with other methods are presented in Table III, which contains absolute errors of the approximate solutions obtained from 3 different methods.

TABLE III
ABSOLUTE ERRORS OF DIFFERENT METHODS FOR EXAMPLE 3

x	VIM $U_{10}(x)$	HPM $\sum_{k=0}^{10} v_k(x)$	OHAM $u^2(x)$
0.1	0	1.10517×10^{-10}	0
0.2	0	2.44281×10^{-10}	0
0.3	0	4.04957×10^{-10}	0
0.4	0	5.96730×10^{-10}	0
0.5	2.22045×10^{-16}	8.24361×10^{-10}	0
0.6	2.22045×10^{-16}	1.09327×10^{-9}	0
0.7	4.44089×10^{-16}	1.40963×10^{-9}	0
0.8	8.88178×10^{-16}	1.78043×10^{-9}	0
0.9	8.88178×10^{-16}	2.21364×10^{-9}	0
1.0	1.33227×10^{-15}	2.71828×10^{-9}	0

Example 4: Consider the third-order integro-differential equation:

$$u'''(x) = \sin x - x - \int_0^{\frac{\pi}{2}} xt\, u'(t)dt$$
$$u(0) = 1, \ u'(0) = 0, \ u''(0) = -1, \quad (21)$$

which has the exact solution $u(x) = \cos x$. Eq.(21) can be written as

$$u(x) - \cos x + \frac{1}{24}x^4$$
$$+ \int_0^x \int_0^\eta \int_0^\tau \int_0^{\frac{\pi}{2}} st\ u'(t)dtdsd\tau d\eta = 0. \quad (22)$$

The OHAM formulation of Eq.(22) is

$$L(u(x;p)) = u(x),$$
$$N(u(x;p)) = \int_0^x \int_0^\eta \int_0^\tau \int_0^{\frac{\pi}{2}} st\ u'(t)dtdsd\tau d\eta,$$
$$g(x) = -\cos x + \frac{1}{24}x^4,$$

which satisfies

$$(1-p)\left[\left(u_0(x) + u_1(x) + u_2(x) + \cdots\right) - \cos x + \frac{1}{24}x^4\right]$$
$$=\left(c_1 p + c_2 p^2 + \cdots\right)\Big[\left(u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots\right)$$
$$- \cos x + \frac{1}{24}x^4 + \int_0^x \int_0^\eta \int_0^\tau \int_0^{\frac{\pi}{2}} st\ u'(t)dtdsd\tau d\eta\Big].$$

By equating the coefficients of the same powers of p, one obtains

$$p^0 : u_0(x) = -\frac{1}{24}x^4 + \cos x$$
$$p^1 : u_1(x, c_1) = c_1 \int_0^x \int_0^\eta \int_0^\tau \int_0^{\frac{\pi}{2}} st\ u_0'(t)dtdsd\tau d\eta$$
$$= -\left(\frac{1}{24} + \frac{\pi^5}{23040}\right)c_1 x^4$$
$$p^2 : u_2(x, c_1, c_2) = (1 + c_1)u_1(x, c_1)$$
$$+ c_1 \int_0^x \int_0^\eta \int_0^\tau \int_0^{\frac{\pi}{2}} st\ u_1'(t, c_1)dtdsd\tau d\eta$$
$$+ c_2 \int_0^x \int_0^\eta \int_0^\tau \int_0^{\frac{\pi}{2}} st\ u_0'(t)dtdsd\tau d\eta$$
$$= -\frac{\left(960 + \pi^5\right)\left\{960c_1 + 960c_2 + \left(960 + \pi^5\right)c_1^2\right\}}{22118400}x^4.$$

TABLE IV
ABSOLUTE ERRORS OF DIFFERENT METHODS FOR EXAMPLE 4

x	VIM $U_{10}(x)$	HPM $\sum_{k=0}^{10} v_k(x)$	OHAM $u^2(x)$
0.1	4.13397×10^{-11}	4.13397×10^{-11}	0
0.2	6.61436×10^{-10}	6.61436×10^{-10}	0
0.3	3.34852×10^{-9}	3.34852×10^{-9}	0
0.4	1.05830×10^{-8}	1.05830×10^{-8}	0
0.5	2.58374×10^{-8}	2.58374×10^{-8}	0
0.6	5.35763×10^{-8}	5.35763×10^{-8}	0
0.7	9.92568×10^{-8}	9.92568×10^{-8}	0
0.8	1.69328×10^{-7}	1.69328×10^{-7}	0
0.9	2.71230×10^{-7}	2.71230×10^{-7}	0
1.0	4.13398×10^{-7}	4.13398×10^{-7}	0

This will give us the second order approximate solution of Eq.(21)

$$u^2(x,c_1,c_2)=\cos x-\frac{1}{24}x^4 -\frac{\left(960+\pi^5\right)\left\{1920c_1+\left(960+\pi^5\right)c_1^2+960c_2\right\}}{21233664000}x^4. \tag{23}$$

Substituting Eq.(23) into Eq.(22), one obtains the following residual

$$R(x,c_1,c_2)=u^2(x,c_1,c_2)-\cos x+\frac{1}{24}x^4 +\int_0^x\int_0^\eta\int_0^\tau\int_0^{\frac{\pi}{2}} st\, u^{2\,\prime}(t,c_1,c_2)dtdsd\tau d\eta=-\left[\frac{960+\pi^5}{24} +\frac{\left(960+\pi^5\right)^2\left\{1920c_1+\left(960+\pi^5\right)c_1^2+960c_2\right\}}{22118400}\right]x^4.$$

Next we consider the following functional

$$J(c_1,c_2)=\int_0^1 R^2(x,c_1,c_2)dx=\frac{1}{9}\left[\frac{960+\pi^5}{24} +\frac{\left(960+\pi^5\right)^2\left\{1920c_1+\left(960+\pi^5\right)c_1^2+960c_2\right\}}{22118400}\right]^2.$$

From the conditions $\frac{\partial J}{\partial c_1}=\frac{\partial J}{\partial c_2}=0$, we obtain

$$c_1=-\frac{960}{960+\pi^5},\quad c_2=0. \tag{24}$$

Substituting Eq.(25) into Eq.(23), we obtain the second order approximate solution $u^2(x,c_1,c_2)=\cos x$ which is the exact solution of Eq.(21). Comparison results with other methods are presented in Table IV, which contains absolute errors of the approximate solutions obtained from 3 different methods.

Remark 1: As can be seen in Tables I to IV, OHAM provides more accurate solutions than VIM and HPM. Moreover, OHAM yields the exact solutions of 4 integro-differential equations.

Example 5: Consider the second-order integro-differential equation:

$$u''(x)=x+e^x-xe^x+xu(x)-\int_0^1 xt\, u(t)dt, \qquad u(0)=1,\ \ u'(0)=1. \tag{25}$$

which has the exact solution $u(x)=e^x$. Eq.(25) can be written as

$$u(x)-\frac{1}{6}x^3+x+(x-3)e^x+2 -\int_0^x\int_0^\tau\left(su(s)-\int_0^1 st\, u(t)dt\right)dsd\tau=0. \tag{26}$$

The OHAM formulation of Eq.(26) is

$$L(u(x;p))=u(x),$$
$$N(u(x;p))=-\int_0^x\int_0^\tau\left(su(s)-\int_0^1 st\, u(t)dt\right)dsd\tau,$$
$$g(x)=-\frac{1}{6}x^3+x+(x-3)e^x+2,$$

which satisfies

$$(1-p)\left[(u_0(x)+u_1(x)+u_2(x)+\cdots)-\frac{1}{6}x^3+x+(x-3)e^x+2\right] =(c_1p+c_2p^2+\cdots)\left[(u_0(x)+u_1(x)p+u_2(x)p^2+\cdots)-\frac{1}{6}x^3+x +(x-3)e^x+2-\int_0^x\int_0^\tau\left(su(s)-\int_0^1 st\, u(t)dt\right)dsd\tau\right].$$

By equating the coefficients of the same powers of p, we obtain

$$p^0: u_0(x)=\frac{1}{6}x^3-x-(x-3)e^x-2,$$
$$p^1: u_1(x,c_1)=-c_1\int_0^x\int_0^\tau\left(su_0(s)-\int_0^1 st\, u_0(t)dt\right)dsd\tau,$$
$$p^2: u_2(x,c_1,c_2)=(1+c_1)u_1(x,c_1) -c_2\int_0^x\int_0^\tau\left(su_0(s)-\int_0^1 st\, u_0(t)dt\right)dsd\tau -c_1\int_0^x\int_0^\tau\left(su_1(s,c_1)-\int_0^1 st\, u_1(t,c_1)dt\right)dsd\tau.$$

By solving the above equation, we can easily obtain $u_0(x), u_1(x,c_1)$ and $u_2(x,c_1,c_2)$ which are as follows:

$$u_0(x)=\frac{1}{6}x^3-x-(x-3)e^x-2,$$
$$u_1(x,c_1)=-\left\{\frac{e}{6}x^3-(12-7x+x^2)e^x +\frac{2160+900x-171x^3-15x^4+x^6}{180}\right\}c_1$$
$$u_2(x,c_1,c_2)=-\left\{\frac{2160+900x+(30e-171)x^3-15x^4+x^6}{180} -(12-7x+x^2)e^x\right\}(c_1+c_2)-\left\{102+37x+\frac{(x^3-301)e}{180}x^3 +\frac{6386163+453600x-33768x^3-1800x^4+70x^6}{907200}x^3 +(102-65x+14x^2-x^3)e^x\right\}c_1^2.$$

This will give us the second order approximate solution of Eq.(25)

$$u^2(x,c_1,c_2)=-2+(3-x)e^x-x+\frac{1}{6}x^3-\frac{(2c_1+c_2)}{180}\left\{2160 +900x-15x^4+x^6+(30e-171)x^3-180(12-7x+x^2)e^x\right\} -c_1^2\left\{102+37x-(102-65x+14x^2-x^3)e^x-\frac{(x^3-301)e}{180}x^3 -\frac{6386163+453600x-33768x^3-1800x^4+70x^6}{907200}x^4\right\}. \tag{27}$$

Substituting Eq.(27) into Eq.(26), one obtains the following residual

$$
\begin{aligned}
R(x,c_1,c_2) &= u^2(x,c_1,c_2) - \frac{1}{6}x^3 + x + (x-3)e^x + 2 \\
&- \int_0^x \int_0^\tau \left(su^2(s,c_1,c_2) - \int_0^1 st\, u^2(t,c_1,c_2)dt \right) ds d\tau \\
= &- \frac{2160 - 180(12 - 7x + x^2)e^x + 900x + (30e - 171)x^3 - 15x^4 + x^6}{180} \\
&- \left\{ 102 + 37x - (102 - 65x + 14x^2 - x^3)e^x - \frac{(x^3 - 301)e}{180}x^3 \right. \\
&\left. - \frac{6386163 + 453600x - 33768x^3 - 1800x^4 + 70x^6}{907200}x^3 \right\} (2c_1 + c_2) \\
&- \left\{ \frac{231(2427129 - 7944x^3 + 10x^6)e}{29936600}x^3 - (1152 - 775x + 199x^2 \right. \\
&- 23x^3 + x^4)e^x + 1152 + 337x - \frac{1170863821}{16632000}x^3 - \frac{43}{12}x^4 \\
&\left. + \frac{352343}{1296000}x^6 + \frac{1}{72}x^7 - \frac{77}{129600}x^9 - \frac{1}{45360}x^{10} + \frac{1}{1710720}x^{12} \right\} c_1^2.
\end{aligned}
$$

Next we consider the following functional

$$
\begin{aligned}
J(c_1,c_2) &= \int_0^1 R^2(x,c_1,c_2)dx \\
&= \frac{1013265229541}{648648000} + \frac{66861054696442043}{817296480000}a \\
&+ \frac{27841838851376910661223}{14620805337600000}a^2 \\
&+ \frac{5772215438200688758943892229}{250893019593216000000}a^3 \\
&+ \frac{72992871975652851854821735160653}{6059066423176166400000000}a^4 \\
&+ \frac{66861054696442043}{1634592960000}b \\
&+ \frac{592665551175606496204449}{47517617347200000}ab \\
&+ \frac{5772215438200688758943892229}{501786039186432000000}a^2b \\
&+ \frac{592665551175606496204449}{190070469388800000}b^2 \\
&- \frac{124053497e}{151200} - \frac{5078990728703e}{121621500}a \\
&- \frac{20514291558562410 97e}{2179457280000}a^2 \\
&- \frac{1044494971283348984072911e}{95035234694400000}a^3 \\
&- \frac{56965560823031777167346 2801e}{10240531411968000000}a^4 \\
&- \frac{5078990728703e}{243243000}b - \frac{167971263315422167e}{272432160000}ab \\
&- \frac{1044494971283348984072911e}{190070469388800000}a^2b \\
&- \frac{167971263315422167e}{1089728640000}b^2 + \frac{11393e^2}{126} \\
&+ \frac{3862283e^2}{900}a + \frac{3133156163881e^2}{35380800}a^2 \\
&+ \frac{939717091997093e^2}{1010880000}a^3 \\
&+ \frac{604122757613350873721e^2}{145202803200000}a^4 + \frac{3862283e^2}{1800}b \\
&+ \frac{3394339093e^2}{58500}ab + \frac{939717091997093e^2}{2021760000}a^2b \\
&+ \frac{3394339093e^2}{234000}b^2.
\end{aligned}
$$

From the conditions $\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = 0$, we obtain

$$
c_1 = 0.9915805215562866, \quad c_2 = -3.9661219572506776. \tag{28}
$$

Substituting Eq.(28) into Eq.(27), we obtain the second order approximate solution of Eq.(25)

TABLE V

ABSOLUTE ERRORS OF DIFFERENT METHODS FOR EXAMPLE 5

x	VIM $U_5(x)$	HPM $\sum_{k=0}^{5} v_k(x)$	OHAM $u^2(x)$
0.1	7.56062×10^{-13}	2.21377×10^{-10}	9.99099×10^{-9}
0.2	8.82561×10^{-12}	4.97696×10^{-10}	7.75976×10^{-8}
0.3	2.89562×10^{-11}	8.37023×10^{-10}	2.76824×10^{-7}
0.4	5.82011×10^{-11}	1.25213×10^{-9}	6.31636×10^{-7}
0.5	5.54410×10^{-11}	1.73329×10^{-9}	1.02533×10^{-6}
0.6	7.02374×10^{-11}	2.26272×10^{-9}	1.14733×10^{-6}
0.7	2.95177×10^{-11}	2.79340×10^{-9}	5.87191×10^{-7}
0.8	2.70028×10^{-10}	3.28422×10^{-9}	7.19874×10^{-7}
0.9	6.63621×10^{-10}	3.76336×10^{-9}	1.33844×10^{-6}
1.0	1.09054×10^{-9}	4.34348×10^{-9}	4.12072×10^{-6}

$$
\begin{aligned}
u^2(x,c_1,c_2) &= -78.4941 - 27.4648x + 1.63326x^3 + 0.326369x^4 \\
&- 0.0107333x^6 - 0.00195086x^7 + 0.0000758667x^9 \\
&- 0.983232e^x(-4.24001 + x)(19.0683 - 7.74321x + x^2).
\end{aligned}
$$

Comparison results with other methods are presented in Table V, which contains absolute errors of the approximate solutions obtained from 3 different methods.

IV. Conclusion

In this paper, we presented the OHAM for solving linear and nonlinear integro-differential equations. Several examples, including some well known problems, showed that OHAM compared with VIM and HPM is a reliable, efficient and powerful method for solving linear and nonlinear integro-differential equations. Therefore, we believe that the OHAM is an expectable technique for solving linear and nonlinear integro-differential equations. The computations associated with examples in this paper were performed using Mathematica.

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